**Weak positivity and the stability of certain Hilbert points**

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The notion of weakly positive sheaves was originally developed by the author in order to express positivity of direct images of powers of dualizing sheaves, needed to study the generalized Itaka – conjecture $C^*_k,m$ (see [17], [18], [20] and the excellent survey articles [2] and [13]). Beside of “weak positivity” applied to families of complex projective varieties over certain projective bundles, in all cases where one was able to prove $C^*_k,m$ one had to use some moduli theory. For example, in order to show $C^*_k,m$ for families of manifolds of general type one could use the existence of quasi-projective moduli schemes (as in [18] for curves and surfaces), or local Torelli theorems for cyclic covers ([19] or in a more general situation: Kawamata [9]) or, as Kollár did in [12], Hodge theoretic estimates on the kernel of the multiplication map.

Reconsidering the link between moduli theory and $C^*_k,m$ for complex manifolds of general type, we tried to use “weak positivity” and other methods from classification theory to construct quasi projective coarse moduli spaces for certain moduli functors (1.1). This aim is not achieved, due to a technical statement (1.10) which I am not able to prove in a sufficiently general situation. We have to resign ourselves to a partial result, saying that smooth points of the reduced Hilbert scheme of canonically polarized manifolds are stable (in the sense of Mumford [14]) under the usual group action and with respect to some ample sheaf (1.7). If, by accident, the reduced Hilbert scheme for one of our moduli problems turns out to be non singular, then we obtain a coarse quasiprojective moduli scheme.

Since we hope that the “gap” 1.10 can be filled some day, may be using more advanced technics from Hodge theory, we formulate our article such that we can state as well:

“**An affirmative answer to 1.10 implies that quasi-projective moduli spaces exist for complex canonically polarized manifolds.**”

Of course our proof is based on Mumford’s geometric invariant theory [14].

Sometimes it is easier to obtain coarse moduli spaces $M$ in the category of Moishezon spaces or algebraic spaces (see [10], [14], [15] and [16]). If $M$ happens to be a fine moduli space, our approach to construct an ample invertible
sheaf on $M$ becomes quite elementary. The same method works for arbitrary families of Gorenstein varieties of general type provided the map to the moduli functor is finite over some open set (see 1.18 and 1.19 for the exact statements) and it shows the existence of natural sheaves on the base having lots of sections.

As an obvious corollary we obtain an elementary proof of $C_{n,m}^+$ for morphisms whose general fibre is of general type (see 1.20), a result obtained by Kollár [12] before.

After finishing this manuscript I received a copy of János Kollár's paper on "Projectivity of complete moduli". A motivation for me to add 4.9 and a short discussion of his and my method in 5.18.

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Hélène Esnault had a great influence on the content of this paper. She pointed out several ambiguities in the first version and some of the methods and some improvements are due to her. The approach presented here is partly based on our common work, especially on [3].

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Leitfaden

The proof that certain Hilbert points are stable (1.7) uses only § 1, A and B, § 3 and § 5.

The existence of ample sheaves on the base of a family of certain varieties (1.18 and 1.19) is based on § 1, B and C, § 2, § 3 and § 4.

The reader just interested in a "simple" proof of $C_{n,m}^+$ for fibre spaces whose general fibre is a manifold of general type (1.20) should read § 1, C and D, § 2 and § 4. The weak positivity results contained in [17] or [18] are strong enough for those applications (see remark 1.21).

§ 1. Notations and discussion of the main results

All varieties and schemes are supposed to be defined over the field $\mathbb{C}$ of complex numbers. We try to use the notations of [6].
A. Moduli and Hilbert schemes

1.1. Let \( h(T) \) be a polynomial of degree \( n \). As in [14] we consider the moduli functor \( \mathcal{M}_n \) of complex projective normal irreducible varieties \( X \) with at most rational Gorenstein singularities and with an ample canonical sheaf \( \omega_X \) satisfying \( \chi(X, \omega_X^\vee) = h(v) \). In order to have “nice” Hilbert schemes we need further restrictions and therefore we define:

a) If \( n \) (which is nothing but \( \dim X \) for \( X \in \mathcal{M}_n(\mathbb{C}) \)) is one or two we define \( \mathcal{M}_n = \mathcal{M}_n' \).

b) If \( n > 2 \) we define \( \mathcal{M}_n \) by \( \mathcal{M}_n(S) = \{ f: \mathcal{X} \to S \in \mathcal{M}_n'(S); f \text{ smooth or } h^0(X, \omega_X) > 0 \text{ for all singular fibers } X \text{ of } f \} \).

Results due to Matsusaka, Tankeev and Kollár (see for example [10]) show that the families in \( \mathcal{M}_n(S) \) are bounded with respect to the canonical polarization and that for \( v \gg 0 \) the \( v \)-canonical embedded \( X \in \mathcal{M}_n(\mathbb{C}) \) are parametrized by a scheme \( H \):

**Theorem 1.2.** Let \( \mathcal{M}_n \) be as in 1.1.

i) There exists some number \( v \) such that for all \( X \in \mathcal{M}_n(\mathbb{C}) \) \( \omega_X^v \) is very ample.

ii) There exists a “Hilbert scheme” \( H \) and a universal family \( h: \mathcal{X} \to H \in \mathcal{M}_n(H) \) together with an embedding \( i: \mathcal{X} \to \mathbb{P}(h_* \omega_{\mathcal{X}/H}^v) \cong \mathbb{P}^{r-1} \times H \) such that all \( v \)-canonical embedded

\[
\begin{array}{ccc}
\mathcal{X}_v & \longrightarrow & \mathbb{P}^{r-1} \times H \\
\downarrow f_{v} & & \downarrow \\
S & = & S
\end{array}
\]

with \( f_v \in \mathcal{M}_n(S) \) are obtained as pullback of \( \mathcal{X} \to \mathbb{P}^{r-1} \times H \) under a unique morphism \( S \to H \).

iii) The action of \( S_l(r, \mathbb{C}) \) on \( H \) corresponding to “change of coordinates in \( \mathbb{P}^{r-1} \)” is proper.

**Notations 1.3.** If \( f: X \to Y \) is a proper flat Gorenstein morphism (respectively a proper surjective morphism between Gorenstein schemes) we denote by \( \omega_{X/Y} \) the relative dualizing sheaf (respectively the difference of the dualizing sheaves \( \omega_X \otimes f^* \omega_Y^{-1} \)). If \( Y \) is irreducible we will always try to use:

\[
r(\eta) = \text{rank} (f_* \omega_{X/Y}) \quad \text{and} \quad \chi(\eta) = \det (f_* \omega_{X/Y}).
\]

1.4. For \( \mu \gg 0 \) the sheaf \( \mathcal{L}_{0} = \lambda_\eta^{\mu(v)} \otimes \lambda_\omega^{r(v, \eta)(v)} \) is ample on the Hilbert scheme \( H \) introduced in 1.2. Moreover there exists a \( S_l(r(v), \mathbb{C}) \)-linearization of \( \mathcal{L}_{0} \) ([14], Def. 1.6).

In fact, \( \mathcal{L}_{0} \) is the ample sheaf arising from the Plücker coordinates on \( H \) and \( \mu \) has to be chosen such that for all \( X \in \mathcal{M}_n(\mathbb{C}) \) the ideal of \( X \) in \( \mathbb{P}(H^0(X, \omega_X^v)) \) is generated by polynomials of degree \( \mu \) ([1], in 4.3 we will use a similar construction).
1.5. Mumford introduced in [14] the notion of a stable point under a group action and with respect to any linearized invertible sheaf $\mathcal{L}$ (see 5.3). If we consider the Hilbert scheme and the $Sl(r(v), \mathbb{C})$ action we will denote the set of stable points by $H(\mathcal{L})$. We freely use the notations and results from [14].
Remark, that there it is shown that:

i) $H(\mathcal{L})$ is an open subscheme of $H$, the quotient $H(\mathcal{L})/Sl(r(v), \mathbb{C})$ exists as a quasi-projective scheme and the $Sl(r(v), \mathbb{C})$-invariant sections of some power of $\mathcal{L}$ define an embedding of this quotient into some projective space

ii) If $H = H(\mathcal{L})$ then $M = H/Sl(r(v), \mathbb{C})$ is a quasi-projective moduli scheme for $\mathcal{M}_h$.

1.6. Mumford [14] for curves and Gieseker [4] for surfaces verified the stability for all points of $H$ and showed that for $v, \mu > 0$ and $\mathcal{L}_0$ as in 1.4 one has $H = H(\mathcal{L}_0)^\mu$. Regarding Gieseker’s proof one finds that

(*) $\mathcal{L}_0$ is ample on $H$ by construction.

(**) It is difficult to decide whether a given point lies in $H(\mathcal{L}_0)^\mu$.

The approach presented in this paper shifts the difficulty from (**) to (*).

We will observe in § 5:

Let $\mathcal{L}_n = \mathcal{L}_0 \otimes \mathcal{L}_n^\mu$. Then

(*) It is difficult to show that $\mathcal{L}_n$ is ample on $H$.

(**) If one knows that for $\eta > 0 \mathcal{L}_n$ is ample on some open $Sl(r(v), \mathbb{C})$ invariant subscheme $H_0$ of $H$, then it is easy to show that $H_0 \subseteq H(\mathcal{L}_0)^\mu$.

The precise statement, which will be shown in § 5, is:

**Theorem 1.7.** Let $\mathcal{M}_h$ be one of the moduli functors considered in 1.1 and let $H$ be the Hilbert scheme (for some $v > 0$ as in 1.2).

a) Let $H_0 \subseteq H$ be the largest open subscheme of $H$ such that $(H_0)_{\text{red}}$ is non singular. Then $H_0 = H_0(\mathcal{L}_0^\mu \otimes \mathcal{L}_n^\mu)^\mu$ for $a, b, \mu > 0$.

b) An affirmative answer to problem 1.10 implies that for $a, b, \mu > 0$ one has $H = H(\mathcal{L}_0^\mu \otimes \mathcal{L}_n^\mu)^\mu$ and hence that a coarse quasi-projective moduli scheme exists for $\mathcal{M}_h$.

c) If $H_{\text{red}}$ is non singular then a coarse quasi-projective moduli scheme exists for $\mathcal{M}_h$.

**B. Weak positivity and some open problems**

**Definition and Notation 1.8.**

i) Let $Y$ be a scheme (or an analytic space) and $U \subseteq Y$ an open subscheme (or Zariski open subspace). We say that an $\mathcal{O}_Y$-module $\mathcal{F}$ is globally generated over $U$, if the natural map $H^0(Y, \mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_U \rightarrow \mathcal{F}$ is surjective over $U$ (respectively: $H^0_Y(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_U \rightarrow \mathcal{F}$, where $H^0_Y$ denotes the meromorphic sections, as in 1.16).

ii) If $\mathcal{F}$ is a coherent sheaf on $Y$ and $i: U \rightarrow Y$ is the maximal open subscheme (or space) where $\mathcal{F}$ is locally free, then we define $S'(i* \mathcal{F}) = i_* S'(i* \mathcal{F})$, $A'(i* \mathcal{F}) = i_* A'(i* \mathcal{F})$ and $\det(i* \mathcal{F}) = i_* \det(i* \mathcal{F})$. In § 2 we will introduce tensor-bundles $T(i* \mathcal{F})$ and then $T(i* \mathcal{F})$ is supposed to be $i_* T(i* \mathcal{F})$. For simplicity we write $\mathcal{F}'$ instead of $S'(i* \mathcal{F})$ when $\mathcal{F}$ is of rank one.
1.9. There are several slightly different definitions of weakly positive sheaves in the literature (See [13] for a discussion). We will return here to the original one and in order to formulate 1.10 – we have to extend this notion to sheaves on arbitrary reduced quasi projective schemes. In all applications (even when we will sometimes forget to mention it) we will assume that the open set $U$ meets all components of $Y$ and that the sheaf $\mathcal{F}$ is locally free on some neighbourhood of the non normal locus of $Y$.

**Definition.** Let $\mathcal{F}$ be a coherent torsion free sheaf on a reduced quasi-projective scheme $Y$ and $U \subseteq Y$ be an open subscheme. Let $\mathcal{H}$ be an ample invertible sheaf on $Y$. Then $\mathcal{F}$ is called weakly positive over $U$ if $\mathcal{F}|_U$ is locally free and if for all $a>0$ there exists some $b>0$ such that $S^b(\mathcal{F}) \otimes \mathcal{H}^b$ is globally generated over $U$.

Obviously this definition is independent of the ample invertible sheaf $\mathcal{H}$ chosen. Moreover, we can say that for all $a>0$ there exists some $b_0>0$ such that $S^{b_0}(\mathcal{F}) \otimes \mathcal{H}^{b_0}$ is globally generated over $U$ for all $b \geq b_0$ (see [19], 3.2.i). Weakly positive sheaves have properties similar to those of ample sheaves. Some are recalled in §3.

**Problem 1.10.** Let $Y_0$ be a reduced quasi projective scheme and $(f_0 : X_0 \to Y_0) \in \mathcal{M}_a(Y_0)$ for one of the moduli functors $\mathcal{M}_a$ considered in 1.1. Can one find for $v > 1$ a projective compactification $Y$ of $Y_0$ and a coherent extension $\mathcal{F}$ of $f_0^* \omega^X_{X_0/Y_0}$ to $Y$ such that $\mathcal{F}$ is weakly positive over $Y_0$?

In fact, 1.10 should not depend on the ampleness of $\omega^X_{X_0/Y_0}$. Moreover the extension $\mathcal{F}$ should come from some compactification. Therefore, being more optimistic, we could ask:

**Problem 1.11.** Let $f : X \to Y$ be a surjective projective flat Gorenstein morphism of reduced quasi projective schemes. Assume that for some $v > 0, f_* \omega^X_{X/Y}$ is locally free. Let $Y_0 \subseteq Y$ be an open subscheme, such that for all $y \in Y_0, f^{-1}(y)$ is normal with at most rational singularities. Is $f_* \omega^X_{X/Y}$ weakly positive over $Y_0$?

As we will show in 3.7, 1.11 has an affirmative answer if one assumes in addition that $Y_0$ is non singular. If moreover $f_0 : X_0 = f^{-1}(Y_0) \to Y_0$ is smooth, then this has been obtained in [17] and [18] building up on Kawamata’s positivity theorem ($v = 1$, see [8]). This last mentioned theorem has also been obtained by Kollár [11] as a corollary of his vanishing theorem for semi ample sheaves. Using his idea and some of [3], one can give now a quite simple “algebraic” proof (see [20], 6 and 8).

It is easy to see that for non singular $Y_0$ 3.7 implies an affirmative answer to 1.10 as well. In fact, in order to find $\mathcal{F}$ in 1.10 we may leave out subvarieties of a smooth compactification $Y$ as long as their codimension is bigger than one. If $f_0$ is smooth we can take as well a smooth compactification $X$ of $X_0$ such that $f_0$ extends to $f : X \to Y$. Leaving out the non flat locus of $f$ we can apply 1.11 and find $\mathcal{F} = f_* \omega^X_{X/Y}$ to be weakly positive over $Y_0$. If $Y_0$ is smooth and $f_0 \in \mathcal{M}_a(Y_0)$ we take for $X$ any normal compactification. Leaving out some small subvarieties of $Y$ we may assume that $f$ as well as $f|_{\text{sing}(X)}$ are flat. The remark after 3.7 implies that $\mathcal{F} = f_* \omega^X_{X/Y}$ is weakly positive over $Y_0$. 

An affirmative answer to the following problem 1.12 would allow to use 3.7 in order to show that 1.11 holds true.

**Problem 1.12.** Let $\mathcal{L}$ be an invertible sheaf on $Y$ and $\tau: Y' \to Y$ a desingularization. Assume $\tau^* \mathcal{L}$ is weakly positive over $\tau^{-1}(U)$. Is then $\mathcal{L}$ weakly positive over $U$?

Several unsuccessful attempts to answer this question let me doubt however whether the answer to 1.12 is yes.

1.11 has also an affirmative answer if $Y$ is projective and if the fibres of $f$ are not too bad. In some way our problems have to do with the problem how to find “good” compactifications of morphisms. Instead one could try to compactify the bundles coming from Hodge theory.

1.13. Let $f_0: X_0 \to Y_0$ be a smooth equidimensional projective morphism of quasi projective schemes, let $\tau_0: Y'_0 \to Y_0$ be a desingularization and $i: Y'_0 \to Y'$ be a compactification. We call $Y'$ a good compactification if $Y'$ is non singular, projective and if $Y' - Y'_0$ is a normal crossing divisor. If $f_0: X_0 \to Y'_0$ is the morphism obtained as pullback of $f_0$, we assume that (for $k = \dim X_0 - \dim Y_0$) the monodromy of $R^k f_{0*} \mathcal{C}_{X_0}$ around the components of $Y' - Y'_0$ is unipotent. By W. Schmid’s nilpotent orbit theorem one has a natural locally free sheaf $\mathcal{H}$ on $Y'$ such that $\mathcal{H}|_{Y'_0} = (R^k f_{0*} \mathcal{C}_{X_0}) \otimes_{\mathcal{O}_{Y'_0}} \mathcal{O}_{Y'_0}$ and the subbundle $f_{0*} \omega_{X_0/Y'_0}$ extends to a subbundle $\mathcal{F}'$ of $\mathcal{H}'$. Both sheaves are compatible with further blowing ups of $Y'$.

**Problem.** Can one find a compactification $Y$ of $Y_0$ and a locally free sheaf $\mathcal{F}$ on $Y$ (or even a locally free sheaf $\mathcal{H}$ on $Y$) such that for any good compactification $Y'$ of $Y_0$ with a morphism $\tau: Y' \to Y$ one has $\mathcal{F} = \tau^* \mathcal{F}$ (or $\mathcal{H} = \tau^* \mathcal{H}$)?

In 3.12 we will indicate how an affirmative answer to 1.13 implies one to 1.11 and 1.10, at least under the additional assumption that $f_0$ is smooth. Studying base change properties of powers of dualizing sheaves more carefully than we will do, one should also be able to deduce 1.10 and 1.11 as stated.

**Convention 1.14.** Throughout this article we formulate the proofs such that an affirmative answer to 1.11 allows to erase the words “non singular locus” in all statements of the form “... is weakly positive over the non singular locus of ...” or “... is ample with respect to the non singular locus of ...” Especially this holds for 1.18 and 1.19.

C. Application to fibre spaces

**Convention 1.15.** i) All analytic spaces $Z$ occurring should be Zariski open subspaces of a reduced irreducible separated compact analytic space $\bar{Z}$. We fix a bimeromorphic equivalence class of compactifications.

ii) All coherent sheaves $\mathcal{F}$ on $Z$ should extend as coherent sheaves to $\bar{Z}$. Thereby it makes sense to talk about meromorphic sections of $\mathcal{F}$ and those are denoted by $H^0(Z, \mathcal{F})$.

iii) Each morphism between analytic spaces should extend to some compactification in the equivalence classes choosen.
**Definition 1.16.** Let $Y$ be an analytic space, $U \subseteq Y$ be a Zariski open subspace and $\mathcal{L}$ be a coherent torsionfree sheaf of rank 1 on $Y$. We call $\mathcal{L}$ ample with respect to $U$ if $\mathcal{L}|_U$ is invertible and if for some $a > 0$ there exists a finite dimensional subspace $V \subseteq H^a_0(Y, \mathcal{L}^a)$ such that $V \otimes \mathcal{O}_U \to \mathcal{L}^a$ is surjective over $U$ and the natural morphism $U \to \mathbb{P}(V)$ an embedding.

Of course, if $U \neq \emptyset$, this implies that $Y$ is Moishezon and $U$ quasiprojective. As promised in the introduction we will show that a fine moduli space carries a rank one sheaf, ample with respect to its non-singular points. In fact, we can weaken the assumptions:

**Assumptions 1.17.** Let $f: X \to Y$ be a flat surjective projective Gorenstein morphism of analytic spaces (remember 1.15) and $Y_0 \subseteq Y$ be a non empty Zariski open subspace. Assume that:

i) All fibres of $f|_{X_0}: X_0 \to Y_0$ are irreducible normal varieties of general type with at most rational singularities.

ii) For all $y \in Y_0$, there exists only a finite set of $y' \in Y_0$ such that $f^{-1}(y')$ is birational to $f^{-1}(y)$.

iii) $Y$ is normal (or at least $f_\ast \omega_{X/Y}$ and the sheaf $\mathcal{G}$ in 1.19 are both locally free in some neighbourhood of the non normal locus of $Y$).

**Theorem 1.18.** Under the assumptions made in 1.17 assume that for some $v > 1$ the sheaf $\omega_{X/Y}$ is very ample for each fibre $F$ of $f_0: X_0 \to Y_0$. Then for some $a, b, \mu > 0$ the sheaf $\mathcal{L} = \det(f_\ast \omega_{X/Y}^a) \otimes \det(f_\ast \omega_{X/Y}^b)^\mu$ is ample with respect to the non singular locus of $Y_0$.

**Theorem 1.19.** Under the assumptions made in 1.17 assume that

$$f_\ast \omega_{X/Y} \otimes \mathcal{O}(y) \subset H^0(f^{-1}(y), \omega_{X^{-1}(y)})$$

defines a birational map of $f^{-1}(y)$ for all $y \in Y_0$. Then for some $a, b, \mu > 0$ and

$$\mathcal{G} = \text{Im}(S^a/(f_\ast \omega_{X/Y}^a) \to f_\ast \omega_{X/Y}^a)$$

the sheaf

$$\mathcal{L} = \det(\mathcal{G}) \otimes \det(f_\ast \omega_{X/Y}^b)^\mu$$

is ample with respect to the non singular locus of the open subset $U \subseteq Y_0$ where both $f_\ast \omega_{X/Y}$ and $\mathcal{G}$ are locally free.

Of course 1.18 is just a special case of 1.19. In fact, since $\omega_{X_0/Y_0}$ is ample on each fibre of $f_0$ one can use the Grauert-Riemenschneider vanishing theorem (see [3]) to show that $f_\ast \omega_{X_0/Y_0}$ is locally free for $v > 1$. Moreover, for $\mu > 0$ the multiplication map is surjective over $Y_0$. Therefore we can assume that the inclusion $\mathcal{G} \subset f_\ast \omega_{X/Y}$ is an isomorphism over $Y_0$ and choose $U = Y_0$ in 1.19.

In fact we will see in 4.9, that under the assumptions of 1.18, $\mathcal{L}$ is weakly positive over $Y_0$ if the singular locus of $Y_0$ is compact.

The reader interested in stability of Hilbert points and familiar with [14] can use the proof of Theorem 1.19 in § 4 as an illustration how the proof of
the stability Theorem 1.7 will work. In some way, the Reynolds-operator used in [14] can be replaced by the splitting obtained in 2.6, and in 4.5 the Hilbert-Mumford criterium is hidden behind the curtain (see Remark 4.4 and 4.6).

D. Proof of \( C_{n,m} \) for certain fibre spaces

We will use the notations coming from classification theory and the reader not familiar with this theory should consult the excellent survey articles [2] and [13] for the exact definitions, references and historical remarks.

**Theorem 1.20** (Kollár [12], Kawamata and myself, under more restrictive assumptions [9, 19]). Let \( f: X \to Y \) be a morphism of projective manifolds with an irreducible general fibre \( X_\omega \) of general type.

i) \( C_{n,m} \) \( \kappa(Y) \geq 0 \) then

\[
\kappa(X) \geq \max \{ \kappa(Y) + \kappa(X_\omega), \operatorname{Var}(f) + \kappa(X_\omega) \}
\]

ii) If \( \operatorname{Var}(f) = \dim Y \), then for \( \mu, \nu \gg 0 \) the sheaf \( S^\mu (f_* \omega_X^\nu) \) contains an ample subsheaf of full rank.

iii) If \( \operatorname{Var}(f) = \dim Y \), then for some \( \eta \gg 0 \) the reflexive hull of \( f_* \omega_X^\eta \) contains an ample invertible sheaf.

**Proof.** As explained in [19], 3.4ii) and iii) are equivalent. In [18] it was shown that ii) implies \( C_{n,m} \) for the corresponding type of morphism. To prove ii) we use 1.19 together with the constructions developed in [18]:

We can replace \( Y \) by the complement of a codimension two subvariety (as in 3.3.d) and thereby we may assume that \( f: X \to Y \) satisfies the assumptions made in 1.17. As in [18], 6.1 we can make semistable reduction in codimension one and, leaving again out a codimension two subvariety, we may assume that \( f \) is semi-stable. 1.19 tells us that for some \( v, \mu, a, b \gg 0 \) the sheaf \( \mathcal{L} := \det(\mathcal{O}^\bigotimes \det(f_* \omega_X^\nu) \bigotimes \omega_Y^\nu) \) has maximal Itaka dimension \( \kappa(\mathcal{L}) \). Therefore, choosing \( a \) and \( b \) big enough, we can assume \( \mathcal{L} \) to contain an ample invertible sheaf. Since \( \det(\mathcal{O}) \) is contained in some wedge product of \( f_* \omega_X^\nu \) we may find some \( \eta_1 \) and \( \eta_2 \) such that \( \mathcal{L} \) lies in

\[
(\bigotimes^\nu f_* \omega_X^\nu) \bigotimes (\bigotimes^\nu f_* \omega_X^\nu)
\]

(see § 2, for example, or [7]).

By [18], 3.4 and 3.5 \( \bigotimes f_* \omega_X^\nu \) is nothing but \( f_*^{(a)} \omega_X^{(a)} \) where \( X^{(a)} \) is a desingularization of the \( a \)-fold fibre product \( X \times_Y X \times_Y \ldots \times_Y X \). The product map for \( X^{(a)} \) induces

\[
S^\mu \bigotimes f_* \omega_X^\nu \to \bigotimes f_* \omega_X^\nu
\]

and therefore \( \mathcal{L}^\mu \) is a subsheaf of \( \bigotimes f_* \omega_X^\nu \) for \( \eta = \mu \cdot \eta_1 + \eta_2 \). The equivalence of ii) and iii), applied to the fibre space \( X^{(a)} \) shows that for some \( \gamma \gg 0 \)

\[
S^\gamma \bigotimes f_* \omega_X^\nu
\]

contains an ample subsheaf of full rank. Then the same must hold for the quotient \( S^{\gamma-n}(f_* \omega_X^\nu) \).
Remark 1.21. The proof of $C_{n,m}^+$ given above does not use anymore analytic methods from Hodge theory, except of the degeneration of the Hodge-Deligne spectral sequence, hidden behind the vanishing theorem of Kollár (see [3], §3). Since the degeneration of this spectral sequence has been shown by Deligne and Illusie using characteristic $p$ methods we can say that the proof of $C_{n,m}^+$ for families of manifolds of general type, presented here, is algebraic and “easier” than the ones given before.

If $Y$ is a curve, the necessary tools from “weak positivity” are quite trivial, and the proof of $C_{n,1}^+$ obtained from 1.19 is quite simple.

§ 2. Tensor bundles

2.1. Throughout this section we consider an algebraic scheme $X$ or an analytic space $X$ together with a locally free sheaf $\mathcal{E}$ which is of rank $r$ on all components of $X$. As described in [7] for example, a finite dimensional representation $T: Gl(r, \mathbb{C}) \to Gl(n, \mathbb{C})$ gives rise to a bundle $T(\mathcal{E})$.

Definition. We call $T(\mathcal{E})$ the tensor bundle (of $T$). If $T$ is an irreducible representation we call $T(\mathcal{E})$ an irreducible tensor bundle.

2.2. Let $T$ be an irreducible representation. Then the irreducible tensor bundle $T(\mathcal{E})$ is, up to isomorphism, uniquely determined by the “upper weight” $c(T) = (n_1, \ldots, n_r)$. This, as well as the following construction of $c(T)$, can be found in [7], A.6: Let $P$ be the group of upper triangular matrices. There is a unique one dimensional subspace of $\mathbb{C}^n$ consisting of eigenvectors of $T|_P$. If $\lambda: P \to \mathbb{C}^*$ is the corresponding character, then $\lambda$ applied to a diagonal matrix $(h_{ii})$ gives $\prod_i h_{ii}^k$ if $c(T) = (n_1, \ldots, n_r)$. One has $n_1 \geq \cdots \geq n_r$.

Definition. We call $c(T)$ the upper weight of the irreducible tensor bundle $T(\mathcal{E})$. $T(\mathcal{E})$ is called positive if $n_r \geq 0$ and $n_1 > 0$ (for all irreducible summands if $T$ is reducible).

2.3. Examples of tensor bundles are the symmetric products $S^r(\mathcal{E})$, the tensor products $\otimes^r(\mathcal{E})$. If $T_1(\mathcal{E})$ are tensor bundles, for $i = 1, 2$, then the same holds for $T_1(\mathcal{E}) \oplus T_2(\mathcal{E})$ and $T_1(\mathcal{E}) \otimes T_2(\mathcal{E})$. The determinant $\det(\mathcal{E})$, as well as $\det(\mathcal{E})^\eta$ for $\eta \in \mathbb{Z}$, are irreducible tensor bundles of upper weight $(\eta, \ldots, \eta)$.

Lemma 2.4. Let $\rho: \det(\mathcal{E}) \to \bigotimes^r \mathcal{E}$ be the map

$$\rho(x_1 \wedge \ldots \wedge x_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sign}(\sigma) x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(r)}.$$ 

Then for all $\eta \in \mathbb{N}$ the image of $\rho^\eta$ embeds $\det(\mathcal{E})^\eta$ as direct summand in $\bigotimes^r S^\eta(\mathcal{E})$.

Proof. If $e_1, \ldots, e_r$ is a basis of $\mathbb{C}^r$ then, with respect to the standard representation,

$$u = \frac{1}{r!} \sum_{\sigma \in S_r} \text{sign}(\sigma) e_{\sigma(1)} \otimes \ldots \otimes e_{\sigma(r)} \in \bigotimes^r \mathbb{C}^r.$$
is an eigenvector of $P$. The induced irreducible subrepresentation has upper weight $(1, \ldots, 1)$. As in [7], p. 75, let $u^\eta$ be the image of $u^{\otimes \eta}$ under the multiplication map $\otimes \to \otimes S^n \mathcal{C}^*$. Then $u^\eta$ is again an eigenvector, and the corresponding upper weight is $(\eta, \ldots, \eta)$. Since the irreducible subrepresentation is uniquely determined by the upper weight it must be $\det(\mathcal{C}^*)^\eta$.

2.5. A more geometric interpretation of the map $\rho$ from (2.4) can be given by considering the projective bundle $\mathbb{P} = \mathbb{P}(\bigoplus \mathcal{E}^\vee) \to X$ of $\bigoplus \mathcal{E}^\vee = \text{Hom}_{\text{aff}}(\mathcal{E}, \bigoplus \mathcal{C}_X)$. We have

$$\pi_* \mathcal{O}_\mathbb{P}(v) = S^r(\bigoplus \mathcal{E}^\vee) = \bigoplus_{i=1}^r S^{x_i}(\mathcal{E}^\vee),$$

where the direct sum is taken over all $(\mu_1, \ldots, \mu_r)$ with $\sum_i \mu_i = v$. Therefore the map $\rho$ gives rise to

$$\rho: \det(\mathcal{E})^{-1} \to \bigotimes \mathcal{E}^\vee \to \pi_* \mathcal{O}_\mathbb{P}(r).$$

We denote the induced section of $\mathcal{O}_\mathbb{P}(r) \otimes \pi^* \det(\mathcal{E})$ again by $\rho$ and write $D$ for its zero divisor. On $\mathbb{P}$ we have the universal map $\pi^* \bigoplus \mathcal{E}^\vee \to \mathcal{O}_\mathbb{P}(1)$ or, taking its dual, a “universal basis”

$$(s_1, \ldots, s_r): \mathcal{O}_\mathbb{P}(-1) \to \bigoplus \pi^* \mathcal{E}.$$ 

The wedge product of $s_1, \ldots, s_r$ factors over

$$\mathcal{O}_\mathbb{P}(-r) \to S^r(\bigoplus \pi^* \mathcal{E}) \to \bigotimes \pi^* \mathcal{E} \to \det(\pi^* \mathcal{E}).$$

Since this is just the dual of $\rho$ we find $D$ to be the degeneration locus of $s_1, \ldots, s_r$.

Altogether we obtain:

**Lemma 2.6.** Let $s: \bigoplus \mathcal{O}_\mathbb{P}(-1) \to \pi^* \mathcal{E}$ be the universal basis and $D$ the degeneration locus of $s$. Then the corresponding section

$$\rho^\eta: \mathcal{O}_\mathbb{P} \to \mathcal{O}_\mathbb{P}(\eta \cdot D) = \mathcal{O}_\mathbb{P}(r \cdot \eta) \otimes \pi^* \det(\mathcal{E})^\eta$$

gives rise to a direct summand $\mathcal{C}_X \to \pi_* \mathcal{O}_\mathbb{P}(\eta \cdot D)$.

The following proposition is the key of the proof of Theorem 1.18 and 1.19 in §4. We remind the reader of the convention 1.15 on analytic spaces and of Definition 1.16.

**Proposition 2.7.** Let $X$ be an analytic space, $\mathcal{L}$ be a coherent torsionfree rank one sheaf on $X$ and $U \subseteq X$ be a Zariski open subspace. Let $\tau: \mathbb{P} \to \mathbb{P}$ be a proper modification of $\mathbb{P}$ with center in $D$ (we keep the notations from 2.5) and $\pi' = \pi \circ \tau$. 

Let $D'$ be an effective divisor of $\mathbb{P}$ with support in $\tau^{-1}(D)$. Assume that $\mathcal{L}' = \pi^* \mathcal{L} \otimes \mathcal{O}_D(D')$ is ample with respect to $\pi'^{-1}(U)$. Then $\mathcal{L}'$ is ample with respect to $U$.

**Proof.** For $v > 0$ we can find some $\eta$ such that $0 \leq vD' \leq \tau^*(\eta \cdot D)$ and such that one has an inclusion $\tau_* \mathcal{O}_\mathbb{P}(vD') \to \mathcal{O}_\mathbb{P}(\eta \cdot D)$. By 2.5 we obtain $\mathcal{O}(\pi^* \mathcal{L}') = \mathcal{L}'$ as a direct summand. The inclusion $\pi^* \mathcal{L}' \to \mathcal{L}'$ gives thereby rise to a natural splitting of

$$H^0_m(X, \mathcal{L}') \to H^0_m(\mathbb{P}', \mathcal{L}').$$

"Natural" means: If $Z$ is a (not necessarily reduced) analytic subspace of $X$ and $\mathbb{P}_Z$ the proper transform of $Z$ in $\mathbb{P}'$, the splitting of

$$H^0_m(Z, \mathcal{L}'|_Z) \to H^0_m(\mathbb{P}_Z, \mathcal{L}'|_{\mathbb{P}_Z})$$

is compatible with the one given above.

We have a commutative diagram

$$\begin{array}{ccc}
H^0_m(\mathbb{P}', \mathcal{L}') & \to & H^0_m(X, \mathcal{L}') \\
\downarrow{z'} & & \downarrow{z} \\
H^0_m(\mathbb{P}_Z, \mathcal{L}'|_{\mathbb{P}_Z}) & \to & H^0_m(Z, \mathcal{L}').
\end{array}$$

If we take $Z = x \cup y$, for two points $x, y \in U$, a finite dimensional subspace of $H^0_m(\mathbb{P}', \mathcal{L}')$ embeds a neighbourhood of $\mathbb{P}_Z$ in a projective space. Choosing $v$ bigger we may assume that $z'$ is surjective. Then $z$ is surjective as well and we find $V \subseteq H^0(X, \mathcal{L}')$ generating $\mathcal{L}'$ in $x$ and $y$ and separating the points $x$ and $y$.

In the same way we can take $Z$ to be the subspace defined by the square of the ideal of $x$, in order to see that (for $v \gg 0$) some subspace $V_v$ separates the tangent directions in $x$. If we define $V_v, \eta \in N$ to be the subspace spanned by monomials in elements of $V_v$, the same holds for $V_v$. Since $U$ is compact in the Zariski topology we can find for $\eta \gg 0$ some larger finite dimensional space $V' \subseteq H^0_m(X, \mathcal{L}'|_{\eta})$ giving an embedding of $U$.

§ 3. Weak positivity, revisited

Since the notation of "weakly positive sheaves over a given open subscheme" introduced in 1.9 is central for this article we recall and extend the properties of weakly positive sheaves (see also [17–19] and [13]). Some of them will be needed in 3.7, where we verify 1.10 and 1.11 under the additional assumption that $Y_0$ is smooth.
Assumptions 3.1. $Y$ is a reduced quasi-projective scheme, $U \subseteq Y$ an open subscheme and $\mathcal{H}$ is an ample invertible sheaf on $Y$. We assume that $U$ meets all components of $Y$. Let $\mathcal{F}$ be a coherent torsion free sheaf which is locally free in some neighbourhood of the non normal locus of $Y$.

Lemma 3.2. Assume that $\mathcal{F}$ is of rank one. Then $\mathcal{F}$ is weakly positive over $U$ if and only if $\mathcal{F}^a \otimes \mathcal{H}$ is ample with respect to $U$ for all $a > 0$.

Proof. If $\mathcal{F}$ is weakly positive over $U$ then $\mathcal{F}^{2ab} \otimes \mathcal{H}^b$ is globally generated over $U$ for $b > 0$, and therefore $\otimes \mathcal{H}^b$ maps to a subsheaf of $\mathcal{F}^{2ab} \otimes \mathcal{H}^{2b}$, surjective over $U$. This implies that $\mathcal{F}^a \otimes \mathcal{H}$ is ample with respect to $U$. The other direction is obvious.

3.3. Some simple properties

a) $\mathcal{F}$ is weakly positive over $U$ if and only if each $u \in U$ has a neighbourhood $V(u)$ such that $\mathcal{F}$ is weakly positive over $V(u)$.

b) If $\mathcal{F}$ and $\mathcal{F}'$ satisfy the properties asked for in (3.1), and if both are weakly positive over $U$ then $\mathcal{F} \otimes \mathcal{F}'$ is weakly positive over $U$.

c) If $\mathcal{F} \to \mathcal{F}'$ is surjective over $U$ and $\mathcal{F}$ weakly positive over $U$, then $\mathcal{F}'$ is weakly positive over $U$ as well.

d) If $U \subseteq Y' \subseteq Y$ are open subschemes and depth$_{Y-Y'} \mathcal{O}_Y \geq 2$ then $\mathcal{F}$ is weakly positive over $U$ if and only if $\mathcal{F}|_{Y'}$ is weakly positive over $U$. Especially we can leave out subvarieties of $Y$ of codimension bigger than or equal to two, as long as they do not meet the non normal locus of $Y$, and thereby we may -- whenever it is convenient -- assume that $\mathcal{F}$ is locally free.

e) If, for some $\eta > 0$, $S^\eta(\mathcal{F})$ or $\otimes^\eta(\mathcal{F})$ is weakly positive over $U$, then the same holds for $\mathcal{F}$.

Proof. a, b and c follow directly from the definition. d) follows from [5], 1.9 and 3.8 and to verify e) one just has to use the natural maps $\otimes^\eta \mathcal{F} \to S^\eta(\mathcal{F})$ and $S^\eta S^\eta(\mathcal{F}) \to S^{\eta+\eta}(\mathcal{F})$, which are both surjective over $U$.

3.4. Functorial properties

a) If $\tau: Y' \to Y$ is a morphism such that $\tau^{-1}(U)$ meets all components of $Y'$, and if $\mathcal{F}$ is weakly positive over $U$, then $\tau^* \mathcal{F}$ is weakly positive over $\tau^{-1}(U)$.

b) Let $\tau: Y' \to Y$ be a projective surjective morphism such that $\tau^{-1}(U)$ meets all components of $Y'$ and such that $\tau^{-1}(U) \to U$ is finite. Assume moreover that $\mathcal{O}_{Y'} \to \tau_* \mathcal{O}_{\tau^{-1}(U)}$ is a direct summand (for example this holds if $U$ is normal). Then $\mathcal{F}$ is weakly positive over $U$ if and only if $\tau^* \mathcal{F}$ is weakly positive over $\tau^{-1}(U)$.

c) $\mathcal{F}$ is weakly positive over $U$ if and only if there exists some $\mu > 0$ such that for all finite surjective morphisms $\tau: Y' \to Y$ and for all ample invertible sheaves $\mathcal{H}'$ on $Y'$ the sheaf $\tau^* \mathcal{F} \otimes \mathcal{H}'^{\mu}$ is weakly positive over $\tau^{-1}(U)$.

d) Assume that $\mathcal{F}$ is locally free, and let $\pi: \mathbb{P}(\mathcal{F}) \to Y$ be the projective bundle of $\mathcal{F}$. Then $\mathcal{F}$ is weakly positive over $U$ if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ is weakly positive over $\pi^{-1}(U)$. 
e) Assume that the singular locus of $U$ is compact. Let $\tau: Y' \to Y$ be a surjective projective generically finite morphism. Then $\mathcal{F}$ is weakly positive over $U$ if and only if $\tau^* \mathcal{F}$ is weakly positive over $\tau^{-1}(U)$.

Proof. a) is obvious. Using it together with 3.3d we may assume that $\mathcal{F}$ is locally free in the sequel.

b) The “only if” follow from a). Let us assume $\tau^* \mathcal{F}$ to be weakly positive over $\tau^{-1}(U)$. We may choose $\mathcal{H}$ such that $\tau_* \mathcal{O}_Y \otimes \mathcal{H}^\nu$ is generated by its global sections for all $\nu \gg 0$. By a) we can replace $Y'$ by any blow up and hence we may find an effective divisor $E$ such that $\mathcal{O}_{Y'}(-E)$ is relative ample for $\tau$ and such that $\tau^{-1}(Y-U) = E_{\text{red}}$. Moreover we may assume $\tau^* \mathcal{H}(-E)$ to be ample on $Y$. By assumption $\mathcal{O}_Y \to \tau_* \mathcal{O}_{Y'}$ splits over $U$. Therefore for $b \gg 0$ we obtain a map $\rho: \tau_* \mathcal{O}_{Y'}(-bE) \to \mathcal{O}_Y$ surjective over $U$. For given $a$ and $b$, we have a map
\[ \bigoplus \mathcal{O}_{Y'} \to S^{2a-b}(\tau^* \mathcal{F}) \otimes \tau^* \mathcal{H}(-E)^b, \]
surjective over $\tau^{-1}(U)$. Then the induced map
\[ \bigoplus \tau_* \mathcal{O}_{Y'} \otimes \mathcal{H}^b \to S^{2a-b}(\mathcal{F}) \otimes \mathcal{H}^{2b} \otimes \tau_* \mathcal{O}_Y(-bE)^b \to S^{2a-b}(\mathcal{F}) \otimes \mathcal{H}^{2b} \]
is also surjective over $U$.

c) The “only if” part follows from a) and the obvious fact, that a weakly positive sheaf keeps this property if it is tensorised by an ample sheaf. For the other direction we choose $\mathcal{H}$ to be very ample on $Y$ and $Y \to \mathbb{P}^N$ to be the corresponding embedding. For a given and $d=1+2a \cdot \mu$ we choose a non singular finite cover $\pi: Z \to \mathbb{P}^N$ such that $\pi^* \mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{H}^d$. If we take $Y' = \pi^{-1}(Y) \to \pi$, then $\mathcal{O}_Y \to \tau_* \mathcal{O}_Y$ splits. By assumption, for $b \gg 0$,
\[ S^{2a-b}(\pi^* \mathcal{F} \otimes \mathcal{H}^b) \otimes \mathcal{H}^b = \tau^* S^{2a-b}(\mathcal{F}) \otimes \mathcal{H}^b \]
is globally generated over $\tau^{-1}(U)$. The same argument as in b) finishes the proof.

d) Since $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^N}(1)$ is a quotient of $\pi^* \mathcal{F}$ the “only if” is obvious. For the other direction we choose $\mathcal{H}$ such that $\tau^* \mathcal{H} \otimes \mathcal{O}(1)$ is very ample. For given $a$ we find some $b$ such that $\mathcal{O}(2a+b-1) \otimes \mathcal{O}(b) \otimes \tau^* \mathcal{H}^b$ is globally generaced over $\tau^{-1}(U)$. Then $\mathcal{O}(2a+b) \otimes \tau^* \mathcal{H}^{2b}$ will have enough global sections to embed all fibres of $\pi$ over $U$ into some projective space. Therefore for $\eta \gg 0$ the multiplication map
\[ \pi_* S^\eta(\bigoplus \mathcal{O}) \to \pi_* \mathcal{O}(2a+b \cdot \eta) \otimes \mathcal{H}^{2b}, \eta \]
is surjective over $U$.

To prove e) we need:

3.5. Applications of vanishing theorems

Let $Z$ be a non singular projective variety and $i: Y \to Z$ an open embedding. Let $\mathcal{G}$ be an invertible sheaf on $Z$ such that $\mathcal{G} \otimes \omega_Z^{-1}$ is numerically effective. Let $\mathcal{F}$ be an invertible sheaf on $Y$ and $\mathcal{H} = t^* \mathcal{G}$.
a) If $\mathcal{F}$ is ample with respect to $U$ we can find a coherent sheaf $\mathcal{L}$, not depending on $U$, and an inclusion $\mathcal{L} \rightarrow t_* \mathcal{F}$, surjective over $U$, such that $H^i(Z, \mathcal{L} \otimes G) = 0$ for $i > 0$.

b) Let $\mathcal{G}$ be very ample on $Y$, and let $\tau: Y' \rightarrow Y$ be a projective birational morphism such that $\tau^* \mathcal{F}$ is weakly positive over $\tau^{-1}(U)$. Then $\mathcal{F} \otimes \mathcal{H}^+\tau^* \mathcal{G}$ is globally generated over $U$ for all $v > \text{dim } Y + 1$.

c) Under the assumptions made in b) $\mathcal{F}$ is weakly positive over $U$.

*Proof of 3.4(e).* By part d) we only have to consider an invertible sheaf $\mathcal{F}$. Using b) we may assume that the singular locus of $Y$ lies in $U$. Moreover we may assume $\tau: Y' \rightarrow Y$ to be a desingularization (The general case then follows from b). Let us consider first the case where $\tau$ is an isomorphism outside of the singular locus $S$ of $Y$. Since $S \subseteq U$, the invertible sheaf $\mathcal{F}|_S$ is numerically effective and (using Seshadri's criterion) $\mathcal{F}^* \otimes \mathcal{H}^+|_S$ is ample for all $a > 0$ and all ample invertible sheaves $\mathcal{H}^+$.

Let $i: Y \rightarrow Z$ be a compactification, non singular outside of $S$. We obtain a desingularization $i^*: Z' \rightarrow Z$ such that $i^*: \tau^{-1}(Y) = Y' \rightarrow Z'$ and $\tau = i|_{Y'}$.

Blowing up $Z'$ a little bit more we can find an exceptional divisor $E$ such that $\mathcal{O}_{Z'}(-E)$ and $\mathcal{O}_{Z'}(-E) \otimes \mathcal{O}_{Z'}$ are both $\tau'$-ample and $\tau_* \mathcal{O}_{Z'}(-E)$ contained in $\mathcal{O}_{Z'}$. By Kodaira vanishing $\mathcal{O}_{Z'}(-\cdot E)$ will have no higher cohomology if we tensor with the pullback of a high power of an ample sheaf on $Z$. Then $R^i \tau_* \mathcal{O}_{Z'}(-\cdot E) = 0$ for $i, v > 0$.

Replacing $\mathcal{H}^+$ by some power we may assume that $\mathcal{H}^+ = \tau^* \mathcal{G}$ for some ample sheaf $\mathcal{G}$ on $Z$ and that $\mathcal{F} = \tau^* \mathcal{G} \otimes \mathcal{O}_{Z'}(-E)$ and $\mathcal{F} \otimes \mathcal{O}_{Z'}(-E)$. are both very ample.

By 3.5(b) $\tau^* (\mathcal{F}^* \otimes \mathcal{H}^+ \otimes \mathcal{G}^*) \otimes \mathcal{O}_{T'}(-v \cdot E)$ is globally generated over $\tau^{-1}(U)$ for $v > \text{dim } Y + 1$ and all $a, b > 0$. Therefore it just remains to verify that $\mathcal{F}^a \otimes \mathcal{H}^+ \mathcal{G}^b$ is generated in points of $S$ by global sections. By 3.5(a) we find some subsheaf $\mathcal{L}'$ of $\tau_* \mathcal{F}^a \otimes \mathcal{H}^+ \mathcal{G}^b$, isomorphic over $\tau^{-1}(U)$, such that $\mathcal{L}' \otimes \mathcal{O}_{T'} \otimes \mathcal{O}_{T'} \otimes \mathcal{H}^+ \mathcal{G}^b$ has no higher cohomology for $v > 0$ and $\mu \geq 0$. Again this implies that $R^i \tau_* \mathcal{L}' \otimes \mathcal{H}^+ \mathcal{G}^b = 0$ and $H^i(Z, \mathcal{L}') = 0$ for $i > 0$ and $\mathcal{L}' = \tau_* (\mathcal{L}' \otimes \mathcal{G}^* \mathcal{H}^+ \mathcal{G}^b \mathcal{H}^+ \mathcal{G}^b)$ contained in $\tau_* \mathcal{O}_{T'}(-v \cdot E) \otimes \mathcal{O}_{T'}(\mathcal{F}^a \otimes \mathcal{H}^+ \mathcal{G}^b \mathcal{H}^+ \mathcal{G}^b)$, isomorphic over $U$.

$\mathcal{C} = \text{coker}(\mathcal{L}' \rightarrow \tau_* (\mathcal{F}^* \otimes \mathcal{H}^+ \mathcal{G}^b \mathcal{H}^+ \mathcal{G}^b))$ is a direct sum of $\mathcal{C}' = \text{coker}(\tau_* \mathcal{O}_{T'}(-v \cdot E) \rightarrow \mathcal{O}_{T'} \otimes \mathcal{F}^* \otimes \mathcal{H}^+ \mathcal{G}^b \mathcal{H}^+ \mathcal{G}^b)$ and of some sheaf supported outside of $Y$. Alltogether we get a surjection

$$H^0(Z, \tau_* (\mathcal{F}^* \otimes \mathcal{H}^+ \mathcal{G}^b)) \rightarrow H^0(Z, \mathcal{C}).$$

$\mathcal{C}'$ is supported in $S$ and -- replacing $a$ and $b$ by some common multiple -- we can assume that $\mathcal{C}'$ is globally generated over $U \cap S$.

Finally, if $Y$ is non singular and $\tau: Y' \rightarrow Y$ any blowing up we obtain e) from 3.5(c).

*Remark.* Obviously the proof of e) shows that an affirmative answer to Problem 1.12 would imply 3.4e) without the assumption on the compactness of the singular locus. If this holds Theorem 3.7 and the usual base change arguments ([13], §3 for example) would imply affirmative answers to 1.11 and 1.10.

*Proof of 3.5.* a) Since $\mathcal{F}$ is ample with respect to $U$ we can find some extension $\mathcal{F}'$ on $Z$ of $\mathcal{F}$ which is again ample with respect to $U$. Therefore it is enough
to consider the case \( Y = Z \). For some \( a > 1 \) the sheaf \( \mathcal{F}^a \) is globally generated over \( U \). Let \( \tau: Y' \to Y \) be a blowing up such that a general divisor \( D \) of \( \tau^* \mathcal{F}^a \) is a normal crossing divisor. \( D \) will be non singular on \( \tau^{-1}(U) \). By the vanishing theorem for integral parts of \( \mathcal{Q} \)-divisors ([3], 2.13 for example) one finds for \( i > 0 \) that

\[
H^i\left( Y', \tau^* (\mathcal{F} \otimes \mathcal{O} \otimes \omega_{Y'}^{-1}) \otimes \omega_Y \left( -\left[ \frac{D}{a} \right] \right) \right) = 0.
\]

Again this implies \( R^i \tau_* \omega_Y \left( -\left[ \frac{D}{a} \right] \right) = 0 \) for \( i > 0 \) and, for

\[
\mathcal{L} = \mathcal{F} \otimes \tau_* \omega_{Y'/Y} \left( -\left[ \frac{D}{a} \right] \right),
\]

we obtain a).

To prove b) and c) we choose a compactification \( i': Y' \to Z \) such that \( \tau \) comes from a morphism \( \tau': Z' \to Z \). By 3.4, a we may assume that \( Z' \) is a sequence of blowing ups with non singular center. Since we will prove b) and c) simultaneously we may assume by induction that \( \tau \) is just one blowing up. Then \( \omega_{Z'/Z}^{1/2} \) is \( \tau' \) - ample and for some \( N > 0 \) \( \tau^* \mathcal{F}^N \otimes \omega_{Z'/Z}^{1/2} \) will be ample. By 3.2 \( \tau^*(\mathcal{F} \otimes \mathcal{H}^N) \otimes \omega_{Y'}^{1/2} \) will be ample with respect to \( U \). For \( v > 0 \) and \( \mathcal{F}' = \tau^* \mathcal{F} \otimes \omega_{Z'/Z} \) we can apply a) to obtain a subsheaf \( \mathcal{L}' \) of \( i'_*(\tau^*(\mathcal{F} \otimes \mathcal{H}^N) \otimes \omega_{Y'/Z}^{1/2}) \), isomorphic over \( U \), with \( H^i(Z', \mathcal{L}' \otimes \mathcal{F}') = 0 \) for \( i > 0 \). Again since \( \mathcal{L}' \) is independent of \( \mathcal{F}' \), this implies that the higher direct images of \( \mathcal{L}' \otimes \mathcal{F}' \) vanish as well. For \( \mathcal{L} = \tau_* (\mathcal{L} \otimes \omega_{Z'/Z}) \otimes \mathcal{F}^{-1} \) we obtain \( H^i(Z, \mathcal{L} \otimes \mathcal{F}') = 0 \) for \( i > 0 \) and \( v > N \).

We claim that this implies that \( \mathcal{L} \otimes \mathcal{F}' \) is globally generated over \( U \) for \( v > \dim Y + N \).

In fact, if \( H \) is a smooth divisor of \( \mathcal{F} \), then \( \mathcal{L} \otimes \mathcal{F}'|_H \) has no higher cohomology for \( v > N + 1 \). By induction on \( \dim Y \) we may assume that \( \mathcal{L} \otimes \mathcal{F}'|_H \) is globally generated over \( H \cap U \) for \( v > \dim H + N + 1 = \dim Y + N \). However \( H^0(Y, \mathcal{L} \otimes \mathcal{F}'|_H) \to H^0(H, \mathcal{L} \otimes \mathcal{F}'|_H) \) is surjective.

Hence \( \mathcal{F} \otimes \mathcal{H}' \) is globally generated over \( U \) for \( v > \dim Y + N \). Since \( N \) did not depend on \( \mathcal{F} \) but just on the center of the blowing up we obtain that \( \mathcal{F} \otimes \mathcal{H}' \) is as well globally generated over \( U \) for \( v > \dim Y + N \). Therefore \( \mathcal{F} \) is weakly positive over \( U \) and the blowing up was not necessary. Hence, \( N = 1 \) will do as well.

**Lemma 3.6.** Let \( \mathcal{F} \) be weakly positive over \( U \), and let \( T(\mathcal{F}) \) be any tensor bundle (see 1.8). If \( T(\mathcal{F}) \) is a positive tensor bundle (see 2.2) then \( T(\mathcal{F}) \) is weakly positive over \( U \).

**Proof.** (see also [19], 3.2). Let \( \mathcal{H} \) be ample on \( Y \). By 3.4c) it is enough to show that \( T(\mathcal{F} \otimes \mathcal{H}) \) is weakly positive over \( U \). By [7], 5.1, \( S^v(T(\mathcal{F} \otimes \mathcal{H})) \) will be a direct summand of

\[
S^{v_1}(\mathcal{F} \otimes \mathcal{H}) \otimes \cdots \otimes S^{v_n}(\mathcal{F} \otimes \mathcal{H})
\]

for some \( v_i \) which are growing like \( \eta \). Therefore \( S^v(T(\mathcal{F} \otimes \mathcal{H})) \) will be globally generated over \( U \) for \( \eta > 0 \) and 3.3c) implies the weak positivity of \( T(\mathcal{F} \otimes \mathcal{H}) \).
Examples of positive tensor bundles are: \( \det(\mathcal{F}) \), \( S^r(\mathcal{F}) \) and \( A^r(\mathcal{F}) \). Especially, if \( r \) is the rank of \( \mathcal{F} \) and \( \mathcal{F}^\vee = \mathcal{Hom}(\mathcal{F}, \mathcal{O}_Y) \) then \( \bigwedge^r(\mathcal{F}) = \mathcal{F}^\vee \otimes \det(\mathcal{F}) \) is a positive tensor bundle. Using 3.3b) and the equality

\[
S^2(\mathcal{F} \oplus \mathcal{F}') = S^2(\mathcal{F}) \oplus S^2(\mathcal{F}') \oplus \mathcal{F} \otimes \mathcal{F}'
\]

one sees that weak positivity is compatible with tensor products.

**Theorem 3.7.** Let \( n > 0 \) and \( f: X \to Y \) be a surjective projective flat Gorenstein morphism of reduced quasi projective schemes. Assume that \( f_* \omega_X^{\cdot} \) is locally free. Let \( Y_0 \subseteq Y \) be an open subscheme meeting all components of \( Y \) such that \( f^{-1}(y_0) \) is normal with at most rational singularities, for \( y_0 \in Y_0 \). Then \( f_* \omega_X^{\cdot} \) is weakly positive over the non singular locus of \( Y_0 \).

**Remark.** As we will see in the proof, we can slightly weaken the assumptions made in Theorem 3.7. If \( f: X \to Y \) is a surjective projective flat morphism of reduced quasi-projective schemes it will be enough that \( f \) is Gorenstein outside of a subvariety \( Z \) of \( X \) such that \( f(Z) \subseteq Y - Y_0 \) and the codimension of \( Z \cap f^{-1}(y) \) in \( f^{-1}(y) \) is bigger than one. In this case we choose \( \omega_X^{\cdot} \) to be the reflexive hull of \( \omega_X^{\cdot} \).

Of course 3.7 will be shown by reducing it to the case \( n = 1 \), where it is nothing but the positivity theorem of Kawamata and Fujita (dim \( Y = 1 \)). Since we really need to keep track of the locus where the sheaves are weakly positive we sketch the proof:

**Proof.** We may assume \( Y_0 \) to be non singular. Let \( \tau: Y' \to Y \) be a morphism. We write \( f': X' \to Y' \) for the fibre product \( X \times_Y Y' \to Y' \) and \( Y_0 = \tau^{-1}(Y_0) \).

**Claim 3.8.** We may assume that \( Y \) is normal.

**Proof.** By 3.4a) and b) we are allowed to replace \( Y \) by some blowing up as long as the center does not meet \( Y_0 \). Therefore we may assume that the non normal locus of \( Y \) lies in some Cartier divisor \( F \) supported in \( Y - Y_0 \). Let \( \tau: Y' \to Y \) be the normalization. For \( \mu \geq 0 \) the invertible sheaf \( \mathcal{F} = \tau^* \mathcal{O}_Y(-\mu \cdot F) \) will satisfy

\[
\tau^* \mathcal{F} = \tau_* \mathcal{O}_{Y'} \otimes \mathcal{O}_Y(-\mu \cdot F) \subset \mathcal{O}_Y.
\]

By flat base change [6], one has

\[
pr_{1*} f'^* \mathcal{F} \subset \mathcal{O}_X.
\]

This implies that \( \tau_*((f'_* \omega_{Y'/Y}) \otimes \mathcal{F}) \) is contained in \( f_* \omega_{X/Y} \). Let \( f^*: X' \to X \times_Y Y \) be the s-fold fibre product \( f^* \) is again a Gorenstein morphism and \( f'_* \omega_{Y'/Y} = \bigotimes f'_* \omega_{X'/Y} \) ([18], 3.4, for example). Repeating our calculation for \( X' \) instead of \( X \), we obtain \( \tau_*((\bigotimes f'_* \omega_{X'/Y}) \otimes \mathcal{F}) \) as a subsheaf of \( \bigotimes f_* \omega_{X/Y} \).

The same holds for \( S^b \mathcal{F} \) instead of \( \mathcal{F} \). Choose the ample sheaf \( \mathcal{A} \) on \( Y \) and \( \mathcal{F} \) such that \( \tau^* \mathcal{A} \otimes \mathcal{F} \) is ample and \( \tau_* \mathcal{O}_Y \otimes \mathcal{A} \) generated by its global sections for all \( b > 0 \).
Let \( Z' \subset Y' \) be the largest open subvariety such that \( f'_* \omega^c_{X'/Y'}|_Z \) is locally free. Then \( f'^{-1}(Z) \to Z \) satisfies again the assumptions made in 3.7. Since \( Y_0' \) is contained in \( Z' \) of 3.7 for \( f'^{-1}(Z) \to Z \) and 3.3(d) would imply that \( S^{x,b}(f'_* \omega^c_{X'/Y'}) \otimes \mathcal{H}^b \otimes \mathcal{F}^b \) is globally generated over \( Y_0' \) for some \( b \gg 0 \). Then \( S^{x,b}(f'_* \omega^c_{X'/Y'}) \otimes \mathcal{H}^{2b} \) is as well globally generated over \( Y_0' \).

**Claim 3.9.** Let \( Y \) be normal, \( Y' \) non singular and \( \tau : Y' \to Y \) a projectively generically finite morphism. Assume that 3.7 holds for \( f' \). Then the base change map \([6], III, 9.3.1\) \( \rho : \tau^* f'_* \omega^c_{X'/Y'} \to f'_* \omega^c_{X'/Y'} \) is an isomorphism over \( Y_0' \). Moreover 3.7 holds for \( f \).

**Proof.** Since \( f'_* \omega^c_{X'/Y'} \) is locally free and \( \tau \) generically flat \( \rho \) is injective. If \( \rho \) were not surjective over \( Y_0' \) we could find some exceptional divisor \( F \) meeting \( Y_0' \) such that \( \tau^* \text{det}(f'_* \omega^c_{X'/Y'}) \otimes \mathcal{O}_Y(F) = \text{det}(f'_* \omega^c_{X'/Y'}) \). Since \( F \) must be an exceptional divisor this contradicts the weak positivity of \( \text{det}(f'_* \omega^c_{X'/Y'}) \) over \( Y_0' \). In order to see that 3.7 holds for \( f \) we just remark that \( f'_* \omega^c_{X'/Y'} \) is a direct summand of \( \tau^* f'_* \omega^c_{X'/Y'} \). The weak positivity of \( f'_* \omega^c_{X'/Y'} \) over \( Y_0 \) follows as in 3.4(e).

Due to 3.8 and 3.9 we may assume \( Y \) to be non singular. Moreover, whenever it is convenient, we may replace \( Y' \) by a generically finite cover. Let \( \delta : Z \to X \) be a desingularization and \( q = f \circ \delta : Z \to Y \). Since \( f^{-1}(Y_0) \) has rational Gorenstein singularities \( g_* \omega^c_Z \to \omega^c_X \) is an isomorphism over \( f^{-1}(Y_0) \) and \( g_* \omega^c_{X'/Y'} \to f'_* \omega^c_{X'/Y'} \) is an isomorphism over \( Y_0', g \) is no longer flat. Nevertheless we get:

**Claim 3.10.** Let \( \tau : Y' \to Y \) be either a finite cover or a blowing up. Let \( g' : Z' \to Y' \) be a desingularization of \( X' \). Then we have an inclusion \( \rho : g'_* \omega^c_{Z'/Y'} \to \tau^* g_* \omega^c_{Z/Y'} \) isomorphic over \( Y_0' \).

**Proof.** The existence of \( \rho \) has been shown in \([17], 1.8 \) and \([18], 3.2\). \( \rho \) is an isomorphism over \( Y_0' \) by 3.9.

For \( \nu = 1 \) 3.7 follows from Kawamata’s positivity theorem \(([8] \text{ or } [11])\). It says that \( g_* \omega^c_{Z/Y} \) is weakly positive over \( Y \), if there exists some \( U \subseteq Y \) such that:

i) \( Y - U \) is a normal crossing divisor.

ii) \( g^{-1}(U) \to U \) smooth

iii) For \( k = \dim Z - \dim Y \) the monodromy of \( R^k g_* \mathcal{C}^1_{\mathcal{O}_Z} \) around the components of \( Y - U \) is unipotent.

Those three conditions hold if one replaces \( Y \) by a finite cover of a blowing up, and 3.7 follows from 3.10 and 3.4(e).

For \( \nu > 1 \) we have to argue as in \([18], §5\):

**Claim 3.11.** Assume that \( S^\nu(f'_* \omega^c_{X'/Y'} \otimes \mathcal{H}^\nu) \) is globally generated over \( Y_0' \) for some \( \mu \gg 0 \). Then \( f'_* \omega^c_{X'/Y'} \otimes \mathcal{H}^{\nu-1} \) is weakly positive.

If \( \mathcal{H} \) is any ample sheaf on \( Y \) one obtains, as in \([18], 5.3\), that \( f'_* \omega^c_{X'/Y'} \otimes \mathcal{H}^{\nu-1} \) is weakly positive over \( Y_0' \). This holds as well for the pullback morphism \( f' \), if \( \tau : Y' \to Y \) is a finite cover. By 3.4(e), we finished the proof of 3.7.

**Proof of 3.11** (see \([17]\)). If \( \tau : Y' \to Y \) is generically finite 3.9 tells us that \( S^\nu(f'_* \omega^c_{X'/Y'} \otimes \tau^* \mathcal{H}^\nu) \) is globally generated over \( Y_0' \). Moreover, adding \( \tau^* \mathcal{H}^{\nu-1} \)
does not change the argument indicated in 3.9 and the weak positivity of $f_* \omega_{X/Y} \otimes \tau^* \mathcal{H}^{-1}$ implies that of $f_* \omega_{X/Y} \otimes \mathcal{H}^{-1}$. Therefore again we can replace $Y$ by a generically finite cover, whenever we want to do so.

Let $\mathcal{L} = \omega_{X/Y} \otimes F^* \mathcal{H}$ and $\mathcal{M}$ the subsheaf of $\mathcal{L}^{\mu \cdot v}$ generated by global sections. Let $\delta: Z \to X$ be a desingularization such that $\mathcal{M}' = \delta^* \mathcal{M}/\text{torsion}$ is invertible and such that for $\mathcal{L}' = \delta^* \mathcal{L}$ and an effective normal crossing divisor $D$ one has $\mathcal{O}_Z(D) = \mathcal{L}'^{\mu \cdot v} \otimes \mathcal{M}'^{-1}$. $\mathcal{M}'$ again is generated by its global sections. Let $g: Z' \to Y$ be the cyclic cover obtained by taking the $v \cdot \mu$-th root out of a general section of $\mathcal{M}'$. As, for example, in [3], §2 or [18], §5

$$g_* \left( \mathcal{L}^{\mu \cdot v - 1} \left( - \left[ \frac{(v-1) \cdot D}{v \cdot \mu} \right] \otimes \omega_{Z/Y} \right) \right)$$

is a direct summand of $g_* \omega_{Z/Y}$.

Replacing $Y$ by some generically finite cover, we may again assume that $g'$ satisfies the assumptions of Kawamata's positivity theorem, and therefore that

$$g_* \left( \mathcal{L}^{\mu \cdot v - 1} \left( - \left[ \frac{(v-1) \cdot D}{v \cdot \mu} \right] \otimes \omega_{Z/Y} \right) \right)$$

is weakly positive over $Y$. By the choice of $\mathcal{M}$ the map $f^* f_* \mathcal{M} \to \mathcal{M}$ is surjective over $f^{-1}(Y_0)$. We have inclusions

$$\mathcal{M} \to \delta_* \mathcal{M}' \to \delta_* \mathcal{L}'^{\mu \cdot v - 1} \left( - \left[ \frac{(v-1) \cdot D}{v \cdot \mu} \right] \otimes \omega_{Z/Y} \otimes f^* \mathcal{H} \right).$$

Then the natural inclusion

$$g_* \left( \mathcal{L}^{\mu \cdot v - 1} \left( - \left[ \frac{(v-1) \cdot D}{v \cdot \mu} \right] \otimes \omega_{Z/Y} \right) \right) \to f_* \mathcal{L}^{\mu \cdot v - 1} \otimes \omega_{X/Y} = f_* \omega_{X/Y} \otimes \mathcal{H}^{v-1}$$

is an isomorphism over $Y_0$ and we obtain 3.11.

**Remark 3.12.** Let us assume in addition to the assumptions made in 3.7 that $X_0 = f^{-1}(Y_0) \to Y_0$ is smooth. Then an affirmative answer to problem 1.13 implies that $f_* \omega_{X/Y}$ is weakly positive over $Y_0$.

"Proof." Even if $Y_0$ is not normal we can find a finite cover $\tau: Y_0 \to Y_0$ such that $\mathcal{O}_{Y_0}$ is a direct summand of $\tau_* \mathcal{O}_{Y_0}$ and such that the morphism $X_0 \to Y_0$ obtained as pullback of $f$ satisfies the assumptions made in 1.13. By 3.4(b) we may assume that $f_0 = f|_{X_0} \to Y_0$ satisfies those assumptions. Blowing up the boundary $Y \to Y_0$ and using 1.13 we can extend $f_* \omega_{X_0/Y_0}$ to a locally free sheaf $\mathcal{F}$ on some compactification $\overline{Y}$ of $Y_0$. If $\overline{Y}'$ is a good desingularization of $\overline{Y}$ the pullback $\overline{\mathcal{F}}'$ of $\mathcal{F}$ to $\overline{Y}'$ is weakly positive over $\overline{Y}'$. Then by 3.4(c) $\mathcal{F}$ is weakly positive over $\overline{Y}$. It is well known that $\overline{\mathcal{F}}'$ is the direct image of the relative dualizing sheaf of some desingularization of $\overline{Y}' \times_Y X$. Therefore one has a natural map from $\overline{\mathcal{F}}'$ to $f_* \omega_{X/Y}$ isomorphic over $Y_0$. If $v > 1$ one has to repeat the arguments used in 3.11.
A similar argument should work as well if \( X_0 \to Y_0 \) is not smooth. However, one has to try to study the necessary base change properties more carefully. Since, anyway, we do not know an answer to 1.13 we do not insist on this implication.

§ 4. Fibre spaces

We want to prove 1.19. As we have seen already in §1, C 1.19 implies 1.18 as well.

4.1. Let \( f: X \to Y \) and \( Y_0 \subseteq Y \) satisfy the assumptions made in 1.17 and 1.19. As in 3.3.d) it is easy to see that we can assume that \( f_* \omega_{X/Y} \) is locally free. Let \( s = f_* \omega_{X/Y} \) and \( r \) be the rank of \( s \). Let \( \pi: \mathbb{P}(s) \to Y \) be the projective bundle and \( \rho: X \to \mathbb{P}(s) \) the rational map.

For \( y \in Y_0 \) we have assumed that \( \rho|_{\rho^{-1}(y)} \) is birational. If \( s \) is the ideal sheaf of \( \rho(X) \) we can find some \( \mu \gg 0 \) such that \( \pi^* \pi_* (s \otimes \mathcal{O}_{\mathbb{P}(s)}(\mu)) \to s \otimes \mathcal{O}_{\mathbb{P}(s)}(\mu) \) is surjective. For simplicity we assume that \( r \) divides \( \mu \). Let us consider \( m: S^\mu(f_* \omega_{X/Y}) \to f_* \omega_{X/Y}^\mu \). By our assumption \( s = \text{Im}(m) \) is locally free over \( U \), and, leaving out some codimension two subspace of \( Y - U \), we may again assume that \( s \) is locally free over \( Y \). For simplicity we write \( U = Y_0 \) and assume \( Y_0 \) to be already non singular.

4.2. Recall that we have to show that \( \mathcal{L} = \text{det}(s)^a \otimes \text{det}(s)^b \) is ample with respect to \( Y_0 \) for some \( a, b > 0 \).

Let \( r' = \text{rank}(s) \) and consider

\[
\wedge^{r'} m: \wedge^{r'} S^\mu(f_* \omega_{X/Y}) \to \wedge^{r'} f_* \omega_{X/Y}^\mu.
\]

The image of \( \wedge^{r'} m \) is \( \text{det}(s) \). For

\[
\mathcal{L}_0 = \text{det}(s) \otimes \text{det}(s) \xrightarrow{r'-\mu} r
\]

we obtain from \( \wedge^{r'} m \) a surjection

\[
\gamma: \wedge^{r'} S^\mu(s) \otimes \text{det}(s) \xrightarrow{r'-\mu} r \to \mathcal{L}_0.
\]

Let us return to the construction made in 2.5, i.e.: Let \( \pi: \mathbb{P} = \mathbb{P}(\bigoplus \mathbb{P}^r \otimes s) \to Y \) be the projective bundle. \( s: \bigoplus \mathbb{P}(1) \to \pi^* s \) the universal basis and \( D \) the degeneration locus of \( s \).

Claim 4.3. \( \pi^* \mathcal{L}_0|_{\mathbb{P} - D} \) is ample with respect to \( \pi^{-1}(Y_0) \cap \mathbb{P} - D \).

Proof. By definitions \( s|_{\mathbb{P} - D} \) is an isomorphism. Therefore

\[
\mathcal{O}_{\mathbb{P}}(-r)|_{\mathbb{P} - D} = \pi^* \text{det}(s)|_{\mathbb{P} - D}.
\]
\( \pi^*(\gamma) \) is a surjection
\[ \wedge^r S^n(C^r) \otimes \mathcal{O}_D \to \pi^* \mathcal{L}_0|_{\mathbb{P}^r} \cdot \]

This implies that \( \pi^* \mathcal{L}_0|_{\mathbb{P}^r} \) is globally generated over \( \mathbb{P}^r \). Let \( \mathcal{H}^i \mathcal{H}(\mathbb{P}^{r-1}) \) be the Hilbert scheme of subschemes of \( \mathbb{P}^{r-1} \). Since \( \pi^* \mathcal{L}_0|_{\mathbb{P}^r} \) is a direct sum of \( r \) copies of \( \mathcal{O}_D(-1) \) the \( r \)-canonical rational map gives rise to a morphism
\[ h: \pi^{-1}(Y_0) \cap \mathbb{P}^r \to \mathcal{H}^i \mathcal{H}(\mathbb{P}^{r-1}) \]. Let \( H \) be the component containing the image of \( h \). \( H \) can be embedded in a projective space by the Plücker coordinates ([1], 2.6 for example). Composing with \( h \) we obtain a rational map \( \tilde{h} \) from \( \mathbb{P}^r \) to \( \mathbb{P}^r \) in some projective space. By [1], 2.6, this map is given by the surjection \( \pi^*(\gamma) \). Especially \( \tilde{h}: \mathbb{P}^r \to \mathbb{P}(\wedge^r S^r(C^r)) \) is a morphism and \( \tilde{h}^* \mathcal{O}(1) = \pi^* \mathcal{L}_0|_{\mathbb{P}^r} \).

We have assumed that only finitely many fibres \( f^{-1}(y) \) are birational, for \( y \in Y_0 \). Since the map \( X \times_Y \mathbb{P}^r \to \mathbb{P}(\pi^* \mathcal{E}) \approx \mathbb{P}^{r-1} \times \mathbb{P} \) is given by a universal basis this implies that the fibres of \( h: (\mathbb{P}^r \cap \pi^{-1}(Y_0)) \to H \) are finite. Then \( \pi^* \mathcal{L}_0|_{\mathbb{P}^r} \) must be ample with respect to \( (\mathbb{P}^r \cap \pi^{-1}(Y_0)) \).

**Remark 4.4.** If \( \mathcal{G} = f_{*} \mathcal{O}_Y^0 \) the sheaf \( \mathcal{L}_0 \) is just the same as the sheaf considered in 1.4. If \( Y = Y_0 \), in addition, the map \( h \) already appeared in 1.2 and 1.3. The bundle \( \mathbb{P}^r \to Y \) has \( \mathcal{H}^i \mathcal{H}(r, \mathbb{C}) \) as fibres. If one considers the corresponding group action the invariant sections of \( \pi^* \mathcal{L}_0 \) are those coming from \( Y \).

4.5. Let \( \tau: \mathbb{P}^r \to \mathbb{P}^r \) be a proper modification with center in \( D \) such that the rational map
\[ h' = \tilde{h} \circ \tau: \mathbb{P}^r \to \mathbb{P}(\wedge^r S^r(C^r)) \]
is a morphism. We can as well assume that there exists an exceptional divisor \( F \) of \( \tau \) and a divisor \( E \) supported outside of \( \pi^{-1}(Y_0) \), where \( \pi^* = \pi \circ \tau \), such that \( \mathcal{O}_\tau(-E - F) \) is relatively ample for \( h' \). If we write \( (\pi^* D) \) as \( \sum D_i \). We have \( F_{\text{red}} \subseteq \sum D_i \). Therefore we can find some \( \alpha > 0 \) and \( \gamma_i \in \mathbb{Z} \) such that \( \pi^* \mathcal{L}_0 \otimes \mathcal{O}_\mathbb{P}(\sum \gamma_i D_i - E) \) is ample on \( \mathbb{P}^r \).

**Remark 4.6.** Under the assumptions made in 4.4, the Hilbert-Mumford criterion ([14], Ch. II, §1) for the \( \mathcal{H}^i \mathcal{H}(r, \mathbb{C}) \) action on \( \mathbb{P}^r \) seems to say that one can choose \( \gamma_i > 0 \). Since we can not verify this criterium we just add some effective divisor supported in \( \sum D_i \) and use “weak positivity” to show that this does not effect “ampleness”.

4.7. By 3.7 \( \mathcal{E} \) is weakly positive over \( Y_0 \) and from 3.4a) and 3.6 it follows that \( \bigoplus \pi^* (\mathcal{E} \otimes \det(\mathcal{E})) \) is weakly positive over \( \pi^{-1}(Y_0) \). Then the quotient sheaf \( \tau^* \mathcal{O}_\mathbb{P}(1) \otimes \pi^* \det(\mathcal{E}) \) and its \( r \)-th power are again weakly positive over \( \pi^{-1}(Y_0) \). By definition of \( D \) (see 2.5) this sheaf is nothing but
\[ \mathcal{O}_\mathbb{P}(\tau^* D) \otimes \pi^* \det(\mathcal{E})^r \]

For some \( \eta \geq 0 \) the divisor \( \eta \cdot \tau^* D + \sum \gamma_i D_i = D' \) will be effective. By 3.2
\[ \pi^* \mathcal{L}_0 \otimes \mathcal{O}_\mathbb{P}(\sum \gamma_i D_i - E) \otimes (\mathcal{O}_\mathbb{P}(\tau^* D) \otimes \pi^* \det(\mathcal{E})^r) \]
\[ = \pi^* (\mathcal{L}_0 \otimes \det(\mathcal{E})^r \otimes \mathcal{O}_\mathbb{P}(D) \otimes \mathcal{O}_\mathbb{P}(-E)) \]
is ample with respect to $\pi^{-1}(Y_0)$. For $b = -\frac{a \cdot r' \cdot \mu}{r} + (r-1) \cdot \eta$, this sheaf is $\pi^*(L') \otimes \mathcal{O}_{\mathbb{P}^r}(D') \otimes \mathcal{O}_{\mathbb{P}^r}(-E)$. Since $E$ is not meeting $Y_0$, the sheaf $\pi^*(L') \otimes \mathcal{O}_{\mathbb{P}^r}(D')$ is also ample with respect to $\pi^{-1}(Y_0)$. We can apply 2.7 and find $L'$ to be ample with respect to $U$.

**Remark 4.8.** Some ingredients used before in the proofs of $C^*_{\mathbb{P}^r}$ are reappearing in this chapter: The Hilbert-Mumford criterion and stability (see 4.6) which was used in [18]. The multiplication map, used by Kollár in [12]. The "mysterious covering trick" used in [12] and [19] is of course hidden in the construction of $\mathbb{P}$ and 2.7.

As promised, starting from 4.2 we used the assumption "the singular locus of $Y_0$ compact" just in 4.7 to obtain the weak positivity over $Y_0$ of $\mathcal{E}$.

If $Y_0$ is singular and the singular locus of $Y_0$ compact we can consider a desingularization $\tau: Y' \to Y$ and $f' = pr_2: X \times_Y Y' \to Y'$. If $f: X \to Y$ satisfies the assumptions made in 1.18 then, for $v>1$ the base change map $\tau^* f^* \omega_{X,Y} \to f'_* \omega_{X',Y'}$ is an isomorphism. Therefore $f'$ satisfies the assumptions made in 3.7 and $\tau^* \mathcal{E}$ is weakly positive over $\tau^{-1}(Y_0)$. By 3.4(e) we obtain the weak positivity of $\mathcal{E}$ over $Y_0$. Therefore we can state as well:

**Theorem 4.9.** Under the assumptions made in 1.17 and 1.18 assume that the singular locus of $Y_0$ is compact. Then for some $a, b, \mu > 0$ the sheaf $L' = \text{det}(f^* \omega_{X,Y})^a \otimes \text{det}(f'_* \omega_{X',Y'})^b$ is ample with respect to $Y_0$.

§ 5. Stability of certain Hilbert points

In this section we want to prove 1.7. Recall that for each of our moduli functors $\mathcal{M}_n$ we have some $v>0$ and the Hilbert scheme $H$ of $v$-canonical embedded varieties of $\mathcal{M}_n$. We have a universal family $h: \mathcal{X} \to H$, $\mathcal{E} = h_* \omega_{\mathcal{X}/H}$ is a direct sum of $r$ copies of an invertible sheaf $\mathcal{N}$ and $\lambda_v = \text{det}(\mathcal{E}) = \mathcal{N}^v$ (see 1.2). Moreover the sheaf $\mathcal{L}_n = \mathcal{L}_{n,v} \otimes \lambda_v^{-r(v-b)}$ introduced in 1.4 is ample on $H$.

Similar to our arguments in § 4 we will use the results on weak positivity (§ 3) to show that: Replacing $\mathcal{L}_n$ by $\mathcal{L}_n' = \mathcal{L}_{n,v} \otimes \lambda_v^b$ for $\eta \gg 0$ we get enough sections of $\mathcal{L}_n'$, "positive" at the boundary of an orbit, and this implies that all points, where $\mathcal{L}_n'$ is ample, are stable with respect to the $G = SL(r, \mathbb{C})$ action on $H$.

Due to our "gap" 1.10 we are only able to prove that $\mathcal{L}_n'$ is ample with respect to the largest open subscheme $H_0$ of $H$ with $(H_0)_{\text{red}}$ smooth.

**5.1.** Let us start by recalling some tools from D. Mumford’s geometric invariant theory [14].

Let $G = \text{GL}(r, \mathbb{C})$ be acting on an algebraic scheme $X$ and let $\mathcal{L}$ be an invertible $G$-linearized sheaf on $X$ ([14], Def. 1.6).

**Definition 5.2** ([14], Def. 1.7)

(i) A geometric point $x \in X$ is called stable (with respect to $\mathcal{L}$) if there exists, for some $N > 0$, a $G$-invariant section $s \in H^0(X, \mathcal{L}^N)$ such that: $s(X) \neq 0$, $X_s = X - \{\text{zero set of } s\}$ is affine and the action of $G$ on $X_s$ is closed.

(ii) We write $X(\mathcal{L})^s$ for the set of stable points with finite stabilizers.
Remark 5.3. $X(\mathcal{L})^r$ is an open subscheme and $\mathcal{L}|_{X(\mathcal{L})^r}$ is ample. In [14] $X(\mathcal{L})^r$ is denoted by $X_{(0)}$ (We changed the notation since $H^*_0(\mathcal{L})$ looks too much like a cohomology group). The subscheme $X(\mathcal{L})^r$ is independent of the $G$-linearization choosen ([14], Cor. 1.17).

Lemma 5.4. Assume that $X$ is a projective variety containing a dense orbit $X_0$ on which $G$ acts with finite stabilizers. Assume moreover that the $N$-th power of the sheaf $\mathcal{L}$ has a section $s$ with zero set $D$ such that $X - X_0 = D_{\text{red}}$. Then $X(\mathcal{L})^r \supseteq X_0$.

Proof. $D_{\text{red}}$ is invariant under $G$. Therefore $D$ has only finitely many conjugates under $G$ and taking the product of the corresponding sections we may assume $D$ to be $G$-invariant. Therefore we have one $G$-linearization of $\mathcal{L}^N = \mathcal{O}_X(D)$ such that $S$ is a $G$-invariant section. By [14], Prop. 1.4, there exists at most one $G$-linearization of $\mathcal{L}^N$. So we found a $G$-invariant section $s$ with $X_0 = X_s$. Since $G$ is affine and acts on $X_0$ with finite stabilizer, $X_0$ is affine.

Proposition 5.5 ([14], Prop. 1.18, 1.16 and Thm. 1.19). Let $i: Y \rightarrow X$ be a $G$-linear embedding. Then:

a) $Y \cap X(\mathcal{L})^r \subset Y(i^* \mathcal{L})^r$.
b) If $Y = X_{\text{red}}$, then $(X(\mathcal{L})^r)_{\text{red}} = Y(i^* \mathcal{L})^r$.
c) If $Y$ is proper and $\mathcal{L}$ ample, then $Y \cap X(\mathcal{L})^r = Y(i^* \mathcal{L})^r$.

Remark 5.6. The open subscheme $X(\mathcal{L})^r$ of $X$ depends on the $G$-linearized sheaf $\mathcal{L}$ choosen. By [14], converse 1.13, one knows however that for an open $G$-invariant subscheme $U \subseteq X$ one has an equivalence of:

(i) For some $G$-linearized invertible sheaf $\mathcal{L}'$ one has $U \subseteq X(\mathcal{L}')^r$.

(ii) The action of $G$ on $U$ is proper and a geometric quotient of $U$ by $G$ exists as a quasi-projective scheme.

5.7. Let us return to the notations introduced in §1, A) and recalled in the beginning of this chapter. We know from 3.7, 3.5 and 3.2 that for all $\eta \geq 0$ the sheaf $\mathcal{L}_\eta = \mathcal{L}_0 \otimes \lambda^\eta$ is ample with respect to $(H_0)_{\text{red}}$. Obviously $\mathcal{L}_\eta$ has a $G$-linearization.

In order, not to distinguish between part a and b of 1.7 we write $H = H_0$ in case a) and leave $H$ unchanged in case b. Then $\mathcal{L}_\eta$ is ample with respect to $(H)_{\text{red}}$. We have seen that stability is a Zariski-open condition. Therefore 1.7 follows if we show that:

Claim 5.8. For a given point $x \in H$ there exists some $\eta_0$, such that $x \in H(\mathcal{L}_\eta)^r$ for all multiples $\eta_{\text{of}} \eta_0$.

Proof. By 5.5 b) we can replace $H$ by $(H)_{\text{red}}$ and – again by abuse of notations – we assume $\mathcal{L}$ and $H$ to be reduced. Let $H_x$ be the $G$-orbit of $x$. In order to prove 5.8 we will need the existence of a “good” partial compactification of $H$:

Claim 5.9. There exists an inclusion $i: H \rightarrow H'$ such that:

a) The closure $H'_x$ of $H_x$ in $H'$ is a projective variety and $H'_x = H'_x - H_x$ a Cartier divisor.
b) There exists some $\eta_0 > 0$ such that for all multiples $\eta$ of $\eta_0$ we can find, for some $\beta > 0$, a coherent subsheaf $\mathcal{L}_\eta^\beta$ of $t_*(L^\beta_\eta)$ generated by global sections, and an inclusion $\mathcal{O}_{H_\eta}(I_\eta) \to \mathcal{L}_\eta^\beta|_{H_\eta}$, surjective over $H_\eta$.

We postpone the proof of 5.9 and finish first the one of 5.8:

The morphism $h^{-1}(H_\eta) \to H_\eta$ is isotrivial. In fact, if $\tau: G \to H_\eta$ is given by the group operation, $\mathcal{X} \times_{H_\eta} G \to G$ is the trivial family. Hence, replacing $\beta$ by some multiple, we may assume that $\mathcal{L}_\eta^\beta = \mathcal{O}_{H_\eta}$. Since $\mathcal{L}_\eta^\beta$ is ample we may assume that there exists some finite dimensional subspace $V \subset H^0(H', \mathcal{L}_\eta^\beta)$, generating $\mathcal{L}_\eta^\beta$, such that the natural map $H \to \mathbb{P}(V)$ is an embedding. By [14], Ch. I, §1 we can find a $G$-invariant subspace of $H^0(H, \mathcal{L}_\eta^\beta)$ containing $V$ and by abuse of notations, we may assume $V$ to be $G$-invariant.

Moreover, we can replace $H'$ by its image in $\mathbb{P}(V)$ and choose $\mathcal{L}_\eta^\beta = \mathcal{O}_{H'}(1)$.

By construction $\mathcal{O}_{H_\eta}(1)$ still has a section whose zero divisor $D_\eta$ satisfies $(D_\eta)_{red} = t_\eta$. By 5.4 we know that $H_\eta \simeq (\mathcal{O}_{H_\eta}(1))^{\tau}$ and 5.5(c) tells us that

$$H_\eta(\mathcal{O}_{H_\eta}(1))^{\tau} = H_\eta \cap H(\mathcal{O}_{H'}(1))^{\tau}.$$

Therefore $x \in H'(\mathcal{O}_{H'}(1))^{\tau} \cap H$. Using 5.5(a) we find $x$ to be a point of $H(\mathcal{O}_{H'}(1))^{\tau}$.

However $\mathcal{O}_{H'}(1) = \mathcal{L}_\eta^\beta$ and $H(\mathcal{O}_{H'}(1))^{\tau} = H(\mathcal{L}_\eta^\beta)^{\tau}$.

Remark 5.10. We will give two different proofs of 5.9. The first one has the advantage that it is closer to the arguments used in §4 and more “down to earth”. Also, it is not using the full strength of 1.10 in case b) of 1.7. However, as Y. Kawamata and N. Nakayama pointed out, it has the disadvantage that one needs in 5.11 the additional assumption that $h^{-1}(x)$ has no non-trivial automorphism. The second proof has the disadvantage to be quite complicated, but (hopefully) it works in general.

Lemma 5.11. Assume that $\text{Aut}(h^{-1}(x)) = \{\text{id.}\}$. Then there exists a quasi-projective scheme $H'$ containing $H$ as an open subscheme and a prolongation $h': \mathcal{X}' \to H' \in \mathcal{M}_h(H')$ of $h$ such that the closure $H'_x$ of $H_x$ in $H'$ is projective and $h'^{-1}(H'_x) \to H'_x$ is isomorphic to $H'_x \times h'^{-1}(x) \to H'_x$.

Proof. Let us start with an arbitrary projective $H'$ such that $H' \to H$ is a divisor. Let $\mathcal{E}'$ be a coherent extension of $\mathcal{E}$ to $H'$, which we can assume to be locally free (blowing up $H'$ a little bit). Since $h^{-1}(H_x) \to H_x$ is trivial

$$\mathcal{E}'|_{H_x} \cong \bigoplus_{x \in H_x} \mathcal{O}_{H_x} = \mathcal{O}_{H_x} \otimes \mathcal{O}_{H_x}(h^{-1}(x), \omega_{H-x}^{-1}(x)).$$

Let $s_1, \ldots, s_\delta$ be the “trivial sections” of $\mathcal{E}'|_{H_x}$ coming from a basis of $H^0(h^{-1}(x), \omega_{H-x}^{-1}(x))$. Adding some multiple of $H' \to H$ we may assume that $s_1, \ldots, s_\delta$ give rise to sections of $\mathcal{E}'|_{H_x}$, which we denote again by $s_i$. Let $\mathcal{A}$ be a very ample sheaf on $H'$ such that $\mathcal{E}' \otimes \mathcal{O}(H_x(A_1 \cap H_x))$ is generated by its sections and has no higher cohomology, where $\mathcal{O}$ is the ideal sheaf of $H_x$.

Let $A_1, \ldots, A_\delta$ be divisors of $\mathcal{A}$ in general position such that $\bigcap A_j = \emptyset$. For each $j$ we have sections $s_j \in H^0(H_x, \mathcal{E}' \otimes \mathcal{O}_{H_x}(A_j \cap H_x))$ and we can find a tuple of sections

$$S_j^{(i)}: H \to \mathcal{E}' \otimes \mathcal{O}_{H}(A_j).$$
such that each of the sections, restricted to $H_x'$, gives $s^i_j$ and such that $S^i_j$ is surjective over $H' - H_x$. We denote the induced map $\bigoplus_i \mathcal{A}^{-1} \to \mathcal{E}'$ again by $S^i_j$. Adding up over all $i$ and $j$ we obtain $\mathfrak{E}: \bigoplus_{i,j} \mathcal{A}^{-1} \to \mathcal{E}'$, surjective over $H' - H_x$, and $\text{Im}(\mathfrak{E}|_{H_x'})$ is just the subsheaf generated by $s_1, \ldots, s_r$. Blowing up with center in $H_x' - H_x$, we obtain a similar map where the image is locally free.

Therefore we may assume that we have chosen $\mathcal{E}'$ from the beginning to be a locally free sheaf such that $s_1, \ldots, s_r$ generate $\mathcal{E}'|_{H_x'}$. We have a diagram

\[
\begin{array}{ccc}
\mathcal{A} & \longrightarrow & \mathbb{P}(\mathcal{E}) \\
\downarrow & & \downarrow \\
H & = & H' \\
\end{array}
\]

We choose $h': \mathcal{A}' \to H'$ by taking the compactification of $\mathcal{A}$ in $\mathbb{P}(\mathcal{E}')$. $h'$ need not be flat, but (as we did in [18] p. 345) we can blow up $H'$ with center in $H' - H$ such that the dominating component of the pullback family becomes flat. So we may assume that $\mathcal{A}'$ is a subscheme of $\mathbb{P}(\mathcal{E}')$, flat over $H'$.

Restricting everything to $H_x'$ we have

\[
\begin{array}{ccc}
h^{-1}(H_x') & \longrightarrow & \mathbb{P}(\mathcal{E}'|_{H_x'}) = \mathbb{P}^{-1} \times H_x' \\
\downarrow & & \downarrow \\
H_x' & = & H_x'.
\end{array}
\]

The isomorphism $\mathbb{P}(\mathcal{E}'|_{H_x'}) = \mathbb{P}^{-1} \times H_x'$ is given by the basis $s_1, \ldots, s_r$. Over $H_x'$ the family $h^{-1}(H_x) \to H_x'$ is trivial and its image in $\mathbb{P}^{-1}$ under $p \circ r_1$ is just $h^{-1}(x)$, embedded by $s_1, \ldots, s_r$. Therefore $h^{-1}(H_x')$ must be the product of $H_x'$ with $h^{-1}(x)$ as well.

The condition that a fibre of $h'$ belongs to $\mathcal{M}_b(\mathcal{C})$ is open (otherwise we would not have a Hilbert scheme, see [10]). Therefore, replacing $H'$ by some open neighbourhood of $H_x'$ which contains $H$ we are done.

5.12. The proof of 5.9 if $h^{-1}(x)$ has no non-trivial automorphism

We keep the notations introduced in 5.11. If $H$ was non singular we can choose $H'$ to be non singular as well. Moreover we may assume that $H' - H$ is a Cartier divisor. The orbit $H_x'$ is a quotient of $G$. We can compactify $G$ by $\mathbb{P} = \mathbb{P}(\bigoplus \mathcal{C})$. Blowing up centers in $\mathbb{P} - G$ we obtain another compactification $\mathbb{P}'$ of $G$ together with a generically finite morphism $\tau: \mathbb{P}' \to H_x'$.

Remember that $\mathcal{E} = \mathcal{E}|_{H_x'} \simeq \bigoplus \mathcal{N}$. We can choose an extension $\mathcal{N}'$ of $\mathcal{N}$ such that this isomorphism gives an inclusion $\mathfrak{E}: \bigoplus \mathcal{N}' \to \mathcal{E}'$ or $\mathcal{N}' \to \bigoplus \mathcal{E}'$.  

Blowing up centers in $H' - H$, we may assume that $\mathcal{N}'$ is a subbundle. As in 2.5 we obtain natural maps

\[ \mathcal{N}' \rightarrow S'((\oplus \mathcal{E}') \rightarrow \bigotimes \mathcal{E}' \rightarrow \det(\mathcal{E}'). \]

Let $\Delta$ be the divisor of the corresponding section of $\det(\mathcal{E}') \otimes \mathcal{N}'$.\(\star\)

Regarding the dual construction we get surjections $S'((\oplus \mathcal{E}' \bigotimes \det(\mathcal{E}')) \rightarrow \mathcal{N}' \rightarrow \mathcal{E}' \bigotimes \det(\mathcal{E}').$ From 3.6 and 3.3c) we find that $\det(\mathcal{E}' \bigotimes \det(\mathcal{E}'))$ is weakly positive over $H'$. If $r-1$ divides $\chi$ we write

\[ \mathcal{L}_{\eta}^{(\tau)} = \mathcal{L}_{\eta}^{(\tau_0)} \otimes (\mathcal{O}_H(\Delta) \otimes \det(\mathcal{E}'))^{-1}. \]

One has $\mathcal{L}_{\eta}^{(\tau)} \bigotimes H = \mathcal{L}_{\eta}^{(\tau)}$. As a tensor product of an ample and a weakly positive sheaf $\mathcal{L}_{\eta}^{(\tau)}$ is ample (see 3.2).

By definition $\mathcal{L}_{\eta}^{(\tau)}$ is a subbundle of $\mathcal{L}_{\eta}^{(\tau)}$. Hence to finish the proof of 5.9 (under the additional assumption) we just have to verify that for some $\gamma \geq 0$ the sheaf $(\mathcal{L}_{\eta}^{(\tau)})^{\bigotimes \gamma}$ has a section whose zero divisor has $I_x$ as support. Since $\tau$ is generically finite this follows from

**Claim 5.13.** There exists some $n_0$ such that, for all $n_0 \geq n_0$, $\tau^* \mathcal{L}_{\eta}^{(\tau)} = \mathcal{O}_H(D')$ for an effective divisor $D'$ with $(D')_{\text{red}} = \mathbb{P} - \mathbb{P} G l(r, \mathbb{C}).$

**Proof.** Since $\tau^* \mathcal{L}_{\eta}^{(\tau)} = \mathcal{O}_H(\sum n_0 D_i)$ for $n_0 \in \mathbb{Z}$ and $\sum D_i = \mathbb{P} - \mathbb{P} G l(r, \mathbb{C})$, we just have to verify that $(\tau^* \mathcal{D})_{\text{red}} = \mathbb{P} - \mathbb{P} G l(r, \mathbb{C})$. This is however contained in 2.5:

The inclusion $\tau^* \mathcal{N}' \rightarrow \bigotimes \tau^* \mathcal{E}' = \bigotimes \mathbb{C}' \bigotimes \mathbb{C} \bigotimes \mathcal{O}_H$ was induced by $\mathcal{O}_H \hookrightarrow \bigotimes \mathcal{E}' \bigotimes \mathcal{O}_H$. Restricted to $\mathbb{P} G l(r, \mathbb{C})$ $\mathcal{O}_H$ is just given by the action of $\mathbb{P} G l(r, \mathbb{C})$ on $\mathbb{P}(H^0(H^{-1}(x), \mathcal{O}_H(-1)(x)))$. Therefore $\mathcal{O}_H$ coincides with the universal basis considered in 2.5 and $\tau^* \mathcal{N}'$ is the pullback of $\mathcal{O}_H(-1)$ to $\mathbb{P}$. Therefore $\tau^* \mathcal{D}$ is the pullback of the degeneration locus of $\mathcal{O}_H$ and $\mathbb{P} - (\tau^* \Delta)_{\text{red}} = \mathbb{P} G l(r, \mathbb{C})$.

5.14. The proof of 5.9

The group action is given by a morphism $\sigma: G \times H \rightarrow H$. Moreover we have the two projections $pr_1$ and $pr_2$ from $G \times H$ to $G$ and $H$. Since the action of $G$ lifts to $X$ the two morphisms $X \times_H (G \times H) \rightarrow G \times H$ obtained by pullback under $\sigma$ and $pr_2$ coincide.

Let $H'$ be a projective compactification of $H$ and $\mathbb{P} = \mathbb{P}((\bigoplus \mathbb{C})^r)$ the usual compactification of $G$. We can choose a projective compactification $Z$ of $G \times H$ such that $\sigma$, $pr_2$, and $pr_1$ induce morphisms from $Z$ to $H'$ and $\mathbb{P}$, respectively. We denote them by $\phi$, $p_2$, and $p_1$. Let $U = \phi^{-1}(H)$.

**Claim 5.15.** We can choose $H'$ and $Z$ such that $p_2(U)$ is open.
Proof. The action of \( G \) on \( H \) is proper. By definition that means that \((\sigma, p_2)\): \( G \times H \rightarrow H \times H \) is proper. Let \( \Psi: Z \rightarrow H' \times H' \) be the induced map. We have \( U = \Psi^{-1}(H' \times H' \times H' \times H' \times H') \). Especially \( Z - U \) contains the proper transform \( T \) of \((H' - H) \times H'). \( p_2: Z_0 = Z - T \rightarrow H' \) is flat over \( H \) and of finite type. Therefore we can blow up centers in \( H' \rightarrow H \) to obtain another compactification \( H'' \) of \( H \) such that the main component \( Z_0'' \) of \( Z_0 \times_H H'' \) is flat over \( H'' \). If \( U'' \) is the pullback of \( U \) to \( Z_0'' \), the image of \( U'' \) in \( H'' \) will be open ([6], p. 266 for example).

Let us assume therefore that (5.15) holds. For any \( y \in H \) let \( U_y = \sigma^{-1}(y) \subseteq G \times H \) and \( U_y' \) its closure in \( Z \). Since \( \sigma \circ \pi_2(U_y) = H' \), one finds \( \sigma \circ \pi_2(U_y') = H' \), and, since \( U_y' \subseteq U \), the open subvariety \( p_2(U) \) contains the closure of each of the orbits. Since, anyway, we are just interested in partial compactifications we may – by abuse of notations – replace \( H' \) by \( p_2(U) \) and \( Z \) by \( p_2^{-1}(p_2(U)) \).

Altogether we obtained a projective morphism \( p_2: Z \rightarrow H' \), an open sub- \( \mathcal{U} \) \( \mathcal{Z} \) of \( \mathcal{U} \), such that \( p_2(U) = H' \), a proper surjective morphism \( \sigma: U \rightarrow H \) and a morphism \( p_1: Z \rightarrow \mathcal{U} \). All are prolongations of the corresponding morphisms of \( G \times H \).

We may blow up \( H' \) and \( Z \) as long as the centers stay away from \( H \) and \( G \times H \). Therefore we can assume that \( H' = H' \) is a Cartier divisor and that \( H' \) is a singular for our given point \( x \). Especially 5.9a) holds true.

If \( H \) is smooth we can also assume that \( Z \) is smooth.

The morphism \( \mathcal{X} \times_H (G \times H) \rightarrow G \times H \) extends to a smooth morphism over \( U \) as well as over \( p_2^{-1}(H) \). Therefore we can find a common smooth extension \( h_0: \mathcal{Z}_0 \rightarrow \mathcal{Z}_0 = U \cup p_2^{-1}(H) \) of the pullback of \( h: \mathcal{X} \rightarrow H \) under \( \sigma \) and \( p_2 \).

We have assumed that 1.10 holds true in case b) and we have verified 1.10 in cases a) of 1.7. Therefore we know that \( h_0 \cdot \omega^\mathcal{Z}_{\mathcal{Z}_0} \) extends to a coherent sheaf \( \mathfrak{F}' \), weakly positive over \( \mathcal{Z}_0 \). Of course we can assume \( \mathfrak{F}' \) to be locally free. We write \( \lambda' = \det(\mathfrak{F}') \).

Claim 5.16. After further blowing ups, we can find an effective Cartier divisor \( D \) on \( Z \) such that \( \lambda''^{-1} \otimes \mathcal{O}_Z(D) \) is weakly positive over \( Z_0 \) and such that \( D_{red} = p_2^{-1}(\Gamma) \).

Proof. On \( p_2^{-1}(H) \) the sheaf \( \mathfrak{F}' \) is nothing but \( p_2^* \mathfrak{F} = \bigoplus p_2^* \mathfrak{N}' \). As in 5.12 we choose some extension \( \mathfrak{N}'' \) of \( \mathfrak{N}' \) to \( Z \) such that the inclusion \( p_2^* \mathfrak{N}'' \rightarrow \bigoplus p_2^* \mathfrak{F} \) extends to \( \varepsilon: \mathfrak{N}'' \rightarrow \bigoplus \mathfrak{F} \). Blowing up \( Z \) we may assume that \( \mathfrak{N}'' \) is invertible and that \( \varepsilon \) splits locally. \( \varepsilon \) induces \( \varepsilon: \bigoplus \mathfrak{N}'' \rightarrow \mathfrak{F} \). We choose \( D \) to be the zero divisor of \( \det(\varepsilon) \). \( D \) is, by construction, supported in \( p_2^{-1}(\Gamma) \). Moreover \( \mathcal{O}_Z(D) = \lambda' \otimes \mathfrak{N}''^{-1} \). The dual of \( \varepsilon \) induces a surjection

\[ S'(\bigoplus \mathfrak{F}'' \otimes \lambda') \rightarrow \lambda''^{-1} \otimes \mathcal{O}_Z(D). \]

From 3.6 and 3.3c) we obtain (as in 5.12) the first part of Claim 5.16.

For any \( y \in H \) we have \( (A \cap U_y)'_{red} = U_y' \rightarrow U_y' \). Therefore on \( Z_0 \) the divisors \( D_{red} \) and \( p_2^{-1}(\Gamma) \) coincide. Adding some divisor contained in \( Z - Z_0 \) to \( D \) does not effect the weak positivity over \( Z_0 \) and hence we find some \( D \) which satisfies both conclusions of 5.16.
5.17. By construction we have a morphism \( \rho : Z \to \mathbb{P} \times H' \) such that \( p_2 = pr_2 \circ \rho \). For some \( \gamma > 0 \) we choose \( \mathcal{H} = pr_1^* \mathcal{O}_P(\gamma) \). We can assume that there is some exceptional divisor \( F \) for \( \rho \) such that \( p_* \mathcal{O}_Z(-F) \) is contained in \( \mathcal{O}_{\mathbb{P} \times H'} \), and such that \( \mathcal{O}_Z(-F) \) is \( \rho \)-ample. If \( \gamma \) is big enough \( \mathcal{H} = p^* \mathcal{H} \otimes \mathcal{O}_Z(-F) \) will be ample for \( p_2 \).

\( F_x = F|_{U_x} \) is supported outside of \( U_x \). Since \( U_x = (pr_1 \circ \rho)^{-1}(G) \cap U_x' \) we can find some \( A_x \) with support in \( U_x' - U_x \) such that \( \mathcal{H}|_{U_x} = \mathcal{O}_{U_x}(A_x) \).

For \( \beta > 0 \) we have natural inclusions

\[
p_{2*} \mathcal{H}^\beta \to p_{2*} \mathcal{H}^\gamma = \bigoplus \mathcal{O}_{U_x^\beta} \to p_{2*} \mathcal{H}^\beta
\]

and a restriction map

\[
p_{2*} \mathcal{H}^\beta \to p_{2*} \mathcal{O}_{U_x^\beta}(\beta \cdot (A_x + F_x)).
\]

Let \( B_x \) be a divisor supported in \( H_x' - H_x \) with \( p_2^* B_x|_{U_x} \cong A_x + F_x \).

We have a trace map

\[
p_{2*} \mathcal{O}_{U_x^\beta}(\beta \cdot (A_x + F_x)) \to \mathcal{O}_{H_x}(\beta \cdot B_x).
\]

Combining those maps we obtain

\[
\phi(\beta) : \bigoplus \mathcal{O}_{H_x} \to \mathcal{O}_{H_x}(\beta \cdot B_x).
\]

We may assume that for some \( \alpha > 0 \) the ample sheaf \( \mathcal{L}_0^\alpha \) on \( H \) extends to an ample invertible sheaf \( \mathcal{L}_0^{\alpha(\beta)} \) on \( H' \). For \( \alpha \gg 0 \) \( \mathcal{H} \otimes p_2^* \mathcal{L}_0^{\alpha(\beta)} \) is ample. Again \( p_2^* \mathcal{L}_0^{\alpha(\beta)}|_{U_x} = \mathcal{O}_{U_x} \). Since \( A_x = A|_{U_x} \) is effective and \( (A_x)_{red} = U_x' - U_x \) we can find some \( \eta_0 \) such that for \( \eta \geq \eta_0 \) the sheaf

\[
p_2^* \mathcal{L}_0^{\alpha(\beta)} \otimes (\lambda^{\alpha - 1} \otimes \mathcal{O}_Z(\lambda))^\beta|_{U_x}
\]

has a section whose zero divisor is supported in \( U_x' - U_x \) and is larger than \( p_2^* (H_x + B_x)|_{U_x} \).

By 5.16 the sheaf

\[
\mathcal{M} = \mathcal{H} \otimes p_2^* \mathcal{L}_0^{\lambda(\beta)} \otimes (\lambda^{\alpha - 1} \otimes \mathcal{O}_Z(\lambda))^\beta
\]

is ample with respect to \( Z_\alpha \). Therefore we can find for some \( \beta > 0 \) a finite number of generating sections of \( \mathcal{M}^\beta \). Applying \( p_{2*} \) we find a coherent subsheaf \( \mathcal{P} \) of

\[
p_{2*} \mathcal{M}^\beta = p_{2*} \mathcal{H}^\beta \otimes i_*(\mathcal{L}_0^\beta \otimes \lambda^{(r-1)\eta^\beta})
\]

\[
= \bigoplus i_*(\mathcal{L}_0^\beta \otimes \lambda^{(r-1)\eta^\beta})
\]

which is generated by its global sections, such that \( \phi(\beta)(\mathcal{P}) \) contains \( \mathcal{O}_{H_x}(H_x + B_x)^\beta \), both isomorphic over \( H_x \). Let us write \( \eta = \frac{(r-1)\beta}{\alpha} \) and \( \beta' = \beta \cdot \alpha \). Then, if \( \alpha \) divides \( \eta \), \( \mathcal{P} \) is contained in \( \bigoplus i_*(\mathcal{L}_0^\beta \otimes \lambda^{(r-1)\eta^\beta}) = \bigoplus \mathcal{L}_0^\beta \). Therefore we can
find some coherent subsheaf $\mathcal{L}^{(p)}_n$ of $i_\ast \mathcal{L}^{(p)}_n$, which is globally generated on $H'$, such $\mathcal{I} \subseteq \mathcal{L}^{(p)}_n$. We still have an inclusion of $\mathcal{E}_{H_n}(\beta \cdot \mathcal{I}_n + \beta \cdot \mathcal{B}_n)$ in

$$\phi(\beta)(\mathcal{L}^{(p)}_n) \subset \mathcal{E}_{H_n}(\beta \cdot \mathcal{B}_n) \otimes \mathcal{L}^{(p)}_1,$$

which is surjective over $H_n$. Then $\mathcal{L}^{(p)}_n$ contains $\mathcal{E}_{H_n}(\beta \cdot \mathcal{I}_n)$ and satisfies the conditions asked for in 5.9b).

**Remark 5.18.** In his recent paper “Projectivity of complete moduli” János Kollár proves that projective moduli schemes exist for certain moduli functores $\mathcal{M}$ (including those considered in 1.1) if there exists a compact algebraic coarse moduli space $M$. He shows that the determinant sheaves $\lambda_n$ are defined on $M$, for $v \geq 0$, and he uses “positivity” to verify that $\lambda_n$ satisfies the Nakai-Moishezon criterion for ampleness.

A special case of his “Proposition 2.7” is:

**Proposition J.K.** Let $\mathcal{M}_n$ be one of the moduli functors considered in 1.1 and $M$ the corresponding coarse analytic moduli space (see [10] 4.1.1 or [14] p. 172). Let $Z$ be a subspace of $M$ of finite type. Then there is a scheme $Y$ of finite type, a surjective finite, morphism $p: Y \to Z$ and a family $f: X \to \mathcal{M}_n(Y)$, such that for all ye $Y$ the moduli point of $f^{-1}(y)$ is $p(y)$.

This proposition together with the analytic construction of moduli (see [10] and [14]) allows to use 1.18 directly to show that certain open subspaces of $M$ are quasi projective and to give a proof of 1.7 without using § 5. In fact, using 4.9 instead of 1.18 one even can get from the proposition J.K.:

**Corollary.** For $\mathcal{M}_n$ as in 1.1 let $Z \subset M$ be an open subspace of finite type. If the singular locus of $Z$ is compact, then $Z$ is a quasi projective variety.

**References**


Note added in proof
A second part of this paper is in preparation. It (hopefully) will give an affirmative answer to problem 1.10 and, using 1.7b, finish the proof of the existence of quasi-projective moduli spaces for complex canonically polarised manifolds. Unfortunately I forgot to give the reference for the first construction of the analytic moduli space for canonically polarised manifolds: