Contemporary Mathematics
Volume 131, 1992 (Part 3)

Positivity of sheaves and geometric invariant theory

Eckart Viehweg
FB6 Mathematik
Universität-GHS Essen
D-4300 Essen 1, Germany

1 Introduction

T. Fujita, Y. Kawamata and the author showed that direct images of powers of dualizing sheaves have a certain positivity property, called now "weak positivity". For families of compact complex manifolds with non trivial moduli, one hopes that those sheaves satisfy stronger positivity properties (see [4] or [8] for the exact statements and references). If the fibres are of general type or if they have minimal models this has been shown by Y. Kawamata, J. Kollár and the author (loc.cit.), and in those proofs, global or local moduli or Hodge theoretic estimates of the kernel of the multiplication map played a central role.

It turned out, however, that for families of manifolds of general type, one can avoid those methods by some universal bundle construction and Plücker coordinates on Grassmann manifolds. May be, it is not too surprising that the better understanding of positive sheaves, induced by families of manifolds, had also some implications for the construction of moduli schemes (see [9]):

Theorem 1.1 *Let h be a polynomial of degree n and*

$$\mathcal{M}_h : Schemes/\mathbb{C} \rightarrow Sets$$

be one of the following moduli functors:
a) The moduli functor of compact complex canonically polarized manifolds

1991 *Mathematics Subject Classification.* Primary 14D22; Secondary 14J10.
This paper is in final form and no version of it will be submitted for publication elsewhere.

F with Hilbert polynomial h, up to isomorphisms.
b) The moduli functor of compact complex inhomogeneously polarized manifolds $(F, \mathcal{L})$ with Hilbert polynomical h and ω_F numerically effective, up to isomorphisms and linear equivalence. (i.e.: $(F, \mathcal{L}) \sim (F', \mathcal{L}')$ if there exist isomorphisms $\tau : F \to F'$ and $\tau^* \mathcal{L}' \simeq \mathcal{L}$.)
Then there exists a quasi-projective coarse moduli space M_h for $\mathcal{M}_h$ (Here we are cheating a little bit, see 5.3 for the exact definition of the moduli functor M_h in case b).

Of course, the moduli functor considered in b) is not the "right" one. One wants usually to classify polarized manifolds $(F, \mathcal{L})$ up to $\mathbb{Q}$ - numerical equivalence, as explained in [6] or [7]. (i.e. $(F, \mathcal{L}) \sim (F', \mathcal{L}')$ if there is an isomorphism $\tau : F \to F'$ such that $\tau^* \mathcal{L}'^a$ is numerically equivalent to $\mathcal{L}^b$ for some $a, b \in \mathbb{Z}$). Those two moduli problems coincide only if one has $h^0(F, \Omega^1_F) = 0$.

In fact, in Theorem 1.1 one can allow F to have normal rational Gorenstein singularities, as long as the moduli functor stays bounded and separated. Since this holds true for $n = 2$ (see [2] for other cases) we obtain as a corollary:

Corollary 1.2
a) (D. Mumford, D. Gieseker) *There exist quasi-projective moduli schemes for curves and surfaces of general type.*
b) (I. I. Pjatetskij-Šapiro, I. R. Šafarevich) *There exist quasi-projective moduli spaces for polarized $K3$ surfaces.*

However, we have to confess, that the ample sheaves obtained by the authors quoted in 1.2 are much nicer and more natural than the ones obtained in [9].

The proof of 1.1 uses D. Mumford's geometric invariant theory. A second possible approach, due to J. Kollár, consists in constructing M_h as an analytic space first and then to use the positivity theorems mentioned above to construct an ample sheaf on M_h (see [3], [9],I,5.18 and II,§5). The ampleness criterion, used to this aim, is quite close to the methods which will be discussed in §3. However, one can only apply it if the moduli problem has a reasonable compactification or if M_h is normal.

In this survey, we will try to work out in §2 and §3 the stability criterion used to prove 1.1. Hence the formulation of 2.8 and 3.2 will be new, but not their proofs. In §4 we present the two "hard positivity theorems" (4.1 and 4.4) and sketch the minimizing principle used to obtain "weakly positive polarizations". Then, §5 gives a short outline, how to finish the proof of 1.1.

So, §4 and §5 are just supposed to be a very coarse guide- line to [9] and the word "proof" just means "idea of the proof" in both sections. Nevertheless we hope that the slightly more general point of view will help to clarify some parts of [9] and convince some of the readers to use similar methods for different moduli problems.

We keep the conventions from [9], II. Especially the reader should keep in mind that all schemes are supposed to be separated and of finite type over $\mathbb{C}$. Moreover, locally free sheaves are always supposed to be of constant finite rank (It seems however, that the stability criteria of §1 and §2 work as well in characteristic $p > 0$).

We refere to the three parts of [9] by I, II and III.

I would like to thank the organizers of the conference in honor of Prof. A.I. Maltsev for the invitation to participate.

2 Stability and weak positivity

Notations 2.1 Let $G = \mathbb{P}Gl(r, \mathbb{C})$ be acting properly on a scheme H. Hence we have maps $\sigma : G \times H \to H$ and $pr_2 : G \times H \to H$ such that

$$(\sigma, pr_2) : G \times H \to H \times H$$

is proper. Recall that a locally free sheaf $\mathcal{L}$ on H is called G-linearized, as in [5], Def. 1.6, if one has an isomorphism $\sigma^*\mathcal{L} \to pr_2^*\mathcal{L}$. Let $\mathcal{L}$ be an invertible G-linearized sheaf on H. In fact, it will be enough, if $\mathcal{L}^p$ is G-linearized for some $p > 0$. We will not destinguish those two cases. If $x \in H$, then H_x will denote the G-orbit of x in H.

Definition 2.2 ([5], Def. 1.7).
i) A geometric point $x \in H$ is called *stable* (with respect to $\mathcal{L}$ and σ) if there exist for some $N > 0$ a G-invariant section $s \in H^0(H, \mathcal{L}^N)$ such that: $H - V(s)$ is affine, $x \in H - V(s)$ and G acts on $H - V(s)$ with closed orbits.
ii) We write $H(\mathcal{L})^s$ for the set of stable points with finite stabilizers.

Assumptions 2.3 Let $\mathcal{L}_0$ and $\mathcal{N}$ be invertible sheaves on H, both G-linearized. Assume that $\mathcal{L}_0$ is ample. We write $\mathcal{L}_\eta = \mathcal{L}_0 \otimes \mathcal{N}^\eta$. If $j : H \to H'$ is a projective compactification, we will write H'_x for the closure of the orbit H_x. Let us assume that G acts with finite stabilizers.

Stability criterion 2.4 Let x be a point of H. Assume that there is a projective compactification $j : H \to H'$ and for some $\alpha > 0$ a coherent subsheaf $\mathcal{L}'$ of $j_*(\mathcal{L}_\eta)^\alpha$ such that:
a) The sheaf $\mathcal{L}'$ is generated by some subspace $V \subset H^0(H', \mathcal{L}')$, one has $\mathcal{L}' \mid_H = (\mathcal{L}_\eta)^\alpha$ and the induced mophism $H \to \mathbb{P}(V)$ is an embedding.
b) There is an effective Cartier divisor D_x on H'_x with $H'_x - D_x = H_x$ and an inclusion $\mathcal{O}_{H'_x}(D_x) \to \mathcal{L}' \mid_{H'_x}$ surjective over H_x.
Then $x \in H(\mathcal{L}_\eta)^s$.

PROOF. We may assume $\alpha = 1$. By [5],I,§1, we can find a G-invariant subspace V' of $H^0(H, \mathcal{L}_\eta)$ which contains V. Choosing $\mathcal{L}'$ even bigger we may assume that V is G-invariant. Replacing H' by the closure of H in $\mathbb{P}(V)$ and $\mathcal{L}'$ by $\mathcal{O}_{\mathbb{P}(V)}(1) \mid_{H'}$ we may assume that G acts on H' and that $\mathcal{L}'$ is G-linearized. Moreover we can choose D_x in b such that $\mathcal{L}' \mid_{H'_x} = \mathcal{O}_{H'_x}(D_x)$. Then $x \in H'_x(\mathcal{L}' \mid_{H'_x})^s$. In fact, since $(D_x)_{red}$ is G-invariant, some sum over finitly many conjugates of D_x will be G-invariant. Hence, the corresponding section satisfies the assumptions asked for in 2.2 By [5], Prop. 1.18, we have

$$x \in H'_x(\mathcal{L}' \mid_{H'_x})^s = H'(\mathcal{L}')^s \cap H'_x$$

and hence by [5], Prop. 1.19,

$$x \in H'(\mathcal{L}')^s \cap H \subseteq H(\mathcal{L}_\eta)^s.$$

Remarks 2.5
a) Assume that for all $x \in H$ we can find some η_0 such that the assumptions made in 2.4 hold true for all multiples η of η_0. Then, since $H(\mathcal{L}_\eta)^s$ is Zariski open, we will get $H = H(\mathcal{L}_\eta)^s$ for some $\eta >> o$.
b) By [5], Prop. 1.16, we know that $H_{red}(\mathcal{L}_\eta)^s = (H(\mathcal{L}_\eta)^s)_{red}$. Therefore we may assume in the sequel that H is a reduced scheme.

In order to be able to formulate a second stability criterion we have to introduce some more notations:

Definition 2.6 Let Y be a quasi-projective reduced complexe scheme, and $j : Y_0 \to Y$ a dense open subscheme. Let $\mathcal{F}$ be a coherent sheaf on Y such that $\mathcal{F}_0 = j^*\mathcal{F}$ is locally free. Then we call $\mathcal{F}$ *weakly positive over* Y_0 if for all (or one) ample invertible sheaves $\mathcal{H}$ on Y and all $a > 0$ there is some $b > 0$ such that the sheaf $\mathcal{G} = S^{a \cdot b}(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated over Y_0.

Recall that $\mathcal{G}$ is globally generated over Y_0 if the map

$$H^0(Y, \mathcal{G}) \otimes_{\mathbb{C}} \mathcal{O}_Y \to \mathcal{G}$$

is surjective over Y_0.

Moreover, we will use the convention that for all finite dimensional representations T the tensor bundle $T(\mathcal{F})$ is the reflexive hull of $T(\mathcal{F} \mid_U)$, where U is the largest open subscheme where $\mathcal{F}$ is locally free.

Convention 2.7 In this note we will call $\mathcal{F}_0$ weakly positive over Y_0 in a bounded way, if we can find some Y projective compactification Y of Y_0 and some extension $\mathcal{F}$ of $\mathcal{F}_0$ to Y and $\mathcal{F}$ satisfying 2.6.

The boundednes will be needed later in order to get H' and some extension $\mathcal{N}'$ of $\mathcal{N}$ to H. However, in order to prove that the sheaves considered in the proof of theorem 1.1 have this property one better uses a more explicite definition "weakly positive with respect to some desingularization of Y". Since we will not carry out all the nescessary calculations we will not use this definition (see II, 2.2).

Let us return to the situation considered in 2.1 and 2.4:
The group G is an open subvariety of $\mathbb{P} = \mathbb{P}(\oplus^r \mathbb{C}^r)$. Let H' and Y be projective compactifications of H and $G \times H$ respectively, such that the three morphisms σ, pr_2 and pr_1 extend to

$$\varphi : Y \to H' \ , \ p_2 : Y \to H' \text{ and } p_1 : Y \to \mathbb{P} \ .$$

Let us write $U = \varphi^{-1}(H), V = p_2^{-1}(H)$ and $Y_0 = U \cup V$.

Since we assumed G to act properly we have $U \cap V = G \times H$. Let $\mathcal{L}'_0$ and $\mathcal{N}'$ be invertible sheaves on H' such that $\mathcal{L}'_0 \mid_H = \mathcal{L}_0^{\alpha}$ and such that $\mathcal{N}' \mid_H = \mathcal{N}^{\alpha}$. We will assume that $\mathcal{L}'_0$ is ample on H'. Again we write $\mathcal{L}'_\eta = \mathcal{L}'_0 \otimes \mathcal{N}'^{\eta}$.

Stability criterion 2.8 Assume that:
a) $\mathcal{N}$ is weakly positive over H_{red} in a bounded way
b) There exist an invertible sheaf $\mathcal{M}'$ on Y which satisfies:
i) $\mathcal{M}'$ is weakly positive over $(Y_0)_{red}$.
ii) For some effective Cartier divisor D' on Y with $Y - D' = p_2^{-1}(H)$ and for some $\mu > 0$ one has

$$\mathcal{M}'(-D') \mid_{\varphi^{-1}(H)} = \varphi^* \mathcal{N}'^{\mu} \mid_{\varphi^{-1}(H)}$$

$$\text{and}\quad \mathcal{M}' \mid_{p_2^{-1}(H)} = p_2^* \mathcal{N}'^{\mu} \mid_{p_2^{-1}(H)} .$$

Then, for all multiples η of some given $\eta_0 > 0$ one has $H = H(\mathcal{L}_\eta)^s$.

PROOF Since 2.8, without being stated, is proved in I §5 and III, §4 we just sketch the arguments needed:

First of all, as we remarked in 2.5, we can assume H to be reduced and we will just show that some given x lies in $H(\mathcal{L}_\eta)^s$. We can assume that $\alpha = 1$ and $\mu = 1$ in 2.8, b), and that $H' - H$ is the support of some effective Cartier divisor Γ. Replacing $\mathcal{N}'$ by $\mathcal{N}'(\rho \cdot \Gamma)$ we can assume that $\mathcal{N}'$ is weakly positive over H and that there is an inclusion $\mathcal{M}' \to p_2^* \mathcal{N}'$ surjective over $p_2^{-1}(H)$.

For $\mu, \gamma >> 0$ one can find, blowing up Y if necessary, an ample invertible subsheaf $\mathcal{H}'$ of $p_1^* \mathcal{O}_{\mathbb{P}}(\mu) \otimes p_2^* \mathcal{L}'^{\gamma}_0$, such that both sheaves are isomorphic over $G \times H$. If we choose $\mathcal{H}'$ small enough, then the inclusion $\mathcal{H}'^{\beta} \hookrightarrow p_1^* \mathcal{O}_{\mathbb{P}}(\mu \cdot \beta) \otimes p_2^* \mathcal{L}'^{\gamma \cdot \beta}_0$ will induce an inclusion.

$$\rho : p_{2*} \mathcal{H}'^{\beta} \otimes \mathcal{N}'^{\eta \cdot \beta} \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu \cdot \beta)) \otimes_{\mathbb{C}} \mathcal{L}'^{\gamma \cdot \beta}_0 \otimes \mathcal{N}'^{\eta \cdot \beta}$$

for all $\beta > 0$.

Let $U'_x = \varphi^{-1}(x)$ and $U_x = U'_x \cap G \times H$. Then, for some Cartier divisor Δ_x we will have $\mathcal{H}' \mid_{U'_x} = \mathcal{O}_{U'_x}(\Delta_x)$. By b,ii) the sheaf $\mathcal{M}' \mid_{U'_x}$ has a section whose zero set D_x is exactly supported in $U'_x - U_x$. Hence, if we choose $\eta >> 0$ the sheaf $\mathcal{H}' \otimes \mathcal{M}'^{\eta} \mid_{U'_x}$ will have a section σ whose zero locus $\Delta_x + \eta \cdot D_x$ will be effective and exactly supported in $U'_x - U_x$.

Since $\mathcal{M}'$ is weakly positive over some neighbourhood of U'_x, one can lift for some $\beta > 0$ the section σ^{β} to a section s of $\mathcal{H}'^{\beta} \otimes \mathcal{M}'^{\eta \cdot \beta}$, using I, 3.2 for example. In fact, we can as well assume that $\mathcal{H}'^{\beta} \otimes \mathcal{M}'^{\eta \cdot \beta} \mid_{U'_x}$ is generated by sections lifting to $H^0(Y, \mathcal{H}'^{\beta} \otimes \mathcal{M}'^{\eta \cdot \beta})$. Let us consider the "trace map"

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu \cdot \beta)) \to p_{2*} \mathcal{O}_{U'_x}(* \cdot D_x) \to \mathcal{O}_{H'_x}(* \cdot \Gamma_x)$$

where $\Gamma_x = H'_x \cap \Gamma$. The image will be in some subsheaf of $\mathcal{O}_{H'_x}(\beta \cdot \delta \cdot \Gamma_x)$ for some δ independent of β.

On the other hand the sections of $\mathcal{H}'^{\beta} \otimes \mathcal{M}'^{\beta \cdot \eta}$ generate under the composed map

$$p_{2*} \mathcal{H}'^{\beta} \otimes \mathcal{M}'^{\beta \cdot \eta} \longrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(\mu \cdot \beta)) \otimes_{\mathbb{C}} \mathcal{L}'^{\gamma \cdot \beta}_0 \otimes \mathcal{N}'^{\eta \cdot \beta} \longrightarrow$$

$$\longrightarrow p_{2*} \mathcal{O}_{U'_x}(* \cdot \Delta_x) \longrightarrow \mathcal{O}_{H'_x}(* \cdot \Gamma_x)$$

a subsheaf of $\mathcal{O}_{H'x}(* \cdot \Gamma_x)$ which contains $\mathcal{O}_{H'_x}(\zeta \cdot \Gamma_x)$ where ζ is rising like $\eta \cdot \beta$. This can only happen, if for $\eta >> 0$ the subsheaf $\mathcal{G}$ of $\mathcal{L}'^{\gamma \cdot \beta}_0 \otimes \mathcal{N}'^{\eta \cdot \beta}$ which is generated by global sections satisfies the assumption 2.4.b. The assumption 2.4.a is obvious, since $\mathcal{N}'$ is weakly positive over H.

3 Applications to special group actions

We want to apply 2.8 to group actions induced by coordinate changes in some free bundle. To this aim it is more convenient to lift the action of $\mathbb{P}Gl(r, \mathbb{C})$ to the group $Sl(r, \mathbb{C})$. Hence in this section we write

$$G = Sl(r, \mathbb{C}) .$$

Correspondingly the map $p_1 : Y \longrightarrow \mathbb{P}$ in 2.7 will now be an extension to Y of the composed map

$$G \times H \longrightarrow G \longrightarrow \mathbb{P}Gl(r, \mathbb{C}) .$$

We keep the other notations and assumptions from 2.3 and 2.8. So G acts properly on H with finite stabelizers and $\mathcal{L}_0$ and $\mathcal{N}$ are G-linearized invertible sheaves, $\mathcal{L}_0$ ample. Let us assume that we have a G-linearization

$$\Phi : \sigma^* \oplus^r \mathcal{N} \to pr_2^* \oplus^r \mathcal{N}.$$

Φ allows to lift the G-action σ to

$$\sigma' : G \times H \times \mathbb{P}^{r-1} = \mathbb{P}(\sigma^* \oplus^r \mathcal{N}) \simeq \mathbb{P}(pr_2^* \oplus^r \mathcal{N}) \to H \times \mathbb{P}^{r-1}.$$

The G-linearization Φ induces a G-linearization Φ' of $\mathcal{O}_{\mathbb{P}(\sigma^* \oplus^r \mathcal{N})}(1)$.

Assumption 3.1 σ' is induced by $\sigma : G \times H \to H$ and the usual G-action

$$\tau : G \times \mathbb{P}^{r-1} \longrightarrow \mathbb{P}^{r-1}.$$

(i.e.: $\sigma'(g, h, p) = \sigma(g, h) \times \tau(g, p)$), and Φ' is induced by the G-linearizations of $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$ and $\mathcal{N}$.

For φ, p_1, p_2, U, V and Y_0 as in 1.9 the sheaves $\varphi^*(\oplus^r \mathcal{N})$ and $p_2^*(\oplus^r \mathcal{N})$ coincide on $U \cap V = G \times H$, using Φ. We can glue them together to obtain a locally free sheaf $\mathcal{F}_0$ on Y_0.

The main assumption, replacing b) in 2.8, is just the weak positivity of $\mathcal{F}_0$:

Theorem 3.2 *Let $G, H, \mathcal{L}_0$ and $\mathcal{N}$ be as above and Φ a G-linearization of $\oplus^r \mathcal{N}$ satisfying 3.1. Assume moreover that $\mathcal{N}$ is weakly positive over H_{red} in a bounded way and that $\mathcal{F}_0$ is weakly positive over $(Y_0)_{red}$ in a bounded way. Then, for all multiples η of some $\eta_0 > 0$, and $\mathcal{L}_\eta = \mathcal{L}_0 \otimes \mathcal{N}^\eta$ one has $H = H(\mathcal{L}_\eta)^s$.*

PROOF. (see also I, §5 and III, §4). Again we can assume that all the schemes are reduced. Let us write $\mathcal{B}_0 = (p_2 |_V)^* \mathcal{N}$. Then we have an inclusion $\mathcal{B}_0 \to \oplus^r \mathcal{F}_0 |_V$, induced by $\oplus^r \mathcal{B}_0 \to \mathcal{F}_0 |_V$. Blowing up Y, if nescessary, we can extend $\mathcal{B}_0$ to a subbundle $\varepsilon : \mathcal{B} \to \oplus^r \mathcal{F}_0$.
Let $s : \oplus^r \mathcal{B} \to \mathcal{F}_0$ be the induced map. If D is the degeneration divisor of s, then $\mathcal{B}^r = det(\mathcal{F}_0) \otimes \mathcal{O}_{Y_0}(-D)$. The dual of ε induces a surjection

$$S^r \oplus^r \wedge^{r-1} \mathcal{F}_0 = S^r \oplus^r \mathcal{F}_0^\vee \otimes det(\mathcal{F}_0) \to \mathcal{B}^{-r} \otimes det(\mathcal{F}_0)^r = det(\mathcal{F}_0)^{r-1} \otimes \mathcal{O}_{Y_0}(D).$$

Since weak positivity is compatible with quotients and positive tensor bundles (I,§3), the sheaf $\mathcal{M} = det(\mathcal{F}_0)^{r-1} \otimes \mathcal{O}_{Y_0}(D)$ is weakly positive ove Y_0 in a bounded way.
Blowing up again, we can assume that there is some invertible extension $\mathcal{M}'$ of $\mathcal{M}$ to Y, weakly positive over Y_0. Let D' be the closure of D in Y, enlarged by the components of $Y - Y_0$. Then we have $\mathcal{M}(-D') |_U = (\varphi |_U)^* \mathcal{N}^{r(r-1)}$ and $\mathcal{M} |_V = (p_2 |_V)^* \mathcal{N}^{r(r-1)}$. Hence, in order to be able to apply 2.8 we just have to show that $Y - D' = V$ or, $Y_0 - D = V$.

In fact, it is enough to show this equality on $U'_x = \varphi^{-1}(x)$ for all x. We have a map $p_1 : U'_x \to \mathbb{P}$, and

$$s |_{U'_x} : \oplus^r \mathcal{B} |_{U'_x} \to \oplus^r \mathcal{O}_{U'_x} = \mathcal{F}_0 |_{U'_x}$$

is nothing but the pullback of the universal bases on $\mathbb{P}$, i.e. of the map

$$\oplus^r \mathcal{O}_{\mathbf{P}}(-1) \to \mathcal{O}_{\mathbf{P}} \otimes_{\mathbb{C}} \mathbb{C}^r,$$

induced by the tautological map

$$\oplus^r \mathcal{O}_{\mathbf{P}} \otimes_{\mathbb{C}} \mathbb{C}^r \to \mathcal{O}_{\mathbf{P}}(1).$$

Hence, $D \cap U'_x$ is the pullback of $\mathbb{P} - G$.

Remarks 3.3

a) Up to now we did not really use that we are working over $\mathbb{C}$. I think one should be able to prove 3.2 in characteristic $p > 0$ in a similar way.
b) If H is an open subscheme of some Hilbert scheme or of some scheme

parametrizing quotients of locally free sheaves, one will get some $\oplus^r \mathcal{N}$ and some Φ such that the assumptions made in 3.1 hold true. Hence the cruical assumption in 3.2 is the one, asking $\mathcal{F}_0$ to be weakly positive over Y_0. Without refering to some geometrical situation which allows some "natural choice" of $\mathcal{F}_0$, there seems to be no way, to find some $\mathcal{N}$ such that $\mathcal{F}_0$ satisfies this assumption.

4 Examples of weakly positive sheaves

The starting point is the

Theorem 4.1 *Let $g : Y \to S$ be a smooth surjective projective morphism of reduced quasi-projective schemes. Then $g_*\omega_{Y/S}$ is weakly positive over S in a bounded way.*

This theorem, due to T. Fujita, if S is a smooth curve, and to Y. Kawamata, if S is any smooth quasi-projective scheme, is quite difficult to prove.

One has to find an extension of $g_*\omega_{Y/S}$ to some compactification S' of S, such that over a desingularization of S' this extension pulls back to some canonical extension, coming from Hodge theory. This crucial point in based on an extension criterion of Ofer Gabber, which is reproved in II in a slightly different set up.

We will need a stronger version of 4.1. To this aim, let us introduce some number which is a measure for the singularities of a divisor (see [1]):

Definition 4.2 Let Z be a manifold, $\mathcal{M}$ an invertible sheaf and Γ an effective divisor.
a) Let $\tau : Z' \to Z$ be a blowing up, such that $\Gamma' = \tau^*\Gamma$ is a normal crossing divisor. Then $e(\Gamma) = Min\{N \in \mathbb{N} - \{o\}; \tau_*\omega_{Z'/Z}(-[\frac{\Gamma'}{N}]) = \mathcal{O}_Z\}$.
b) $e(\mathcal{M}) = Max\{e(\Gamma); \Gamma \text{ zero divisor of } s \in H^0(Z, \mathcal{M})\}$.

Of course, one has to show that $e(\Gamma)$ is independent of Z' and that $e(\mathcal{M})$ is finite, at least if Z is compact. The main observation taken from [1] is:

Lemma 4.3 *If $\mathcal{M}$ is very ample and Z compact, then*

$$e(\mathcal{M}^\nu) \leq \nu \cdot c_1(\mathcal{M})^{dim Z} + 1.$$

Moreover, if $X = Z \times \ldots \times Z$ and $\mathcal{H} = \bigotimes_{i=1}^r pr_i^*\mathcal{M}$, then

$$e(\mathcal{H}^\nu) \leq \nu \cdot c_1(\mathcal{M})^{dim Z} + 1$$

as well.

Theorem 4.4 *Let $g : Y \to S$ be a smooth surjective projective morphism, and Γ an effective Cartier divisor and $e \in \mathbb{N}$ such that for all fibres F of g one has $F \cap \Gamma \neq F$ and $e(\Gamma \mid_F) \leq e$. Assume moreover, that $\mathcal{L}$ is an invertible sheaf such that $\mathcal{L}^e(-\Gamma)$ is semi ample and $g_*(\mathcal{L} \otimes \omega_{Y/S})$ locally free and compatible with arbitary base change.*
Then $g_*(\mathcal{L} \otimes \omega_{Y/S})$ is weakly positive in a bounded way.

If $\Gamma = 0$, then 4.4 follows directly from 4.1, applied to finite coverings of Y. In general, one has to repeat part of the proof of 4.1 in this more general set up (see III).

The "minimizing principle" 4.5 which will appear in the proof of 4.6 and 4.7 is based on the following quite simple observation (I, 3.4 or II, 2.4): If $\mathcal{F}$ is a locally free sheaf on S and if for some constant μ , for all finite covers $\tau : S' \to S$ and all ample sheaves $\mathcal{H}'$ on S' one knows that $\tau^*\mathcal{F} \otimes \mathcal{H}'^{\mu}$ is weakly positive over S', then $\mathcal{F}$ is weakly positive (In fact, this was used several times during the last couple of years to show that direct images of powers of dualizing sheaves are weakly positive over some non empty open set).
In order to find such μ one starts with some minimal number m such that $\tau^*\mathcal{F} \otimes \mathcal{H}'^{m}$ is weakly positive. If $\mathcal{F}$ is some direct image sheave one tries to use 4.4 to get another number m' , depending on m , such that $\tau^*\mathcal{F} \otimes \mathcal{H}'^{m'}$ is weakly positive as well. If one is lucky, the corresponding inequality $m' > m - 1$ gives some bound for m , independent of τ.

Theorem 4.6 *Let $g : Y \to S$ be a smooth surjective projective morphism of reduced quasi-projective schemes and assume that for some $N > 0$ the map $g^* g_* \omega^N_{Y/S} \to \omega^N_{Y/S}$ is surjecitve.*
Then $g_*\omega^\nu_{Y/S}$ is weakly positive over S in a bounded way for all $\nu > 0$.

Theorem 4.7 *Let $g : Y \to S$ be a smooth surjective projective morphism of reduced quasi-projective schemes , $\mathcal{M}$ an invertible sheaf on Y, relatively ample for the morphism g. Assume moreover that*
a) $r = h^0(F, \mathcal{M} \mid_F)$ and $e = c_1(\mathcal{M} \mid_F)^n + 1$ are the same for all fibres F of g.
b) $g^* g_*(\mathcal{M} \otimes \omega^e_{Y/S}) \to \mathcal{M} \otimes \omega^e_{Y/S}$ is surjective.
c) $g_*(\mathcal{M} \otimes \omega^e_{Y/S})$ and $g_*\mathcal{M}$ are both locally free and compatible with arbitary base change.
Then $(\bigotimes^r g_*(\mathcal{M} \otimes \omega^e_{Y/S})) \otimes det(g_*\mathcal{M})^{-1}$ is weakly positive over S in a bounded way.

Let me sketch the proof of 4.7. The proof of 4.6 is similar. Replacing S by some finite cover one can assume that $det(g_*\mathcal{M})$ is an $r-th$ power and then we can assume as well that it is $\mathcal{O}_S$. For simplicity, let us drop the "bounded way". If $\mathcal{H}$ is any ample invertible sheaf on S, define

$$m = Min\{\mu \in \mathbb{N}; \mathcal{H}^{\mu\cdot e-1} \otimes g_*(\mathcal{M} \otimes \omega^e_{Y/S}) \text{ is weakly positive over} S\}.$$

Then for some $N > 0$ the sheaf $S^N(\mathcal{H}^{m\cdot e-1} \otimes g_*(\mathcal{M} \otimes \omega^e_{Y/S})) \otimes \mathcal{H}^N$ will be generated by global sections over S and, by b we find $\mathcal{M} \otimes \omega^e_{Y/S} \otimes g^*\mathcal{H}^{m\cdot e}$ to be semi-ample.

Consider the r-fold product

$$f : Y^{(r)} = Y \times_S \ldots \times_S Y \to S \text{ and} \mathcal{M}^{(r)} = \bigotimes_{i=1}^{r} pr_i^*\mathcal{M} .$$

Then $\mathcal{M}^{(r)} \otimes \omega^e_{Y^{(r)}/S} \otimes f^*\mathcal{H}^{m\cdot e\cdot r}$ is semi-ample as well. One has a section

$$\mathcal{O}_S = det(g_*\mathcal{M}) \to \bigotimes^{r} g_*\mathcal{M} = f_*\mathcal{M}^{(r)}.$$

Let Γ be the zero divisor of the induced section of $\mathcal{M}^{(r)}$. If we write

$$\mathcal{L} = \mathcal{M}^{(r)} \otimes \omega^{e-1}_{Y^{(r)}/S} \otimes f^*\mathcal{H}^{m(e-1)\cdot r}$$

then $\mathcal{L}^e(-\Gamma)$ is semi-ample and 4.3 allows to apply 4.4 in this situation. Hence

$$f_*\mathcal{L} \otimes \omega_{Y^{(r)}/S} = (\bigotimes^{r} g_*(\mathcal{M} \otimes \omega^e_{Y/S})) \otimes \mathcal{H}^{m(e-1)\cdot r}$$

is weakly positive over S and therefore $g_*(\mathcal{M} \otimes \omega^e_{Y/S}) \otimes \mathcal{H}^{m(e-1)}$ as well. This is only possible by our choice of m, if

$$m(e-1) > (m-1)\cdot e - 1 \text{ or } m < e+1.$$

Therefore, independently of $\mathcal{H}$, we get that $g_*(\mathcal{M} \otimes \omega^e_{Y/S}) \otimes \mathcal{H}^{(e+1)\cdot e-1}$ is weakly positive over S and by 4.5 $g_*(\mathcal{M} \otimes \omega^e_{Y/S})$ will have the same property.

5 Moduli functors and Hilbert schemes

In this last chapter we just want to sketch how to use 3.2, 4.6 and 4.7 to prove 1.1.

Let h be a polynomial of degree n and $\mathcal{M}'_h$ the moduli functor of inhomogeneously polarized compact complex manifolds $(F, \mathcal{H})$ with Hilbert polynomial h. let $\mathcal{M}_h$ be any submoduli functor. Recall that $(g : Y \to S, \mathcal{H})$ and $(g' : Y' \to S, \mathcal{H}') \in \mathcal{M}_h(S)$ are equivalent if one has an invertible sheaf $\mathcal{B}$ on S an a pair consisiting of an S-isomorphism $\tau : Y \to Y'$ and an isomorphism $\tau^*\mathcal{H}' \simeq g^*\mathcal{B} \otimes \mathcal{H}$. We need, first of all, some kind of rigidification:

Assumption 5.1
a) Assume that for all $(g : Y \to S, \mathcal{H}) \in \mathcal{M}_h(S)$ one has a unique "good" $(g : Y \to S, \mathcal{H}_g) \in \mathcal{M}_h(S)$ such that $(g : Y \to S, \mathcal{H}) \sim (g : Y \to S, \mathcal{H}_g)$.
b) If $\zeta : S' \to S$ is any morphism,then $pr_2^*\mathcal{H}_g$ is the "good" polarization of $pr_1 : S' \times_S Y \to S'$.
c) For all $(g : Y \to S, \mathcal{H}) \in \mathcal{M}_h(S)$ and all $\nu > 0$ the sheaves $g_*\mathcal{H}_g^\nu$ are weakly positive over S in a bounded way.

Example 5.2 If $\mathcal{M}_h$ is the moduli functor of canonically polarized manifolds, then 4.6 tells us that for $g : Y \to S$ the sheaf $\omega_{Y/S}$ is a "good" polarization satisfying b) and c).

In the situation considered in 1.1.b) we do not know how to choose a good polarization, and in fact, we do not believe that it exists. Therefore we proceed in a slightly different way:

Example 5.3 Let $\mathcal{M}'_h$ be the functor of inhomogeneously polarized compact complex non uniruled manifolds . If $(g : Y \to S, \mathcal{H}) \in \mathcal{M}'_h(S)$ we can find some $\nu >> 0$, independent of $(g, \mathcal{H})$ such that $\mathcal{H}^\nu$ is very ample on the fibres. Replacing ν by $\nu \cdot (n + 2)$, we can assume that $\mathcal{H}^\nu \otimes \mathcal{L} \otimes \omega_{Y/S}$ is relatively very ample for all invertible sheaves $\mathcal{L}$ which are numerically effective along the fibres (III, §1).
Hence, for $e = c_1(\mathcal{H}^\nu \mid_F)^n + 1$, we can choose $\mathcal{H}^\nu \otimes \omega_{Y/S}^e$ as a new polarization, if $\omega_{Y/S}$ happens to be numerically effective along the fibre. Let us take $\mathcal{M}_h$ to be the subfunctor of all families $(g : Y \to S, \mathcal{H})$ such that $\mathcal{H}^\nu \otimes \omega_{Y/S}^e$ is relatively very ample.
Then for $r = rank(g_*\mathcal{H}^\nu)$. , we define the new good polarization for $(g : Y \to S, \mathcal{H})$ to be $(\mathcal{H}^\nu \otimes \omega_{Y/S}^e)^r \otimes g^*det(g_*\mathcal{H}^\nu)^{-1}$.

During this process we changed the Hilbert polynomical. May be, we have to decompose $\mathcal{M}_h$ into a disjoint sum of moduli functors to get r and the new Hilbert polynomical constant.

It is easy to see that this manipulations still allows to define a Hilbert scheme. Hence the following observation will finish the proof of 1.1 in both cases:

Observation 5.4 If $\mathcal{M}_h$ is a bounded and separated subfunctor of the moduli functor $\mathcal{M}'_h$and if $\mathcal{M}_h$ satisfies the assumptions made in 5.2 (may be after the change of polarization explained in 5.3), then there exists a coarse quasi-projective moduli scheme M_h.

PROOF. As explained in [5], [6]. or [7] we have a Hilbert scheme H and a "universal family"

$$(f : \mathcal{X} \to H, \mathcal{H}) \in \mathcal{M}_n(H) \text{ and an embedding } \mathcal{X} \hookrightarrow \mathbb{P}(f_*\mathcal{H}^\nu) = \mathbb{P}^{r'-1} \times H.$$

We may assume that $\mathcal{H}$ is the good polarization. If we write $f_*\mathcal{H}^\nu = \bigotimes^{r'} \mathcal{N}$, then $\mathcal{N}$ is weakly positive over H. The group $G = \mathbb{P}Gl(r', \mathbb{C})$ acts on H by change of coordinates and 3.1 and 3.2 hold. If, as in 2.8, Y is a compactification of $G \times H$, $V = p_2^{-1}(H)$, $U = \varphi^{-1}(H)$ and $Y_0 = U \cup V$, then we can use the group action and 5.1,b) to extend both pullback families to obtain some
$(f' : \mathcal{X}' \to Y_0, \mathcal{H}'^\nu_g) \in \mathcal{M}_h(Y_0)$. By 5.1,c) we know again that $f'_*\mathcal{H}'^\nu_g = \mathcal{F}_0$ is weakly positive over Y_0 in a bounded way and by 3.2 we obtain $H = H(\mathcal{L}_\eta)^s$ for some sheaf $\mathcal{L}_\eta$. By [5], Th. 1.10 we are done.

Remarks and open questions 5.5
a) Find "good polarizations" for other moduli problems. For example, it would be nice to understand projective bundles over some manifold X in this language or, more generally, to construct quasi-projective moduli schemes for uniruled varieties.
b) If $h^0(F, \Omega^1_F) > 0$ for $(F, \mathcal{L}) \in \mathcal{M}_n(\mathbb{C})$, then, in order to get the "right" moduli scheme one still has to divide out the $Pic^0(F)$-part of the equivalence relation. How can one descend "quasi-projectivity" in this step?
c)Of course, one could dream of a classification of all birational equivalence classes of manifolds. Regarding Mori's results on minimal models of threefolds, it would be necessary to allow the varieties considered to have canonical singularities. The first main problem seems to be, that we do not know, wether the corresponding moduli functor is bounded and separated.
d) It is quite unsatisfactory that we do not know any bound for the η occuring at the end of the proof of 5.4 or, in different terms, that we are unable to say exactly, how the ample sheaves on M_h look like. To be more precise:

Let $\mathcal{L}_0$ be the sheaf coming from the Plücker embedding of H. If we consider canonically polarized manifolds, then, for $\mu >> 0$, write

$$\mathcal{L}_0 = det(f_*\omega^{\nu\cdot\mu}_{\mathcal{X}/H})^{r(\nu)} \otimes det(f_*\omega^{\nu}_{\mathcal{X}/H})^{-\mu\cdot r(\nu\cdot\mu)},$$

where $r(\gamma) = rank(f_*\omega^{\gamma}_{\mathcal{X}/H})$. In the case of curves and surfaces the Hilbert-Mumford Criterion shows that $\mathcal{L}_0$ gives an ample sheaf on M_0. If one can compactify the moduli problem the method of J. Kollár in [3] shows that $det(f_*\omega^{\nu\cdot\mu}_{\mathcal{X}/H})$ is the right sheaf to consider. Here we only get that for some $\eta >> 0$ the sheaf

$$\mathcal{L}_\eta = det(f_*\omega^{\nu\cdot\mu}_{\mathcal{X}/H})^{r(\nu)} \otimes det(f_*\omega^{\nu}_{\mathcal{X}/H})^{\eta-\mu\cdot r(\nu\cdot\mu)}$$

induces an ample sheaf on M_h. It would be nice to have an explicite bound for η. There are two possible starting points:
i) One might be able to use one parameter subgroups of G to bound the poles of an ample extension of $\mathcal{L}_0$ to some compactification H' of H. Then η will depend on those bounds.
ii) Or, one could hope to start with any $\eta >> 0$ and to use methods similar to those sketched in I, §1 D, to get other ample sheaves on M_h, which are of the form $det(f_*\omega^{\gamma}_{\mathcal{X}/H})$. Theorem 1.12 ii) in I and theorem 3.1 in [1] give some hope.

Notes added in proof

I. This note was written about two years ago and since then the questions raised in 5.5, b and d, have been settled:

In [10] we prove the existence of a coarse quasi-projective moduli scheme for polarized manifolds X with ω_X numerically effective, up to isomorphisms and $\mathbb{Q}$-numerical equivalence.

In [11] Hélène Esnault and the author construct "nice" ample sheaves on the different moduli schemes mentioned in this note. For example, if M_h is the moduli scheme for canonically polarized manifolds, then the sheaf $det(f_*\omega^{\nu}_{\mathcal{X}/H})$ on the Hilbert scheme descends to an ample sheaf on M_h.
If M_h is the moduli scheme of polarized manifolds X with $\omega^{\delta}_X = \mathcal{O}_X$, then (independently of the polarization) $f_*\omega^{\delta}_{\mathcal{X}/H}$ will give the ample sheaf on M_h.

II. In [9], III, 2.2, we claim that the proof for the bound of $e(\mathcal{H}^{\nu})$ for non-singular varieties remains valid if one allows rational Gorenstein singularities. Unfortunatelly this is wrong, one has to use a slightly stronger condition on $\mathcal{H}$, as explained in [11], section 2. This condition is satisfied if one chooses the number ν in section 1 of [9], III, in a slightly different way. The exact statement can be found in [11], 2.12.

References

1. Esnault, H., Viehweg, E.: Effective bounds for semi positive sheaves and for the height of points on curves over complex function fields. Compos. Math. 76, 69-85 (1990).

2. Kollár, J.: Towards moduli of singular varieties. Compos. Math. 56, 369-398 (1985).

3. Kollár, J.: Projectivity of complete moduli. Journal Diff. Geom. 32, 235-268 (1990).

4. Mori, S.: Classification of higher-dimensional varieties. Algebraic geometry. Bowdoin 1985. Proc. Symp. Pure Math. 46, 269-331 (1987).

5. Mumford, D., Fogarty, J.: Geometric Invariant Theory, Second Edition. (Ergebnisse der Math., Vol. 34) Berlin-Heidelberg-New York: Springer 1983

6. Popp, H.: Moduli theory and classification theory of algebraic varieties. (Lect. Notes in Math. Vol. 620). Berlin-Heidelberg-New York: Springer 1977.

7. Seshadri, C.S.: Theory of moduli. Algebraic Geometry. Arcata 1974. Proc. of Symp. in Pure Math. 29, 263-304 (1975).

8. Viehweg, E.: Vanishing theorems and positivity in algebraic fibre spaces. Proc. of the Intern. Congr. of Math. Berkeley 1986, 682-688.

9. Viehweg, E.: Weak positivity and the stability of certain Hilbert points.
   I. Invent. math. 96, 639-667 (1989)
   II. Invent. math. 101, 191-223 (1990)
   III. Invent. math. 101, 521-543 (1990)

10. Viehweg, E.: Quasi-projective quotients by compact equivalence relations. Math. Ann. 289, 297-314 (1991).

11. Esnault, H., Viehweg, E.: Ample sheaves on moduli schemes. Proc. of the conf. Algebraic and analytic geometry, Tokyo 1990, to appear.