

RATIONAL SINGULARITIES OF HIGHER DIMENSIONAL SCHEMES

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ABSTRACT. Two examples of rational singularities of schemes over an algebraically closed field of characteristic zero are given: Singularities occurring as the quotient of a regular scheme by a finite group and singularities of the type $u^2 - v^2 - g(t_1, \dots, t_N)$.

Unless otherwise explicitly mentioned, we assume that all schemes are irreducible, reduced and of finite type over an algebraically closed field k of characteristic zero and that all points are k -rational.

Let U be a scheme, X a regular scheme and $g: X \rightarrow U$ a proper, surjective, birational morphism. We call $g: X \rightarrow U$ a *resolution* of U .

DEFINITION 1. Let Y be a scheme and $y \in Y$. Then Y has a rational singularity at y if there exists a neighbourhood U of y in Y , such that for every resolution $g: X \rightarrow U$ we have $g_*\mathcal{O}_X \cong \mathcal{O}_U$ and $R^i g_*\mathcal{O}_X = 0$ for $i \neq 0$ (henceforth we write $\mathbf{R}g_*\mathcal{O}_X \cong \mathcal{O}_U$).

In this paper we want to discuss two examples of rational singularities needed in [5, §5].

REMARKS. (i) Using “flat base change” [1] it is easy to see that the question whether $y \in Y$ is a rational singularity depends only on $\mathcal{O}_{y,Y}$ (“ \wedge ” denotes the completion with respect to the maximal ideal).

(ii) If the base field has positive characteristic, one needs additional conditions to define rational singularities [3].

(iii) Every regular point of a scheme is a rational singularity [2].

(iv) In Definition 1 it is sufficient to consider one resolution of U .

This last statement follows from (iii) and the Leray spectral sequence. Using the same kind of argument one gets

LEMMA 1. Let $h: Y' \rightarrow Y$ be a proper, surjective, birational morphism of schemes. Assume that Y' has only rational singularities; then $\mathbf{R}h_*\mathcal{O}_{Y'} \cong \mathcal{O}_Y$, if and only if Y has only rational singularities.

We first consider quotient singularities:

DEFINITION 2. Let Y be a scheme and $y \in Y$. Then Y has a *quotient singularity* at y if there exist a regular scheme Y' and a finite group G acting

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on Y' such that Y'/G exists and is isomorphic to some neighbourhood of y in Y .

Let W be a regular scheme and $D \subseteq W$ a closed, reduced subscheme of codimension 1. We say that D has *normal crossings* if the irreducible components of D are regular and if for every point $w \in W$ regular parameters x_1, \dots, x_n exist such that D is defined by $x_1 \cdot x_2 \cdot \dots \cdot x_r = 0$.

LEMMA 2. *Let W be a regular scheme and $f: Y \rightarrow W$ a finite morphism of normal schemes. Assume that the ramification locus $\Delta(Y/W)$ (see [4]) has only normal crossings. Then Y has quotient singularities and f is flat.*

PROOF. The first statement follows from Abhyankar's lemma: The proof is exactly that used in [4, pp. 32–33] to prove "Satz 4.1". One must simply add "and hence P_2 is a regular point of X_2 " after the 18th line of p. 33.

Since flatness is a local property, we may assume that for some regular scheme Y' and for some group G we have $h: Y' \rightarrow Y'/G \cong Y$. Then $f \cdot h$ is a finite morphism of regular schemes, and hence $(f \cdot h)_* \mathcal{O}_{Y'}$ is locally free. Let $\eta: \mathcal{O}_Y \rightarrow h_* \mathcal{O}_{Y'}$ be the map "multiplication with $\text{ord}(G)^{-1}$ ", and $\text{Tr}: h_* \mathcal{O}_{Y'} \rightarrow \mathcal{O}_Y$ the trace map. Then $\text{Tr} \cdot \eta$ is an isomorphism and, therefore, $f_* \mathcal{O}_Y$ a direct summand of $(f \cdot h)_* \mathcal{O}_{Y'}$.

PROPOSITION 1. *Every quotient singularity is a rational singularity.*

PROOF. Assume that for some $n \geq 1$ and every scheme Y having quotient singularities we already know:

For every resolution $g: X \rightarrow Y$ we have $R^i g_* \mathcal{O}_X = 0$ for $0 < i < n$.

Let $Y = Y'/G$ (as in Definition 2) and $g: X \rightarrow Y$ be a resolution. Denote the normalization of X in the function field of Y' by X' and choose a resolution $h': W' \rightarrow X'$ of X' . We denote the natural morphisms by:

$$\begin{array}{ccccc} W' & \xrightarrow{h'} & X' & \xrightarrow{g'} & Y' \\ & & \downarrow f' & & \downarrow f \\ & & X & \xrightarrow{g} & Y \end{array}$$

Using "embedded resolution of singularities" [2] we may assume that $\Delta(X'/X)$ has normal crossings. X' also has quotient singularities (Lemma 2) and, by assumption, $R^i h'_* \mathcal{O}_{W'} = 0$ for $0 < i < n$. The Leray spectral sequence gives an injection $R^n g'_*(h'_* \mathcal{O}_{W'}) \rightarrow R^n(g' \cdot h')_* \mathcal{O}_{W'}$. Since $g' \cdot h'$ is a resolution of a regular scheme, $R^n(g' \cdot h')_* \mathcal{O}_{W'} = 0$ and therefore $0 = f_* R^n g'_* \mathcal{O}_{X'} = R^n g_*(f'_* \mathcal{O}_{X'})$. Since \mathcal{O}_X is a direct summand of $f'_* \mathcal{O}_{X'}$ (proof of Lemma 2), we get $R^n g_* \mathcal{O}_X = 0$.

The second example is given by an explicit equation:

PROPOSITION 2. *Let Y be a scheme and $y \in Y$ such that*

$$\hat{\mathcal{O}}_{y,Y} = k[[t_1, \dots, t_n, u, v]] / (u^2 - v^2 - g(t_1, \dots, t_n)),$$

$0 \neq g(t_1, \dots, t_n) \in k[[t_1, \dots, t_n]]$. Then we have:

- (i) $\hat{\mathcal{O}}_{y,Y}$ (and hence $\mathcal{O}_{y,Y}$) is Cohen-Macaulay and normal.
- (ii) $\hat{\mathcal{O}}_{y,Y}$ is flat over $k[[t_1, \dots, t_n]]$.
- (iii) y is a rational singularity of Y .

PROOF. $\hat{\mathcal{O}}_{y,Y}$ is a complete intersection and, hence, Cohen-Macaulay. The normality follows from Serre's criterion.

(ii) is true since the induced morphism of schemes is equidimensional. We may assume that

$$Y = \text{Spec}(k[[t_1, \dots, t_n]][u, v]/(u^2 - v^2 - g(t_1, \dots, t_n)))$$

and $W = \text{Spec}(k[[t_1, \dots, t_n]])$. Let D be the subscheme of W defined by $g(t_1, \dots, t_n) = 0$. Using "embedded resolution of singularities" [2], "flat base change" [1] and Lemma 1, we may assume that D_{red} has normal crossings. After choosing another system of regular parameters in W , we get $g(t_1, \dots, t_n) = t_1^{v_1} \cdot t_2^{v_2} \cdot \dots \cdot t_n^{v_n}$, $v_i \in \mathbb{N}$. Although singularities of this type are known to be rational [3], we prove it directly:

Step 1. Blow up ideals of the form $\langle u, v, t_i \rangle$ to decrease the v_i 's until $v_i = 1$ or $v_i = 0$ for $i = 1, \dots, n$.

Step 2. Blow up ideals of the form $\langle u, v, t_i, t_j \rangle$, $i \neq j$, to decrease the number of variables occurring in $g(t_1, \dots, t_n)$.

Denote one of the morphisms in Step 1 or Step 2 by $h: X \rightarrow Y$ and the exceptional locus by E . It is easy to see $H^q(E, \mathcal{O}_E(p)) = 0$ for $q > 0$ and $p \geq 0$. Using decreasing induction on p and the exact sequence

$$0 \rightarrow \mathcal{O}_X(p+1) \rightarrow \mathcal{O}_X(p) \rightarrow \mathcal{O}_E(p) \rightarrow 0,$$

we obtain $\mathbf{R}h_* \mathcal{O}_X \cong \mathcal{O}_Y$.

REMARKS. (i) The type of singularities discussed in Proposition 2 occurs in stable curves over a regular base scheme. Using flat base change, it follows that a stable curve over a scheme with rational singularities has rational singularities itself. (ii) One example: Surprisingly $u^2 - v^2 - t_1^3 - t_2^7$ defines a rational singularity while $u^2 - v^2 - t_1^3 - t_2^7$ does not.

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