Semistable bundles on curves and irreducible representations of the fundamental group

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Für Friedrich Hirzebruch, mit Zuneigung und Bewunderung.

Abstract. A. Bolibruch showed that every irreducible representation of the fundamental group of the complement of finitely many points in $\mathbb{P}^1_{\mathbb{C}}$ is realizable as the solution of a Fuchsian type differential equation. In this note we give a higher genus analogue of his theorem.

Introduction

In this note, we make an attempt to understand the meaning of Bolibruch's theorem for curves of higher genus.

Theorem 0.1 (Bolibruch [**1**]). *Let*

$$\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} - \Sigma) \longrightarrow GL(N, \mathbb{C})$$

be an irreducible representation of the fundamental group of the complement of finitely many points $\Sigma \neq \emptyset$. Then there is a logarithmic connection

$$\nabla : \mathcal{O}^{\oplus N} \longrightarrow \Omega^1_X(\log \Sigma) \otimes \mathcal{O}^{\oplus N}$$

such that the local system $\ker(\nabla|_{X-\Sigma})$ *on $\mathbb{P}^1_{\mathbb{C}} - \Sigma$ is defined by ρ.*

Bolibruch's proof is very analytic, but Gabber ([**2**]) gave a more algebraic approach, which we recall in section 1 (see also [**4**]). Using his construction, we interpret Bolibruch's theorem in the following way.

Theorem 0.2. *Let X be a curve over an algebraically closed field k of characteristic 0, and let $\emptyset \neq \Sigma \subset X(k)$ consist of finitely many points. Let*

$$\nabla : E \longrightarrow \Omega^1_X(\log \Sigma) \otimes E$$

be a logarithmic connection on a vectorbundle E of rank N such that for all subsheaves $\{0\} \neq F \subset E$ with $\operatorname{rank}(F) < N$,

$$\nabla F \not\subset \Omega^1_X(\log \Sigma) \otimes F.$$

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Then for any $p \in \Sigma$, there is a semistable vectorbundle E' of degree 0 and a logarithmic connection

$$\nabla' : E' \longrightarrow \Omega^1_X(\log \Sigma) \otimes E',$$

with $(E', \nabla')|_{X-\{p\}} = (E, \nabla)|_{X-\{p\}}$.

Any semistable bundle E' of rank N and degree 0, has a canonical filtration (see (3.3)), the graded bundles $gr_i E'$ of which are direct sums of stable ones. Due to the Narasimhan-Seshadri correspondence [**5**] over $\mathbb{C}$, there is a unitary connection d_i on grE'_i which is uniquely defined.

The curious point is that, over $k = \mathbb{C}$, we associate to an irreducible representation of the fundamental group

$$\pi_1(X - \Sigma) \longrightarrow GL(N, \mathbb{C})$$

of the open curve $X - \Sigma$, unitary representations of the fundamental group of the compact curve

$$\pi_1(X) \longrightarrow U(N_i, \mathbb{C}), \quad \text{where} \quad \sum_i N_i = N,$$

via theorem 0.2 and the Narasimhan-Seshadri correspondence.

Conversely it is easy to associated such unitary representations of $\pi_1(X)$ an irreducible representation $\pi_1(X - \Sigma) \to GL(N, \mathbb{C})$:

PROPOSITION 0.3. *Let X be a curve over $\mathbb{C}$ let E be a semistable bundle on X of degree 0 with graded bundles $gr_i(E)$ for the canonical filtration.*

1) *There is a connection $\nabla : E \to \Omega^1_X \otimes E$ respecting the canonical filtration on E, such that $gr_i(\nabla) = d_i$.*
2) *There is a constant $\sigma \leq 3$ depending only on E such that for any reduced divisor Σ with $\deg(\Sigma) \geq \sigma$, there is a connection*

$$\nabla : E \longrightarrow \Omega^1_X(\log \Sigma) \otimes E$$

such that for all subsheaves $\{0\} \neq F \subset E$ with $\text{rank}(F) < N$,

$$\nabla F \not\subset \Omega^1_X(\log \Sigma) \otimes F.$$

This way of going back and forth between representations of the projective and the open curve is very lose. On both sides one has parameters. It is not clear whether one should think of this really as a correspondence. It is also not clear how to interpret this in terms of compactification of the moduli space of stable bundles of degree 0.

1. Gabber's construction

We explain Gabber's construction, transposing it to the algebraic context of theorem 0.2. Hence we consider a projective curve X over k, a divisor $\Sigma > 0$ and a logarithmic connection

$$\nabla : E \longrightarrow \Omega^1_X(\log \Sigma) \otimes E$$

on a vectorbundle E. We fix a point $p \in \Sigma$ and denote by

$$\Gamma = \text{res}_p(\nabla) : E \otimes k(p) \longrightarrow E \otimes k(p)$$

the residue of ∇.

For $0 \neq w \in E \otimes k(p)$ define E'_w to be the inverse image of k_w under the restriction map $E \to E \otimes k(p)$, and $E_w = E'_w(p)$. Then $E \subset E_w \subset E(p)$ and $\deg E_w = \deg(E) + 1$.

The connection ∇ extends to ∇_w on E_w if and only if w is an eigenvector of Γ. More precisely, let $(w, e_2, \ldots, e_N)$ be a basis of $E \otimes k(p)$ in which $\Gamma = (\gamma_{ij})$ is triangular, that is $\gamma_{ij} = 0\ i > j$. Then in the basis $(\frac{w}{t}, e_2, \ldots, e_N)$ of $E_w \otimes k(p)$ the residue $\mathrm{res}_p(\nabla_w) = \Gamma_w = (\gamma'_{ij})$ fulfills:

$$\gamma'_{ij} = \gamma_{ij} \quad \text{for} \quad i \geq 2, j \geq 2$$
$$\gamma'_{11} = \gamma_{11} - 1$$
$$\gamma'_{1i} = 0\ i \geq 2.$$

Thus the roots of the characteristic polynomial of Γ_w, are $\gamma_{11} - 1, \gamma_{22}, \ldots, \gamma_{NN}$.

Definition 1.1. We say that (E', ∇') is obtained from (E, ∇) by an elementary G-transformation at p if there is an eigenvector $0 \neq w \in E \otimes k(p)$ of Γ such that $(E', \nabla') = (E_w, \nabla_w)$.

Theorem 1.2 (Gabber). *Let $\nabla : E \to \Omega^1_X(\log \Sigma) \otimes E$ be any connection, and $M \in \mathbb{N}$. Then there is a connection*

$$\nabla' : E' \longrightarrow \Omega^1_X(\log \Sigma) \otimes E'$$

with $(E', \nabla')|_{X - \{p\}} = (E, \nabla)|_{X - \{p\}}$ such that

1) *the characteristic polynomial of $\Gamma' = \mathrm{res}_p(\nabla')$ has no multiple zeros,*
2) *if λ, μ are 2 eigenvalues of Γ', with $\lambda - \mu \in \mathbb{Z}$, then $|\lambda - \mu| \geq M$,*
3) *(E', ∇') is obtained from (E, ∇) by at most $\frac{N^3 M}{2}$ elementary G-transformations at p.*

Proof. One orders the roots of the characteristic polynomial of Γ in subsets $I_1, \ldots, I_\ell$,

$$I_j = \{\lambda_{j,1}, \ldots, \lambda_{j,mj}\}, \quad \text{where} \quad \sum_{j=1}^{\ell} m_j = N$$

such that $0 \leq \lambda_{j,i+1} - \lambda_{j,i} \in \mathbb{N}$, and $\lambda_{j,s} - \lambda_{j',s'} \notin \mathbb{Z}$ for $j' \neq j$. By taking an eigenvector $e_1 \in E \otimes k(p)$ for λ_{11} and replacing E by E_{e_1}, one transforms I_1 to

$$I_1 = \{\lambda_{1,1} - 1, \lambda_{1,2}, \ldots, \lambda_{1,m_1}\}.$$

Repeating this $m_1 M$ times, one replaces I_1 by

$$I_1 = \{\lambda_{1,1} - m_1 M, \lambda_{1,2}, \ldots, \lambda_{1,m_1}\}.$$

Since $\lambda_{1,1} - m_1 M \neq \lambda_{1,2}$, there exists an eigenvector e_2 with eigenvalue $\lambda_{1,2}$, and repeating the same transformation $(m_1 - 1)M$ times with e_2 instead of e_1 one transforms $\lambda_{1,2}$ to $\lambda_{1,2} - (m_1 - 1)M$, without changing the other roots of the characteristic polynomial. After $\frac{m_1(m_1-1)}{2} M$ steps, one has

$$I_1 = \{\lambda_{1,1} - m_1 M, \lambda_{1,2} - (m_1 - 1)M, \ldots, \lambda_{1,m_1 - 1} - M, \lambda_{1,m_1}\}.$$

Repeating this for $I_2, \ldots, I_\ell$, one needs at most

$$(\sum_{j=1}^{\ell} \frac{m_j(m_j - 1)}{2})M \leq \frac{N^3}{2} \cdot M$$

steps to satisfy the first and second condition in 1.2. □

2. The proof of theorem 0.2

Let $E_0 = 0 \subsetneq E_1 \subset E_2 \subset \ldots \subset E_m = E$ be the Harder-Narasimhan filtration [**3**] of a rank N vector bundle E, uniquely determined by the two conditions:

$$\mu_i = \mu(E_i/E_{i-1}) < \mu_{i-1}$$

and E_i/E_{i-1} semistable, where $\mu(F) = \deg(F)/\mathrm{rank}(F)$ for any vector bundle.

In order to prove theorem 0.2 we are allowed to replace E by $E(\ell p)$ for $\ell \in \mathbb{Z}$. In fact, ∇ stabilizes $E(\ell p)$ and the residue Γ of ∇ in p is replaced by $\Gamma - \ell\mathrm{Id}$. In particular this does not change the difference between two eigenvalues of Γ. Thus, replacing E by $E(\ell p)$, we may assume that $-1 < \mu(E_1) \le 0$ and consequently that $\deg(E) \le 0$.

Lemma 2.1. *If $\nabla : E \to \Omega^1_X(\log \Sigma) \otimes E$ does not stabilize any subbundle, and $-1 < \mu(E_1) \le 0$, then*

$$-N - N^2(2g - 2 + \sigma) \le \deg(E) \le 0$$

where g = genus of X, and $\sigma = |\Sigma|$.

Proof. Let i_0 to be the minimal i such that the map

$$\eta_0 : E_i \longrightarrow \Omega^1_X(\log \Sigma) \otimes E/E_{m-1}$$

is not 0. Since ∇ does not stabilize any subbundle, $i_0 \le m - 1$, thus η_0 is linear and factors through E_{i_0}/E_{i_0-1}. This shows that $\mu_{i_0} \le \mu_m + (2g - 2 + \sigma)$. By assumption ∇ does not stabilize E_{i_0-1}. Hence there exists some minimal number $i_1 \le i_0 - 1$ such that

$$\eta_1 : E_i \longrightarrow \Omega^1_X(\log \Sigma) \otimes E/E_{i_0-1}$$

is not trivial. Then η_1 factors through a linear map

$$E_{i_1}/E_{i_1-1} \longrightarrow \Omega^1_X(\log \Sigma) \otimes E_{j_1}/E_{j_1-1}$$

for some j_1 with $i_0 \le j_1 \le m$. Consequently

$$\mu_{i_1} \le \mu_{j_1} + (2g - 2 + \sigma) \le \mu_{i_0} + (2g - 2 + \sigma) \le \mu_m + 2(2g - 2 + \sigma).$$

One obtains inductively

$$-1 \le \mu_1 \le \mu_m + m(2g - 2 + \sigma),$$

and, since $\mu(E) \ge \mu_m$ and $N \ge m$, the inequality of lemma 2.1. □

Finally, one proves theorem 0.2 in the following more precise form:

Theorem 2.2. *Let (X, E, ∇, Σ) be as in theorem 0.2. Assume that*

$$-1 < \mu(E_1) \le 0,$$

that the characteristic polynomial of $\Gamma = \mathrm{res}_p(\nabla)$ has no multiple zeros, and that

$$|\lambda - \mu| \ge M = N + N^2(2g - 2 + \sigma)$$

for different eigenvalues λ and μ of Γ with $\lambda - \mu \in \mathbb{Z}$.

Then there is a semistable vector bundle E' of degree 0, and an extension ∇' of ∇ to E', such that (E', ∇') is obtained from (E, ∇) by at most M elementary G-transformations at p.

Proof. We argue by induction on $-\deg(E)$ which is smaller than or equal to M by lemma 2.1.

If $\deg(E) = 0$, $\mu(E_1) = \mu(E) = 0$ as $\mu(E_1) \geq \mu(E)$. Thus $E_1 = E$ and E is semistable of degree 0.

Assume now that $\deg(E) < 0$. If $\mu(E_1) < 0$ as well, then for any elementary G transformation at p, and any subsheaf $M \subset E_w$, one has

$$\deg(M) \leq \deg(M \cap E) + 1 \leq 0,$$

thus

$$\deg(E_w) = \deg(E) + 1 \quad \text{and} \quad -1 \leq \mu((E_w)_1) \leq 0.$$

Otherwise, $\mu(E_1) = \deg(E_1) = 0$. We set $F = E_1$ for notational simplicity and denote by Q the quotient $Q = E/F$. We consider an elementary G transformation at p such that the eigenvector $w \in E \otimes k(p)$ maps non-trivially to $Q \otimes k(p)$. One obtains an exact sequence

$$0 \longrightarrow F \longrightarrow E_w \longrightarrow Q_w \longrightarrow 0.$$

Let $(E_w)_1$ be the first bundle in the Harder-Narasimhan filtration of E_w. One certainly has

$$-1 \leq \mu(E_1) \leq \mu((E_w)_1).$$

The inequality $\mu((E_w)_1) \leq 0$ is equivalent to the property that $\deg(M) \leq 0$ for all subsheaves $M \subset E_w$. Consider $M \subset E_w$ and $M \subset M' \subset E_w$, where M' is the inverse image of $M/F \cap M$ under the projection $E_w \to E_w/F \cap M$. As $F \cap M \subset F$, one has $\deg(F \cap M) \leq 0$. Thus

$$\deg(M) \leq \deg(M/F \cap M) = \deg(M') + \deg(F) = \deg(M').$$

By definition of $E_1 = F$, one has $\mu((E/F)_1) < 0$ and

$$\deg((M/F \cap M) \cap Q) \leq -1.$$

This shows that

$$\deg(M) \leq \deg(M/F \cap M) \leq \deg\ ((M/F \cap M) \cap Q) + 1 \leq -1 + 1 \leq 0.$$

Thus again

$$\deg(E_w) = \deg(E) + 1 \quad \text{and} \quad -1 < \mu((E_w)_1) \leq 0.$$

By induction we obtain the theorem. □

3. Existence of connections

In this section we lift the unitary connections of the graded pieces of the canonical filtration.

Lemma 3.1 (Compare with [**6**], lemma 3.5). *Let X be an algebraic variety over a field k,*

$$0 \longrightarrow S \overset{\iota}{\longrightarrow} E \overset{p}{\longrightarrow} Q \longrightarrow 0$$

be an extension of vector bundles given by $u \in H^1(X, \mathcal{H}om(Q, S))$. Let d_S and d_Q be connections on S and Q, respectively.

Then there exists a connection ∇ on E lifting d_S and d_Q if and only if $0 = du \in H^1(X, \Omega^1_X \otimes \mathcal{H}om(Q, S))$, where $d = \mathcal{H}om(d_Q, d_S)$. Two such connections differ by an element in $H^0(X, \Omega^1_X \otimes \mathcal{H}om(Q, S))$.

In particular, if X is projective smooth, $k = \mathbb{C}$, and if d is unitary, then ∇ exists.

Proof. Let $X = \bigcup U_i$ be an affine covering of X,

$$\sigma_i : Q|_{U_i} \longrightarrow E|_{U_i}, \tau_i = \mathrm{Id} - \sigma_i : E|_{U_i} \longrightarrow S|_{U_i}$$

be some splitting of u on U_i. Then

$$\begin{aligned} \tau_j &= \tau_i + u_{ij} \circ \pi \\ \sigma_j &= \sigma_i - \iota \circ u_{ij} \end{aligned} \tag{3.1}$$

on U_{ij}. Define $\nabla_i = d_S \circ \tau_i + \sigma_i \circ d_Q$. Then

$$\nabla_j - \nabla_i \in H^0(U_{ij}, \Omega^1_X \otimes \mathcal{H}om(Q, S))$$

is a cocycle. Another choice of σ_i verifies

$$\begin{aligned} \sigma_i' &= \sigma_i - \iota \circ u_i \\ \tau_i' &= \tau_i + u_i \circ \pi \end{aligned}$$

for some $u_i \in H^0(U_i, \mathcal{H}om(Q, S))$. Thus

$$\begin{aligned} &\nabla_i' - \nabla_i - d_S(u_i \circ \pi) - \iota \circ u_i \circ d_Q \\ &= d(u_i) \in H^0(U_i, \Omega^1_X \otimes \mathcal{H}om(Q, S)), \end{aligned} \tag{3.2}$$

and therefore the class α_{ij} of

$$\nabla_j - \nabla_i \in H^1(X, \Omega^1_X \otimes \mathcal{H}om(Q, S))$$

is well defined. If this class vanishes, then in a refinement of (U_i) there are forms $A_i \in H^0(U_i, \Omega^1_X \otimes \mathcal{H}om(Q, S))$ such that $\nabla_j - \nabla_i = A_i - A_j$, thus $\nabla = \nabla_i + A_i$ is globally defined and α_{ij} is the exact obstruction to the existence of ∇.

On the other hand, the computation in 3.2, with u_i replaced by u_{ij}, shows at the same time that $\alpha_{ij} = du_{ij}$. □

Let X be a projective curve over $\mathbb{C}$ and E be a semistable bundle of degree 0 on X. Then there is a unique filtration, which we call *the canonical filtration* of E, verifying

$$\begin{aligned} &0 = E_0 \subset E_1 \subset \ldots \subset E_m = E \\ &gr_i E = E_i / E_{i-1} = \text{ socle of } E/E_{i-2}. \end{aligned} \tag{3.3}$$

Recall that the *socle* of E is the maximal semistable subbundle of E which splits as a sum $\bigoplus_\nu V_\nu$ of stable ones.

$$\begin{aligned} \mathrm{Hom}(gr_i E, E/E_i) &= \mathrm{Hom}(gr_i E, gr_{i+1} E) \\ &= \bigoplus \delta_{\nu\mu} \mathrm{Id}_{V_\nu} \end{aligned} \tag{3.4}$$

with $gr_i E = \bigoplus_\nu V_\nu$, $gr_{i+1} E = \bigoplus_\mu V_\mu$ for stable bundles V_ν and V_μ.

On the other hand, over $\mathbb{C}$, there is a unique unitary connection d_i on $gr_i E$ by the Narasimhan-Seshadri correspondence [**5**].

Proposition 3.2. *Let E be a semistable bundle of degree 0 on a complex projective curve, and E_i be its canonical filtration. Then there is a connection ∇ on E respecting the canonical filtration and lifting the unitary connections d_i on E_i/E_{i-1}.*

PROOF. Since $\mathcal{H}om(d_m, d_{m-1})$ is unitary, there is a connection $d_{E/E_{m-1}}$ lifting d_m and d_{m-1} by lemma 3.1. Assume inductively that d_{E/E_ℓ} exists. We want to see that

$$d : H^1(X, \mathcal{H}om(E/E_\ell, gr_\ell E)) \longrightarrow H^1(X, \Omega^1_X \otimes \mathcal{H}om(E/E_\ell, gr_\ell E))$$

kills the extension of E/E_ℓ by $gr_\ell E$ given by the canonical filtration, where $d = \mathcal{H}om(d_{E/E_\ell}, d_\ell)$. We show directly that d itself vanishes. Its dual is the differential

$$d^* : H^0(X, \mathcal{H}om(gr_\ell E, E/E_\ell)) \longrightarrow H^0(X, \Omega^1_X \otimes \mathcal{H}om(gr_\ell, E/E_\ell)).$$

By the equation 3.4, and the fact that d^* lifts $\mathcal{H}om(d_\ell, d_{\ell+1})$, one has $d^* = \mathrm{Hom}(d_\ell, d_{\ell+1}) = 0$. □

LEMMA 3.3. *Let X be a smooth projective variety defined over a field k of characteristic zero, D be a smooth irreducible divisor, L be an invertible sheaf L, and let $\nabla : L \to \Omega^1_X(\log D) \otimes L$ be a connection. Then the residue $\mathrm{res}_D(\nabla)$ is $m \cdot \mathrm{id}$ for a rational number m. Moreover, if X is a curve, m is an integer.*

PROOF. Since X is projective, we may write $L = \mathcal{O}(A_1 - A_2)$ where A_i are smooth divisors meeting transversally. Thus L carries the trivial connection d_A with $\mathrm{res}_{A_i}(d_A) = (-1)^i \cdot \mathrm{Id}_{L|_{A_I}}$. Hence $\omega := \nabla - d_A \in H^0(X, \Omega^1_X(\log\ (A_1 + A_2 + D))$ with

$$m := \mathrm{res}_D(\omega) = \mathrm{res}_D \nabla, \quad \mathrm{res}_{A_i}(\omega) = -\mathrm{res}_{A_i}(d_A).$$

Let C be an ample smooth curve, meeting D, A_1 and A_2 transversally. Then

$$-(C.A_1) + (C.A_2) + m \cdot (C.D) = \sum_{q \in C \cap (A_1 \cup A_2 \cup D)} \mathrm{res}_q(\omega) = 0$$

and consequently $m \in \mathbb{Q}$ (or $m \in \mathbb{Z}$, if $\dim(X) = 1$). □

LEMMA 3.4. *Let X be a smooth projective variety over a field k of characteristic zero, let $D = \sum_{i=1}^{\rho} D_i$ be a normal crossing divisor and*

$$\nabla : V \to \Omega^1_X \log\ D) \otimes V$$

a connection on a locally free sheaf V. Assume that the eigenvalues of $\mathrm{res}_{D_i}(\nabla)$ are zero for $i = 2, \ldots, \rho$ and that the sum of the eigenvalues of $\mathrm{res}_{D_1}(\nabla)$ does not lie in $\mathbb{Q} - \{0\}$ (or not in $\mathbb{Z} - \{0\}$, if X is a curve). Then $\bigwedge^{\max} V$ is numerically trivial.

PROOF. ∇ induces a connection

$$\nabla' : \bigwedge^{\max} V \longrightarrow \Omega^1_X(\log D) \otimes \bigwedge^{\max} V.$$

$\mathrm{res}_{D_i}(\nabla') = 0$ for $i = 2, \ldots, \rho$, and the image of ∇' lies in $\Omega^1_X(D_1) \otimes \bigwedge^{\max} V$. By 3.3 $\mathrm{res}_{D_1}(\nabla')$ must be a rational number (or an integer), hence 0, and ∇' induces a connection with values in $\Omega^1_X \otimes \bigwedge^{\max} V$. □

4. Existence of irreducible connections

Let E be a semistable bundle of rank N on the curve X and let

$$\nabla : E \to \Omega^1_X \otimes E$$

be a connection. In this section we want to construct a different connection $\nabla' : E \to \Omega^1_X(\log \Sigma) \otimes E$, where $\Sigma = \sum_{i=1}^{\mu} p_i$ is a reduced divisor in X, such that $\mathrm{Ker}(\nabla'|_{X-\Sigma})$ is an irreducible local system. If X is defined over $\mathbb{C}$ this construction and 3.2 imply proposition 0.3.

Proposition 4.1. *Assume that E is not isomorphic to the direct sum $L^{\oplus N}$ for some $L \in \mathrm{Pic}^0(X)$ and let $p, q \in X$ be two different points. Then there exists $\varphi \in \mathrm{Hom}(E, \Omega^1_X(\log(p+q)) \otimes E)$ such that $\mathrm{Ker}(\nabla'|_{X-p-q})$ is irreducible for $\nabla' = \nabla + \varphi$.*

Proof. By assumption there exists a surjection $\tau : E \to S$ for some bundle S on X of rank $s \geq 2$ such that one of the following properties holds true:

i) S is stable
ii) $S = L_1 \oplus L_2$ for $L_1 \not\cong L_2$ and $L_i \in \mathrm{Pic}^0(X)$
iii) $0 \to T \to S \to L^{\oplus \ell} \to 0$ is an extension of $L^{\oplus \ell}$, for $L \in \mathrm{Pic}^0(X)$ with a stable bundle T, such that the induced map
$$H^0(X, \mathcal{O}_X^{\oplus \ell}) \longrightarrow H^1(X, T \otimes L^{-1})$$
is injective.

In fact, let $F_0^* = \{0\} \subset F_1^* \subset \ldots \subset F_m^* = E^*$ be the canonical filtration of the dual bundle and
$$F_0 = \{0\} \subset F_1 = (E^*/F^*_{m-1})^* \subset \ldots \subset F_{m-1} \subset (E^*/F_1^*)^* \subset F_m = E$$
the dual filtration. If F_m/F_{m-1} contains no semistable bundle S as in i) or ii) it is a direct sum $L^{\oplus \ell'}$, for some $\ell' \geq 1$. In this case,
$$F_{m-1}/F_{m-2} \longrightarrow E/F_{m-2} \longrightarrow L^{\oplus \ell'}$$
is a non-trivial extension and for each direct factor T of F_{m-1}/F_{m-2} one obtains a surjection from E to a non-trivial extension
$$0 \longrightarrow T \longrightarrow S' \longrightarrow L^{\oplus \ell'} \longrightarrow 0.$$
Leaving out direct factors of S', which are isomorphic to L, one obtains S as in iii).

For any bundle F on X write $F_q = F \otimes k(q)$. In order to construct a basis of E_q we fix a basis of S_q, case by case:

i) $\bar{v}_1, \ldots, \bar{v}_{m-1}, \bar{v}_N$ is any basis of S_q.
ii) $\bar{v}_1, \bar{v}_N$ is a basis of S_q with $\bar{v}_1 \notin (L_i)_q$, for $i = 1, 2$.
iii) $\bar{v}_1, \ldots, \bar{v}_{m-1}, \bar{v}_N$ is a basis of S_q, such that $T_q \not\subset < \bar{v}_1, \ldots, \bar{v}_{m-1} >$.

Let $K = \mathrm{Ker}(\tau : E \to S)$ and
$$0 \longrightarrow K_q \longrightarrow E_q \xrightarrow{\tau_q} S_q \longrightarrow 0$$
the induced sequence of vector spaces.

Let $v_m, \ldots, v_{N-1}$ be a basis of K_q, and $v_j \in \tau_q^{-1}(\bar{v}_j)$, for $j = 1, \ldots, m-1, N$. Then $v_1, \ldots, v_N$ is a basis of E_q. By Serre duality
$$h^1(X, \mathcal{E}nd(E) \otimes \Omega^1_X(\log p)) = h^0(X, \mathcal{H}om(E, E(-p)) = 0,$$
hence the residue map
$$H^0(X, \mathcal{E}nd(E) \otimes \Omega^1_X(\log(p+q))) \xrightarrow{\mathrm{res}_q} \mathrm{End}(E_q)$$
is surjective. Choose $\varphi \in \mathrm{End}(E, \Omega^1_X(\log(p+q)) \otimes E)$ such that $\mathrm{res}_q(\varphi)$ is one Jordan block for the eigenvalue 0, with respect to $v_1, \ldots, v_N$. In particular, the only $\mathrm{res}_q(\varphi)$ invariant subspaces of E_q are of the form $\mathrm{Ker}(\mathrm{res}_q(\varphi)^\iota)$.

Let $\lambda_1, \ldots, \lambda_\nu$ be the eigenvalues of $\mathrm{res}_p(\varphi)$. Replacing φ by $\pi \cdot \varphi$ for some $\pi \notin \mathbb{Q}(\lambda_1, \ldots, \lambda_\nu)$ we may assume that no linear combination $\Sigma \rho_i \lambda_i \in \mathbb{Q} - \{0\}$ for $\rho_i \in \mathbb{Q}$.

Let $V \subset E$ be a subbundle such that $\nabla'(V) \subset \Omega^1_X(\log(p+q)) \otimes V$, for $\nabla' = \nabla + \varphi$. By 3.4 deg(V) = 0, hence V is a semistable subbundle of E, and the image B of V in S is zero or a semistable subbundle of S.

Since $\mathrm{res}_q(\nabla') = \mathrm{res}_q(\varphi)$, for some $\iota \geq 1$

$$V_q = \mathrm{Ker}(\mathrm{res}_q(\varphi)^\iota) =< v_1, \ldots, v_\iota >\subset E_q.$$

In particular $B \neq 0$. Obviously $B = S$ in case i). In case ii) we remark that $v_1 \in B_q$ and obtain $B = S$, as well.

If in case iii) $B \neq S$, then $B_q =< v_1, \ldots, v_\iota >$ for $\iota \leq m-1$ and $B \cap T \neq T$. Since the degree of B is zero, and since $B/(B \cap T) \subset L^{\oplus \ell}$, $B \cap T = 0$. Then $B \simeq L^{\oplus \iota}$ and the composite

$$H^0(X, B \otimes L^{-1}) \hookrightarrow H^0(X, \mathcal{O}_X^{\oplus \ell}) \longrightarrow H^1(X, T \otimes L^{-1})$$

zero, contradicting the assumptions made.

Hence $B = S$ in all cases, and $v_n \in V_q$. Therefore $V_q = E_q$ and $V = E$. □

If $E = L^{\oplus N}$, then in order to find some φ, with $\mathrm{Ker}(\nabla + \varphi|_{X-\Sigma})$ irreducible, one needs three points p, q_1, q_2. In fact, choosing the "canonical" basis $v_1^{(i)}, \ldots v_N^{(i)}$ in E_{q_i}, induced by the direct sum decomposition, one has again a surjection

$$\mathrm{End}(E, \Omega^1_X(\log(p + q_1 + q_2))) \longrightarrow M(N \times N, \mathbb{C}) \oplus M(N \times N, \mathbb{C}).$$

Let us choose two nilpotent matrices M_1 and M_2 with $M_i^{N-1} \neq 0$ in such a way, that the (unique) eigenvector of M_1 does not lie in $\mathrm{Ker}(M_2^{N-1})$. Repeating the argument used in the proof of 4.1 one obtains:

Proposition 4.2. *Let $\Sigma = q_1 + q_2 + p$ be a reduced divisor and E be a semistable bundle with connection ∇. Then for some $\varphi \in \mathrm{Hom}(E, \Omega^1_X(\log \Sigma) \otimes E)$ the local system* $\mathrm{Ker}((\nabla + \varphi)|_{X-\Sigma})$ *is irreducible.*

Under stronger condition on the structure of E, it is possible to choose $\Sigma = p$, as we illustrate in two examples on an elliptic curve X.

Example 4.3. Let $L \in \mathrm{Pic}^0(X)$, $L \neq \mathcal{O}$, $E = L \oplus \mathcal{O}$. Take $\Sigma = \{p\}$ a point. Then choose

$$\nabla = d + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where d is the sum of the unitary connections on L and $\mathcal{O}$,

$$\alpha, \delta \in H^0(X, \Omega^1_X(\log \Sigma)) = H^0(X, \Omega^1_X)$$
$$\gamma \in H^0(X, L^{-1} \otimes \Omega^1_X(\log \Sigma)) - H^0(X, L^{-1} \otimes \Omega^1_X)$$
$$\beta \in H^0(X, L \otimes \Omega^1_X(\log \Sigma)) - H^0(X, L^{-1} \otimes \Omega^1_X).$$

Assume $\mathrm{res}_q\gamma = \lambda, \mathrm{res}_q\beta = \mu$ are chosen such that $x^2 - \lambda \cdot \mu$ has no zero in $\mathbb{Q}$. If $V \subset E$ of rank 1 is stabilized by ∇, then $\mathrm{residue}_p(\nabla|_V) \notin \mathbb{Q}$. This contradicts lemma 3.3.

Example 4.4. Let X be an elliptic curve and

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{\iota} E \xrightarrow{\pi} \mathcal{O}_Q \longrightarrow 0$$

be the non-trivial extension of $\mathcal{O}_X$ by $\mathcal{O}_X$. As we have seen in 3.2, there exists a connection ∇ on E, lifting $d : \mathcal{O}_X \to \Omega^1_X$. As

$$h^0(X, \mathcal{E}nd(E)) = h^1(X, \mathcal{E}nd(E)) = 2 \text{ and } h^1(X, \mathcal{E}nd(E(p))) = 0$$

for any point p, whereas $H^0(X, \mathcal{O}_X) = H^0(X, \mathcal{O}(p))$, the image of

$$\mathrm{res}_p : \mathrm{Hom}(E, E(p)) = \mathrm{Hom}(E, E \otimes \Omega^1_X(\log\ p)) \longrightarrow M(2 \times 2, k)$$

is a two-dimensional space of matrices of trace 0. In particular the image contains some lower triangular matrix

$$M = \begin{pmatrix} \alpha & 0 \\ \gamma & -\alpha \end{pmatrix} \neq 0,$$

with respect to a basis v_1, v_2 with $v_1 \in \iota(k(p))$. Choose $\phi \in \mathrm{Hom}(E, E(p))$ and $\lambda \in k$ with $\mathrm{res}_p \phi = \lambda \cdot M$, such that $\lambda\alpha \notin \mathbb{Z} - \{0\}$. By 3.4 a rank 1 subbundle $V \subset E$ with $\nabla(V) \subset \Omega^1_X(\log \Sigma) \otimes V$ is numerically trivial, hence equal to $\iota(O_X)$. Then α and γ are both zero, contradicting the assumption $M \neq 0$.

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