Advanced Studies in Pure Mathematics 1, 1983
Algebraic Varieties and Analytic Varieties
pp. 329–353

Weak Positivity and the Additivity of the Kodaira Dimension for Certain Fibre Spaces

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Contents

Let V and W be non-singular projective varieties over the field of complex numbers $\boldsymbol{C}$, $n=\dim(V)$ and $m=\dim(W)$. Let $f: V\to W$ be a fibre space (this simply means that f is surjective with connected general fibre $V_{\overline{W}}=V\times_W \operatorname{Spec}(\overline{\boldsymbol{C}(W)})$. We denote the canonical sheaves of V and W by ω_V and ω_W, and we write $\omega_{V/W}=\omega_V\otimes f^*\omega_W^{-1}$.

S. Iitaka conjectured the following inequality for the Kodaira dimension to be true:

Conjecture $C_{n,m}$. $\kappa(V)\geqslant\kappa(W)+\kappa(V_w)$.
Being more optimistic, one might ask:

Conjecture $C^+_{n,m}$. If $\kappa(W)\geqslant 0$ then

$$\kappa(V)\geqslant \operatorname{Max}\{\kappa(W)+\kappa(V_w), \operatorname{Var}(f)+\kappa(V_w)\}.$$

$\operatorname{Var}(f)$ is defined to be the minimal number k, such that there exists a subfield L of $\overline{\boldsymbol{C}(W)}$ of transcendental degree k over $\boldsymbol{C}$ and a variety F over L with $F\times_{\operatorname{Spec}(L)}\operatorname{Spec}(\overline{\boldsymbol{C}(W)})\sim V_w$ ($\sim$ means "birational").

Received September 28, 1981.
*) supported during the academic year 1980/81 by a fellowship of the German Research Council, DFG.

In his article [9] Y. Kawamata explained the theory of variations of Hodge structures and period domains giving powerful tools to study these Conjectures. Building up on these results, we want to explain algebraic methods which allow to prove

Theorem I. *$C^+_{n,n-2}$ and $C^+_{n,n-1}$ are true if $\kappa(V_w)=\dim(V_w)$.*

The restriction $\kappa(V_w)=\dim(V_w)$ is quite essential for the method applied to prove this theorem. However, $C^+_{n,n-1}$ can be obtained quite easily if the fibre is an elliptic curve (see for example [18], 9.3) and Y. Kawamata obtained $C_{n,n-2}$ if $\kappa(V_w)\leqslant 1$ and $C^+_{n,n-2}$ if $\kappa(V_w)=0$ (see [8] and [9]). He also finished the proof of $C_{n,1}$ ([7]) and hence we can say now, that $C_{n,m}$ is true for $n\leqslant 4$. The applications in the classification theory are discussed in § 9.

Proofs of $C^+_{n,n-1}$ and $C^+_{3,1}$ can also be found in [16] and [18]. In both papers we first constructed lots of sections of $\det(f_*\omega^\nu_{V/W})^k$, and then we had to find some method in order to get sections of $f_*\omega^{\nu\cdot\eta}_{V/W}$. If $\dim(W)=1$, it is possible to use the Riemann-Roch theorem for vector bundles. If $\dim(V_w)=1$, through the Wronski-determinant we get an inclusion $\det(f_*\omega^\nu_{V/W})\to f_*\omega^{\eta\cdot\nu}_{V/W}$ for $\eta\gg 0$. In this paper we use completely different arguments in order to obtain in general:

Theorem II. *Let $f\colon V\to W$ be a fibre space. Assume that for all fibre spaces $f''\colon V''\to W''$ with $\overline{C(W'')}\subseteq\overline{C(W)}$, $V_w\sim V''_w\times_{\mathrm{Spec}\,\overline{C(W'')}}\mathrm{Spec}\,(\overline{C(W)})$ and* $\mathrm{Var}(f'')=\dim(W'')$, *there exists $k>0$, such that*

$$\kappa(W'', \det(f''_*\omega^k_{V''/W''}))=\dim(W'').$$

Then $C^+_{n,m}$ is true for the fibre space $f\colon V\to W$.

The proof of theorem II and its application to families of curves or surfaces as well are based on slight extensions of the methods introduced in [19]. For the convenience of the reader we recall some of them and even sketch the proof of

Theorem III. (see [19]). *Let $g\colon T\to W$ be any surjective morphism between non-singular projective varieties. Then $g_*\omega^k_{T/W}$ is weakly positive for any $k>0$.*

Corollary IV ([19], with the additional condition that $\kappa(V)\geqslant 0$ see [6]). *$C_{n,m}$ (and hence $C^+_{n,m}$) is true for a fibre space $f\colon V\to W$ if $\kappa(W)=\dim(W)$.*

Theorem II reduces the Conjecture $C^+_{n,m}$ to the following question:

Let $f\colon V\to W$ be a fibre space such that $\mathrm{Var}(f)=\dim(W)$. *Does there exist a number $k>0$, such that*

$$\kappa(W, \det(f_*\omega^k_{V/W})) = \dim(W)?$$

In the case that V_w is a surface or a curve of general type we are going to use theorem III and geometric invariant theory (like in [18]) in order to obtain a positive answer.

The author wants to thank Prof. Shigeru Iitaka sincerely for inviting him to University of Tokyo and the JSPS for supporting his stay at Tokyo.

Moreover he owes thank to Prof. Takao Fujita, and Prof. Kenji Ueno for pointing out several gaps and ambiguities in an earlier version of this paper.

§ 1. Weak positivity

Let W be a non-singular quasi-projective variety and $\mathscr{F}$ a torsion-free coherent sheaf on W. We take $i: \hat{W} \to W$ to be the biggest open subvariety such that $\mathscr{F}|_{\hat{W}}$ is locally free.

Definition 1.1. Let S^k denote the symmetric product and $\bigwedge^k$ the exterior product of locally free sheaves. Then we define $\hat{S}^k(\mathscr{F}) = i_* S^k(i^*\mathscr{F})$, $\bigwedge^k(\mathscr{F}) = i_* \bigwedge^k(i^*\mathscr{F})$ and $\det(\mathscr{F}) = \bigwedge^r(\mathscr{F})$, where $r = \mathrm{rk}(\mathscr{F})$.

Definition 1.2. Let U be an open subvariety of W. We call $\mathscr{F}$ *weakly positive over* U, if for every ample invertible sheaf $\mathscr{H}$ on W and every positive number α there exists some positive number β such that $\hat{S}^{\alpha\cdot\beta}(\mathscr{F}) \otimes \mathscr{H}^\beta$ is generated by global sections over U. This means that the natural map

$$H^0(W, \hat{S}^{\alpha\cdot\beta}(\mathscr{F})\otimes\mathscr{H}^\beta)\otimes_C \mathcal{O}_W \longrightarrow \hat{S}^{\alpha\cdot\beta}(\mathscr{F})\otimes\mathscr{H}^\beta$$

is surjective over U. We call $\mathscr{F}$ *weakly positive*, if there exists some open subvariety U such that $\mathscr{F}$ is weakly positive over U.

(1.3) **Remark.** i) If $\hat{S}^{\alpha\cdot\beta}(\mathscr{F})\otimes\mathscr{H}^\beta$ is generated by global sections over U, then $\hat{S}^{\alpha\cdot\beta\cdot\eta}(\mathscr{F})\otimes\mathscr{H}^{\beta\cdot\eta}$ has the same property for any $\eta>0$.

ii) It is enough to check the condition in (1.2) for one invertible sheaf $\mathscr{H}$, not necessarily ample, and all $\alpha>0$.

iii) If $\mathscr{F}$ is locally free and W projective, then $\mathscr{F}$ is semi-positive if and only if $\mathscr{F}$ is weakly positive over W (see [9] or [19], 1.10).

iv) In order to check whether some sheaf $\mathscr{F}$ is weakly positive, we may replace W by $W-S$ for some closed subvariety S of codimension bigger than or equal to two (for example $W-S=\hat{W}$).

Lemma 1.4. *Let $\mathscr{F}$ and $\mathscr{G}$ be torsion-free coherent sheaves on W.*

1) *If* $\mathscr{F}\to\mathscr{G}$ *is a morphism, surjective over* U, *and if* $\mathscr{F}$ *is weakly positive over* U, *then* $\mathscr{G}$ *is weakly positive over* U.

2) *If* $\tau: W'\to W$ *is a surjective morphism and* $\mathscr{F}$ *weakly positive over* U, *then* $\tau^*\mathscr{F}$ *is weakly positive over* $\tau^{-1}(U)$.

3) *If* $\hat{S}^\eta(\mathscr{F})$ *or* $\otimes^\eta \mathscr{F}$ *is weakly positive over* U *for some* $\eta>0$, *then* $\mathscr{F}$ *is weakly positive over* U.

4) *If* $\delta: W\to W''$ *is a birational morphism,* E *any divisor with support in the exceptional locus of* δ *and* U *an open subvariety such that* $\delta|_U$ *is an isomorphism and* $\mathscr{F}\otimes\mathcal{O}_W(E)$ *weakly positive over* U, *then* $\delta_*\mathscr{F}$ *is weakly positive over* $\delta(U)$.

5) *Let* $\tau: W'\to W$ *be a finite morphism. If* $\tau^*\mathscr{F}$ *is weakly positive over* $\tau^{-1}(U)$, *then* $\mathscr{F}$ *is weakly positive over* U.

6) *If* $\mathscr{F}$ *is weakly positive over* U, *then* $\det(\mathscr{F})$ *is weakly positive over* U.

Proof. 1) and 2) follow directly from the definition and 4) and 5) are proved in [19]. In order to verify 3) one just has to use the natural maps, surjective over $\hat{W}$, $\otimes^\eta \mathscr{F}\to\hat{S}^\eta(\mathscr{F})$ and $\hat{S}^\alpha\hat{S}^\beta(\mathscr{F})\to\hat{S}^{\alpha\cdot\beta}(\mathscr{F})$. In order to prove 6) we use:

Lemma 1.5. *Let* $\mathscr{F}$ *be any local free sheaf of rank* r *over a (not necessarily projective) variety. Then* $\det(S^m(\mathscr{F}))=\det(\mathscr{F})^a$, *where* $a\cdot r=\operatorname{rk}(S^m(\mathscr{F}))\cdot m$.

Proof. Since every one-dimensional linear representation of $GL(r)$ is a power of the determinant, the assertion follows from the fact that $\det(S^m(\lambda\cdot Id))=\lambda^{a\cdot r}$.

Proof of (1.4, 6). Let r be the rank of $\mathscr{F}$. For any α and $\mathscr{H}$ there exists $\beta>0$ such that $\hat{S}^{\alpha\cdot\beta\cdot r}(\mathscr{F})\otimes\mathscr{H}^\beta$ is generated by global sections over U. Hence $\det(\mathscr{F})^{\alpha\cdot b}\otimes\mathscr{H}^b$ has the same property for $b=\operatorname{rk}(\hat{S}^{\alpha\cdot\beta\cdot r}(\mathscr{F}))\cdot\beta$.

§ 2. Coverings

In this section we just recall some results about finite coverings we need later. We assume X to be any quasiprojective non-singular variety. Let D be an effective divisor on X. We call D a *normal crossing divisor,* if D has regular components which intersect tranversally. For any divisor E we write $\mathscr{F}(E)=\mathscr{F}\otimes\mathcal{O}_X(E)$.

Lemma 2.1. *Let* $h: Y\to X$ *be a finite morphism,* Y *normal, and assume that the discriminant* $\Delta(Y/X)\subseteq X$ *is a normal crossing divisor. Take any desingularization* $d: Z\to Y$. *Then we have:*

1) *h is flat and Y has only rational singularities (that means that* $R^i d_* \mathcal{O}_Z = 0$ *for* $i > 0$).

2) *For every effective divisor E with support in the excepional locus of d the dualizing sheaf* ω_Y *is isomorphic to* $d_* \omega_Z(E)$.

3) ω_X *is a direct factor of* $h_* d_* \omega_Z$.

1) is proved in [17]; 2) can be found in [18], A.3; and 3) follows from the duality theory for finite morphisms (see [5], III, Ex. 6.10 and 7.2), saying that

$$h_* d_* \omega_Z = h_* \omega_Y = \mathcal{H}om_X (h_* \mathcal{O}_Y, \omega_X).$$

(2.2) *Taking roots out of sections.* In one special case of (2.1) we are able to give an even better description. Let $\mathcal{L}$ be an invertible sheaf on X and s a global section of $\mathcal{L}^N$, having the divisor D as set of zeros. Let B be the sum over the components of D with multiplicity 1 and $D - B = \sum \nu_j \cdot E_j$ the decomposition into prime-divisors. Define

$$\mathcal{L}^{(i)} = \mathcal{L}^i\left(-\sum \left[\frac{\nu_j \cdot i}{N}\right] \cdot E_j\right),$$

where $[b]$ denotes the integral part of the real number b.

s^{-1}: $\mathcal{L}^{-N} \to \mathcal{O}_X$ defines a structure of an $\mathcal{O}_X$-algebra on $\mathcal{A} = \bigoplus_{i=0}^{N-1} \mathcal{L}^{-i}$. Let Y be the normalization of **Spec** $(\mathcal{A})$ and h: $Y \to X$ the natural morphism. Let d: $Z \to Y$ be any desingularization. We say that Z is obtained, by taking the N-th root out of D.

Lemma 2.3 ([3], § 1). *Assume that D is a normal crossing divisor. Then h is étale outside of the support of D and*

$$h_* d_* \mathcal{O}_Z = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)-1} \quad \textit{and} \quad h_* d_* \omega_Z = \bigoplus_{i=0}^{N-1} \mathcal{L}^{(i)} \otimes \omega_X.$$

Of course, the nicest situation is, if we have a non-singular finite covering. This means that the base and the total space are both non-singular. This situation can be obtained quite often, using the following construction, due to Kawamata:

Lemma 2.4 ([6], Theorem 17, see also [1] and [19]). *Let D be a reduced normal crossing divisor on X with components* D_i, $i = 1, \cdots, r$. *Let* m_i *be positive natural numbers. Then there exists a finite covering* τ: $X' \to X$, X' *non-singular such that* $D'_i = (\tau^* D_i)_{\text{red}}$ *are non-singular irreducible divisors intersecting tranversally and* $\tau^* D = \sum_{i=1}^r m_i \cdot D'_i$. *Moreover* $D \cup \Delta(X'/X)$ *is a normal crossing divisor.*

Using Abhyankar's Lemma ([13], page 22) one gets immediately:

Corollary 2.5. *Let $h\colon Y \to X$ be as in* (2.1). *Then there exists a finite covering $\rho\colon X' \to Y$, such that X' is non-singular and $\Delta(X'/X)$ a normal crossing divisor.*

§ 3. Applications of duality theory

Let $h\colon Y \to X$ be any surjective morphism between varieties. We use three different definitions of $\omega_{Y/X}$: If X is Gorenstein and if Y is Cohen-Macaulay, we write $\omega_{Y/X} = \omega_Y \otimes h^* \omega_X^{-1}$. If h is a flat, locally projective morphism and if all fibres of h are Cohen-Macaulay, let $\omega_{Y/X}$ be the dualizing sheaf (see [11]). If h is any finite morphism, we define $\omega_{Y/X}$ to be the sheaf, determined by $h_* \omega_{Y/X} = \mathcal{H}om_X(h_* \mathcal{O}_Y, \mathcal{O}_X)$. These different definitions are compatible (see [11] or [5], III, Ex. 6.10 and 7.2).

(3.1) Let $g\colon T \to W$ be a surjective projective morphism between non-singular varieties. We write

$$W_2 = \{w \in W;\ g \text{ is flat along } g^{-1}(w)\}$$
$$W_1 = \{w \in W_2;\ g^{-1}(w) \text{ is reduced}\}$$
$$W_0 = \{w \in W_1;\ g^{-1}(w) \text{ is non-singular}\}$$

and $T_i = g^{-1}(W_i)$ and $g_i = g|_{T_i}$.

Let $\tau\colon W' \to W$ be a flat, projective morphism between non-singular varieties, $S = T \times_W W'$ the fibre product, S' the normalization of S and T' a desingularization of S'. We denote the induced morphisms by

$$\begin{array}{ccccccc} T' & \xrightarrow{d} & S' & \xrightarrow{\sigma} & S & \xrightarrow{\tau_2} & T \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ W' & \xrightarrow{id} & W' & \xrightarrow{id} & W' & \xrightarrow{\tau} & W \end{array}$$

and $\tau_1 = \tau_2 \cdot \sigma$ and $\tau' = \tau_1 \cdot d$.

Lemma 3.2. *Assume that S' has only rational singularities. Then for all $k \geqslant 0$ one has a natural inclusion $i\colon g'_* \omega^{k+1}_{T'/W'} \to \tau^* g_* \omega^{k+1}_{T/W}$. Moreover i is an isomorphism over $\tau^{-1}(W_1)$ and over the open subvariety U' of W' where τ is smooth.*

The assumption is fulfilled if τ is finite and if $\Delta(S'/T)$ is a normal crossing divisor (see (2.1)). In this case let I be the set of prime-divisors D of T such that $g(D)$ is a divisor on W. For $D \in I$ we define

$$\delta(D) = \operatorname{Min}\{e(h'(D')) - e(D');\ D' \text{ prime-Weil-divisor of } S' \text{ and } \tau_1(D') = D\}$$

where $e(D')$ (respectively $e(h'(D'))$) denotes the order of ramification of D' over T (respectively $h'(D')$ over W).

Lemma 3.3. *Assume that τ is finite and that $\Delta(S'/T)$ is a normal crossing divisor. Using the notation introduced above, let a be bigger than $e(D')$ for all prime-divisors D' of S' with $\tau_1(D') \in I$. Then, for $k \geqslant 0$ we have a natural inclusion*

$$i': g'_*\Big(\omega_{T'/W'}^{k\cdot a+1} \otimes \tau'^* \mathcal{O}_T(k \cdot \sum_{D \in I} \delta(D) \cdot D)\Big) \longrightarrow \tau^* g_* \omega_{T/W}^{k\cdot a+1}.$$

Proof of (3.2) *and* (3.3) (see also [19], 1.9).

τ being flat, $\omega_{W'/W}$ is a dualizing sheaf, compatible with base change, and $\omega_{S/T} \cong h^* \omega_{W'/W}$ or $\omega_{S/W'} \cong \tau_2^* \omega_{T/W}$. Especially S is Gorenstein. By definition of $\omega_{S'/S}$ we have $\sigma_* \omega_{S'/W'} = \mathscr{H}om_S(\sigma_* \mathcal{O}_{S'}, \omega_{S/W'})$ and hence a natural homomorphism $\alpha: \sigma_* \omega_{S'/W'} \to \omega_{S/W'}$ (in fact, α is injective).

The morphism α being affine and birational, we have a surjection, isomorphic over an open subvariety,

$$\sigma^* \sigma_* \omega_{S'/W'} \longrightarrow \omega_{S'/W'}.$$

$\tau_1^* \omega_{T/W}$ is invertible and

$$\sigma^*(\alpha): \sigma^* \sigma_* \omega_{S'/W'} \longrightarrow \tau_1^* \omega_{T/W}$$

must factorize through

$$\beta: \omega_{S'/W'} \longrightarrow \tau_1^* \omega_{T/W}.$$

This homomorphism is in general not surjective. For the application to (3.3), given below, let a be any positive integer and $\mathscr{L}^{-1}$ any invertible ideal-sheaf on T, such that $\beta^{a\cdot k}$ factorizes through

$$\beta'^k: \omega_{S'/W'}^{a\cdot k} \longrightarrow \tau_1^*(\omega_{T/W}^{a\cdot k} \otimes \mathscr{L}^{-k})$$

for all $k \geqslant 0$ (for (3.2) one may take $\mathscr{L} = \mathcal{O}_T$ and $a = 1$). From (2.1, 2) we get a homomorphism

$$d^* \omega_{S'} = d^* d_* \omega_{T'} \longrightarrow \omega_{T'}.$$

The induced homomorphism

$$d^* \omega_{S'/W'}^{a\cdot k} \longrightarrow \omega_{T'/W'}^{a\cdot k}$$

is isomorphic outside of the exceptional locus of d and for some sufficiently big exceptional divisor E the pullback of β'^k induces

$$\omega_{T'/W'}^{a\cdot k+1} \longrightarrow \tau'^*(\omega_{T/W}^{a\cdot k}\otimes\mathscr{L}^{-k})\otimes\omega_{T'/W'}(E).$$

Applying d_* and using (2.1, 2) again, we get

$$d_*\omega_{T'/W'}^{a\cdot k+1} \longrightarrow \tau_1^*(\omega_{T/W}^{a\cdot k}\otimes\mathscr{L}^{-k})\otimes\omega_{S'/W'}$$

and

$$\sigma_* d_*(\omega_{T'/W'}^{a\cdot k+1}\otimes\tau'^*\mathscr{L}^k) \longrightarrow \tau_2^*(\omega_{T/W}^{a\cdot k})\otimes\sigma_*\omega_{S'/W'}.$$

The homomorphism α induces a homomorphism of the second sheaf to $\tau_2^*\omega_{T/W}^{a\cdot k+1}$. Now we may apply h_* in order to obtain

$$i\colon g'_*(\omega_{T'/W'}^{a\cdot k+1}\otimes\tau'^*\mathscr{L}^k) \longrightarrow h_*\tau_2^*\omega_{T/W}^{a\cdot k+1} \cong \tau^* g_*\omega_{T/W}^{a\cdot k+1}.$$

In fact, the isomorphism exists by flat base change.

By definition $g_1\colon T_1\to W_1$ is flat with reduced fibres. Hence the points of T_1 where g_1 is not smooth are concentrated in a subvariety of codimension bigger than or equal to two and the fibre product $h^{-1}(\tau^{-1}(W_1))$ is non-singular in codimension two. Knowing already that S is Gorenstein, we get that $h^{-1}(\tau^{-1}(W_1))$ is normal. The same argument shows, that $h^{-1}(U')$ is normal, where U' is defined as in (3.2). Let U'' be the maximal open subvariety of W', such that $h^{-1}(U'')$ is normal. Over $h^{-1}(U'')$ the homomorphisms α and β are just isomorphisms and $\mathscr{L}|_{h^{-1}(U'')}$ must be trivial. $h'^{-1}(U'')\cong h^{-1}(U'')$ having rational Gorenstein singularities we know from [19], A.3 that

$$d_*\omega_{T'/W'}^{a\cdot k+1}(E)|_{h^{-1}(U'')} = d_*\omega_{T'/W'}^{a\cdot k+1}|_{h^{-1}(U'')} = \omega_{S'/W'}^{a\cdot k+1}|_{h^{-1}(U'')}.$$

The homomorphism i hence is an isomorphism over U'' and injective everywhere.

The statement of (3.2) follows for $a=1$ and $\mathscr{L}=\mathcal{O}_T$.

From now on we assume τ to be finite and $\Delta(S'/T)$ to be a normal crossing divisor. Let $U=S'_{\mathrm{reg}}\cap\tau_1^{-1}(T_2)$.

By ramification theory we know that

$$\omega_{S'/W'}\otimes\tau_1^*\omega_{T/W}^{-1}|_U \cong \mathcal{O}_U\left(\sum (e(D')-e(h'(D')))\cdot D'\right)$$

where the sum is taken over all prime-Weil-divisors D' of S' such that $h'(D')$ is a divisor. If we take a as in (3.3) and $\mathscr{L}=\mathcal{O}_T(\sum_{D\in I}\delta(D)\cdot D)$ we get the inclusion i'.

(3.4) In order to prove theorem II, we need another quite technical construction. Let $f\colon V\to W$ be a fibre space. We define

$$V^s = V\times_W V\times_W\cdots\times_W V \text{ (s-times)}$$

and we take a desingularization $d^s\colon V^{(s)}\to V^s$. Let $f^s\colon V^s\to W$ be the induced morphism and $f^{(s)}=f^s\cdot d^s$. Assume that (using the notation introduced in (3.1)) $W-W_0$ and $V-V_0$ are normal crossing divisors, and that $\operatorname{codim}(W-W_1)\geqslant 2$.

Lemma 3.5. *There exists an open subvariety U of W such that* $\operatorname{codim}(W-U)\geqslant 2$ *and* $f^{(s)}_*\omega^k_{V^{(s)}/W}|_U=\otimes^s f_*\omega^k_{V/W}|_U$.

Proof. We may replace W by W_1 and V by V_1. Hence we may assume that f is flat and has reduced fibres and the subvariety of V where f is not smooth is of codimension bigger than or equal to two. Hence the singular locus of V^s is of codimension bigger than or equal to two. $\omega_{V/W}$ is a dualizing sheaf of f and hence $\omega_{V^s/W}=\otimes^s_{i=1}pr_i^*\omega_{V/W}$ where pr_i denotes the i-th projection. It follows that V^s is Gorenstein and normal and that $f^s_*\omega^k_{V^s/W}=\otimes^s f_*\omega^k_{V/W}$. The only thing we have to verify, is the existence of U such that over $f^{s^{-1}}(U)$ we have $d^s_*\omega^k_{V^{(s)}/W}=\omega^k_{V^s/W}$. This is true, if $f^{s^{-1}}(U)$ has only rational singularities (for example: [18], A.3).

In order to find U we proceed by induction on s. We may assume that (making W smaller, if necessary) V^{s-1} has only rational singularities. The existence of U follows from:

Lemma 3.6. *Let $g\colon T\to W$ be a surjective morphism between nonsingular varieties. Assume that there exists an open subvariety W_0 of W such that $g|_{g^{-1}(W_0)}$ is smooth and $T-g^{-1}(W_0)$ a normal crossing divisor. Assume moreover that there exists an open subvariety W_1 of W such that* $\operatorname{codim}(W-W_1)\geqslant 2$ *and such that $g|_{g^{-1}(W_1)}$ is flat with reduced fibres. Let T' be a variety having only rational singularities and $g'\colon T'\to W$ a surjective morphism. Then there exists an open subvariety U of W such that* $\operatorname{codim}(W-U)\geqslant 2$ *and such that $g^{-1}(U)\times_W T'$ has only rational singularities.*

Proof. Again we may assume that $W=W_1$.
Let $d\colon T''\to T'$ be any desingularization and

$$\begin{array}{ccc} T\times_W T'' & \longrightarrow & T'' \\ \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d} \\ T\times_W T' & \xrightarrow{\ h\ } & T' \\ \downarrow & & \downarrow \\ T & \xrightarrow{\ g\ } & W \end{array}$$

the fibre products. By flat base change we obtain

$$R^i d'_*\mathcal{O}_{T\times_W T''}=h^*R^i d_*\mathcal{O}_{T''}=\begin{cases}\mathcal{O}_{T\times_W T'} & \text{if } i=0\\ 0 & \text{if } i\neq 0\,.\end{cases}$$

If $d'': Z \to T \times_W T''$ is any desingularization, the Leray spectral sequence

$$R^j d'_*(R^i d''_* \mathcal{O}_Z) \Rightarrow R^{i+j}(d' \cdot d'')_* \mathcal{O}_Z$$

tells us, that $g^{-1}(U) \times_W T'$ has only rational singularities, if $g^{-1}(U) \times_W T''$ has only rational singularities.

Hence we may assume that T' is non-singular and we may always replace it by a blowing up.

Over the general points of the components of $W - W_0$, and hence over the complement U of a subvariety of codimension bigger than or equal to two, the degenerate fibres of g can be described in the following way:

$W - W_0$ is locally defined by one equation $y=0$ and $g^{-1}(W - W_0)$ is locally defined by $y = x_0 \cdot x_1 \cdot \cdots \cdot x_n$ for local parameters $x_0, \cdots, x_n$ on T.

Blowing up T', if necessary, we may assume that the local equation of $g'^{-1}(W - W_0)$ is of the form $y = t_0^{s_0} \cdot \cdots \cdot t_r^{s_r}$.

The singularities of $g^{-1}(U) \times_W T'$ are given by equations of the form

$$x_0 \cdot \cdots \cdot x_n = t_0^{s_0} \cdot \cdots \cdot t_r^{s_r}.$$

Such singularities are rational by [10].

§ 4. The proof of theorem III for $k=1$

Using (1.4, 4) or restricting the morphism considered to $g^{-1}(U)$, where U is the complement of the singularities of $W - W_0$, and using (1.3, iv), we get theorem III for $k=1$ from:

Theorem 4.1. *Let $g: T \to W$ be a surjective projective morphism of quasiprojective non-singular varieties such that $W - W_0$ is a normal crossing divisor. Then $g_* \omega_{T/W}$ is locally free and weakly positive over W_0.*

Theorem (4.1) is just a consequence of the Main Theorem of Y. Kawamata in [6]:

Proof. Choosing a good compactification, we may assume that T and W are projective. Moreover we may replace T by any variety obtained by blowing up non-singular subvarieties of T. Hence if $\tau: W' \to W$ is a finite morphism and S' the normalization of $T \times_W W'$ we may assume that $\Delta(S'/T)$ is a normal crossing divisor. If we choose $g': T' \to W'$ and $\tau': T' \to T$ as in (3.1) it is enough to verify the statement for g'. In fact, if $g'_* \omega_{T'/W'}$ is locally free, the same is true for $g'_* \omega_{T'/W}$ and $\tau_* g'_* \omega_{T'/W}$ (τ is flat). From (2.1, 3) we know, that ω_T is a direct factor of $\tau'_* \omega_{T'}$ and hence $g_* \omega_{T/W}$ is a direct factor of the locally free sheaf $g_* \tau'_* \omega_{T'/W}$. If $g'_* \omega_{T'/W'}$ is weakly positive over W_0' we obtain the weak positivity of $g_*\omega_{T/W}$ from (3.2), (1.4, 1) and (1.4, 5). Let Y be the Stein factorization of g. It must be etale over W_0 and hence there exists a non-singular covering $\rho: W' \to Y$ by (2.5). However T' is then the disjoint union of T_i such that $T_i \to W'$ are fibre spaces. Hence we may assume that g is already a fibre space. Using (2.4) again, we may also assume that the local monodromies around the components of $W - W_0$ are unipotent (see [6], [9]). Under this condition Y. Kawamata has shown (4.1).

§ 5. The main lemmata and the proof of theorem III

Lemma 5.1. *Let $g: T \to W$ be a surjective projective morphism between quasi-projective non-singular varieties. Let $\mathscr{L}$ and $\mathscr{M}$ be invertible sheaves on T and $\sum \nu_j \cdot E_j$ a normal crossing divisor, such that for $N > 0$ $\mathscr{L}^N = \mathscr{M} \otimes \mathcal{O}_T(\sum \nu_j \cdot E_j)$. Assume that there exists an open subvariety U of W such that some power of $\mathscr{M}$ is generated over $g^{-1}(U)$ by global sections. Then, if we write* (*see* (2.2))

$$\mathscr{L}^{(i)} = \mathscr{L}^i\left(-\sum \left[\frac{\nu_j \cdot i}{N}\right] \cdot E_j\right),$$

the sheaf $g_*(\mathscr{L}^{(i)} \otimes \omega_{T/W})$ is weakly positive for $0 \leqslant i \leqslant N-1$.

Proof. The statement being compatible with replacing N by $N \cdot N'$ and ν_j by $\nu_j \cdot N'$, we may assume that $\mathscr{M}$ itself is generated by global sections over $g^{-1}(U)$. Let B be the zero-set of a general section of $\mathscr{M}$. By the theorem of Bertini we know that B, restricted to $g^{-1}(U)$, is non-singular and that $B + \sum \nu_j \cdot E_j$ is a normal crossing divisor. Let $\mathscr{M} = \mathcal{O}_T(B + \sum \eta_k \cdot F_k)$. Blowing up T outside of $g^{-1}(U)$ we may assume that B is non-singular and that $B + \sum \nu_j \cdot E_j + \sum \eta_k \cdot F_k$ is a normal crossing divisor everywhere. In fact, if $\rho: T' \to T$ is a blowing up and if $\mathscr{L}' = \rho^*\mathscr{L}$, $\mathscr{M}' = \rho^*\mathscr{M}$ and $\sum \nu_j' \cdot E_j' = \rho^*(\sum \nu_j \cdot E_j)$ and $\mathscr{L}^{(i)\prime}$ the corresponding integral part, then $\rho_*(\mathscr{L}^{(i)\prime} \otimes \omega_{T'/W})$ is contained in $\mathscr{L}^{(i)} \otimes \omega_{T/W}$. Making $\sum \nu_j \cdot E_j$ bigger and using (1.4, 1), we may assume that $\sum \eta_k \cdot F_k = 0$. As we did in (2.2), we denote by Z a desingularization of **Spec** $(\otimes_{i=0}^{N-1} \mathscr{L}^{-i})$. Let $f: Z \to W$ be the corresponding morphism. From (2.3) we obtain

$$f_*\omega_{Z/W} = \bigoplus_{i=0}^{N-1} g_*(\mathscr{L}^{(i)} \otimes \omega_{T/W}),$$

and (5.1) follows from theorem III for $k=1$.

Corollary 5.2. *Let $g: T \to W$ be as above and $\mathscr{H}$ an ample invertible sheaf on W, such that for given $k > 0$ and some $\nu > 0$ the sheaf $\hat{S}^\nu(g_*\omega_{T/W}^k \otimes \mathscr{H}^k)$*

is generated over an open set U by global sections. Then $g_*\omega^k_{T/W}\otimes\mathscr{H}^{k-1}$ *is weakly positive.*

Proof (see also [19], 3.1). By (1.3, iv) we may replace W by $W-S$, as long as S is a closed subvariety of codimension bigger than or equal to two. Hence we may assume that g is flat. Take $\mathscr{L}=\omega_{T/W}\otimes g^*\mathscr{H}$ and

$$\mathscr{M}=\operatorname{Im}(g^*(g_*\omega^k_{T/W}\otimes\mathscr{H}^k)\longrightarrow\omega^k_{T/W}\otimes g^*\mathscr{H}^k).$$

Blowing up T, if necessary, we may assume that $\mathscr{M}$ is invertible and that $\mathscr{L}^k\otimes\mathscr{M}^{-1}=\mathscr{O}_T(E)$ for a normal crossing divisor E. By definition, $\mathscr{M}^\nu$ is generated over $g^{-1}(U)$ by global sections and (5.1) gives for $i=k-1$ the weak positivity of the subsheaf $g_*(\mathscr{L}^{(k-1)}\otimes\omega_{T/W})$ of $g_*\omega^k_{T/W}\otimes\mathscr{H}^{k-1}$. By definition of $\mathscr{L}^{(k-1)}$ $\mathscr{M}\otimes g^*\mathscr{H}^{-1}$ is contained in $\mathscr{L}^{(k-1)}\otimes\omega_{T/W}$. Hence one gets

$$g_*\mathscr{M}\otimes\mathscr{H}^{-1}=g_*\omega^k_{T/W}\otimes\mathscr{H}^{k-1}=g_*(\mathscr{L}^{(k-1)}\otimes\omega_{T/W}).$$

(5.3) *Proof of theorem* III. Let $\mathscr{H}$ be any invertible ample sheaf on W and

$$r=\operatorname{Min}\{s>0;\ g_*\omega^k_{T/W}\otimes\mathscr{H}^{s\cdot k-1}\text{ weakly positive}\}.$$

By definition we can find $\nu>0$ such that $\hat{S}^\nu(g_*\omega^k_{T/W})\otimes\mathscr{H}^{r\cdot k\cdot\nu-\nu}\otimes\mathscr{H}^\nu$ is generated by global sections over an open set. From (5.2) we obtain the weak positivity of $g_*\omega^k_{T/W}\otimes\mathscr{H}^{r\cdot k-r}$. The choice of r allows this only if $(r-1)\cdot k-1<r\cdot k-r$ or $r\leqslant k$. Hence, for every surjective morphism and every $\mathscr{H}$ we have obtained the weak positivity of $g_*\omega^k_{T/W}\otimes\mathscr{H}^{k^2-k}$.

Replacing W by W', such that $\tau\colon W'\to W$ is finite and such that $\tau^*\mathscr{H}=\mathscr{H}'^d$ for $d\gg 0$ and applying this result to $g'\colon T'\to W'$ for a desingularization T' of $T\times_W W'$, one obtains from (3.2) the weak positivity of $\tau^*g_*\omega^k_{T/W}\otimes\mathscr{H}'^{k^2-k}$. If α is a positive integer we choose $d=(k^2-k)\cdot 2\alpha+1$. For β big enough we know that

$$\hat{S}^{2\alpha\beta}(\tau^*g_*\omega^k_{T/W}\otimes\mathscr{H}'^{k^2-k})\otimes\mathscr{H}'^\beta=\tau^*\hat{S}^{2\alpha\beta}(g_*\omega^k_{T/W})\otimes\tau^*\mathscr{H}^\beta$$

is generated over an open subvariety by global sections. Over the open subvariety $\hat{W}$ of W where $\hat{S}^{2\alpha\beta}(g_*\omega^k_{T/W})$ is locally free, one has a surjection

$$\tau_*\tau^*\hat{S}^{2\alpha\beta}(g_*\omega^k_{T/W})\otimes\mathscr{H}^\beta\longrightarrow\hat{S}^{2\alpha\beta}(g_*\omega^k_{T/W})\otimes\mathscr{H}^\beta.$$

Hence one obtains a homomorphism

$$\mathscr{H}^\beta\oplus\tau_*\mathscr{O}_{W'}\longrightarrow\hat{S}^{2\alpha\beta}(g_*\omega^k_{T/W})\otimes\mathscr{H}^{2\beta}$$

which is surjective over an open subvariety. If one chooses β big enough,

$\tau_* \mathcal{O}_{W'} \otimes \mathcal{H}^{\beta}$ is generated by global sections and $\hat{S}^{2a\beta}(g_* \omega^k_{T/W}) \otimes \mathcal{H}^{2\cdot\beta}$ is generated by global sections over an open subvariety. Hence $g_* \omega^k_{T/W}$ is weakly positive.

In order to obtain theorem II, we need a second application of (5.1):

Lemma 5.4. *Let $g: T \to W$ be a surjective projective morphism of quasi-projective non-singular varieties such that $W - W_0$ is a normal crossing divisor. Let a and k be natural numbers, $k' = a \cdot k > 1$. Assume that for an ample invertible sheaf $\mathcal{H}$ on W, one has an inclusion $\mathcal{H} \to \hat{S}^1 g_* \omega^k_{T/W}$. Let r be a positive number. Then there exists a regular covering*
$\tau: W' \to W$ such that for any desingularization
$g': T' \to W'$ of $T \times_W W'$ one has:
1) $\hat{S}^1 \tau^* g_* \omega^{\nu}_{T/W} = \hat{S}^1 g'_* \omega^{\nu}_{T'/W'}$ for all $\nu > 0$,
2) There exists an ample invertible sheaf $\mathcal{H}'$ on W' such that $g'_* \omega^{k'}_{T'/W'} \otimes \mathcal{H}'^{-r}$ is weakly positive.

Proof. The natural map $\mathcal{H}^a \to \hat{S}^a g_* \omega^k_{T/W} \to \hat{S}^1 g_* \omega^{k'}_{T/W}$ allows to assume that $k = k' > 1$. Blowing up T, if necessary, we may assume that there exist an open subvariety U of W_0, an invertible sheaf $\mathcal{M}_0$ over $g^{-1}(U)$ and a normal crossing divisor $\sum E_j$, such that

$$\mathcal{M}_0 = \mathrm{Im}\,(g^* g_* \omega^k_{T/W} \longrightarrow \omega^k_{T/W})|_{g^{-1}(U)},$$
$$\mathcal{M}_0 \otimes \mathcal{O}_T(\textstyle\sum \mu_j \cdot E_j) = \omega^k_{T/W}|_{g^{-1}(U)},$$
$$\mathcal{M}_0 = g^* \mathcal{H} \otimes \mathcal{O}_T(\textstyle\sum \nu_j \cdot E_j)|_{g^{-1}(U)}$$

and $g(E_j) = W$ for all j. Choose a natural number b such that $b > \nu_j$ for all j. Some power of H has a non-singular section in general position. As described in (2.2), we take the root out of this section and, applying (2.4), we can find a covering $\tau: W' \to W$ such that $\tau^* \mathcal{H} = \mathcal{O}_{W'}((b \cdot k \cdot r + 1) \cdot H')$ for some non-singular ample divisor H'. Moreover we may assume that the discriminant $\Delta(W'/W)$ intersects $W - W_0$ properly. Hence, if we choose $g': T' \to W'$ as above, the inclusion of $g'_* \omega^k_{T'/W'}$ into $\tau^* g_* \omega^k_{T/W}$ is an isomorphism outside of some subvariety of codimension bigger than or equal to two (3.2) and hence 1) is true. $g'^{-1}(\tau^{-1}(U))$ is just the fibre product and no E_j is contained in the discriminant of $\tau': T' \to T$. If

$$\tau'^*(\textstyle\sum \nu_j \cdot E_j) = \sum \nu'_j \cdot E'_j$$

we obtain $b > \nu'_j$.

For simplicity we may assume that $W = W'$ and that $\mathcal{H} = \mathcal{H}'^{b \cdot k \cdot r + 1}$. By theorem III the sheaf $g_* \omega^k_{T/W}$ is weakly positive, and hence there exists some $\nu > 0$ such that $\hat{S}^{\nu}(g_* \omega^k_{T/W}) \otimes \mathcal{H}'^{\nu}$ is generated by global sections over an open set. We may choose a divisor B of T such that codim $(g(B))\geqslant 2$ and such that for $\omega=\omega_{T/W}(B)$ one has $\hat{S}^1(g_*\omega^\eta_{T/W})=g_*\omega^\eta$, for all $\eta\leqslant \nu\cdot(b-1)\cdot k$. Take $\mathscr{L}=\omega\otimes g^*\mathscr{H}''^{-r}$, $N=b\cdot k$, and

$$\mathscr{M}=\mathscr{L}^N(-\sum(b\cdot\mu_j+\nu_j)\cdot E_j).$$

By construction there exists a positive divisor F on T, with support outside of $g^{-1}(U)$, such that

$$\mathcal{O}_T(F)=\omega^k(-\sum(\mu_j+\nu_j)\cdot E_j)\otimes g^*\mathscr{H}^{-1}.$$

Comparing the coefficients, one finds

$$\mathscr{M}=(\omega^k(-\sum\mu_j\cdot E_j))^{b-1}\otimes g^*\mathscr{H}''\otimes_T\mathcal{O}(F).$$

The natural maps

$$g^*\hat{S}^{\nu\cdot(b-1)}(g_*\omega^k_{T/W})\longrightarrow(\omega^k(-\sum\mu_j\cdot E_j))^{\nu\cdot(b-1)}\longrightarrow\mathscr{M}^\nu\otimes g^*\mathscr{H}''^{-\nu}$$

are surjective over $g^{-1}(U)$, and the assumptions of (5.1) are fulfilled. By the choice of b, one has for all j that

$$[(b\cdot k)^{-1}\cdot(k-1)\cdot(b\cdot\mu_j+\nu_j)]\leqslant\mu_j+[b^{-1}\cdot\nu_j]=\mu_j.$$

This means that over $g^{-1}(U)$ the sheaf $\mathscr{L}^{(k-1)}\otimes\omega_{T/W}$ contains

$$\mathscr{M}_0\otimes g^*\mathscr{H}''^{-r(k-1)},$$

and for $\mathscr{H}'=\mathscr{H}''^{k-1}$ the inclusion $g_*(\mathscr{L}^{(k-1)}\otimes\omega_{T/W})\to g_*\omega^k\otimes\mathscr{H}'^{-r}$ is an isomorphism over U. From (5.1) we get (5.4,2).

§ 6. Products of fibre spaces

In order to use (5.4) in the situation considered in theorem II, we need the following interpretation of a result due to D. Mumford. We use the notations introduced in (3.1).

Proposition 6.1. *Let $g\colon T\to W$ be a surjective morphism. Then we can find an open subvariety W_3 of W, such that* codim $(W-W_3)\geqslant 2$, *and a non-singular covering $\tau\colon W_3'\to W_3$, such that $T\times_W W_3'$ has a desingularization T_3' with the following property:*

Let $g_3'\colon T_3'\to W_3'$ be the induced morphism. Then g_3' is flat with semi-stable fibres. Especially $T_3'-g_3'^{-1}(\tau^{-1}(W_0))$ is a normal crossing divisor and all fibres are reduced.

Proof. Let x be a general point of one component of $W-W_0$ and $X\to Y=\operatorname{Spec}(\mathcal{O}_{x,W})$ the restriction of g. In [10], IV it is shown that there exists a covering Y' over Y, such that for all coverings Y'' of Y' the scheme $Y'' \times_Y X$ has a desingularization, for which the statement is true. Everything being defined over a finite extension of $\boldsymbol{C}(W)$, one can find a covering W' such that $T \times_W W'$ has a desingularization of the type wanted, up to codimension two in W'. Leaving out the bad points and the singular points of W', we find W'_3.

Lemma 6.2. *Let $f\colon V \to W$ be a fibre space of projective non-singular varieties. Assume that for every non-singular W' generically finite over W and for every desingularization V' of $V \times_W W'$ one has $\kappa(W', \det(f'_*\omega^k_{V'/W'})) = \dim(W)$ for some $k > 0$. Let $\mathscr{H}$ be an ample sheaf on W. Then there exists $\eta > 0$ such that $\mathscr{H}$ is contained in $\hat{S}^1(f_*\omega^\eta_{V/W})$.*

Proof. As wellknown, Lemma (6.3), proved at the end of this section, says that the assumption just means, that for all ample sheaves $\mathscr{H}'$ on W' we can find $\delta > 0$ such that $\mathscr{H}'$ is contained in $\det(f_*\omega^k_{V'/W'})^\delta$. Using this formulation, we may (by definition of $\hat{S}^1$) restrict ourself to open subvarieties of W as long as the complement is of codimension bigger than or equal to two. Hence we may assume that f is flat.

Moreover we claim, that it is enough to find a non-singular finite covering $\tau\colon W' \to W$ such that $\tau^*\mathscr{H}$ is contained in $\tau^* f_*\omega^\eta_{V/W}$. In fact, making W a little smaller and replacing W' by a bigger covering, we may assume, that W' is Galois over W with Galois group G. Let $S = V \times_W W'$ and $\sigma\colon S' \to S$ the normalization. τ being flat, one has

$$\tau^* f_*\omega^\eta_{V/W} = pr_{2*} pr_1^* \omega^\eta_{V/W}.$$

Assume that $\tau^*(f_*\omega^\eta_{V/W} \otimes \mathscr{H}^{-1})$ has a section. Then $\sigma^* pr_1^*(\omega^\eta_{V/W} \otimes f^*\mathscr{H}^{-1})$ has a section s. Let ν be the order of G. Then $\prod_{g \in G} g(s)$ defines a G-invariant section of $\sigma^* pr_1^*(\omega^{\eta\cdot\nu}_{V/W} \otimes f^*\mathscr{H}^{-\nu})$. Hence $\omega^{\eta\cdot\nu}_{V/W} \otimes f^*\mathscr{H}^{-\nu}$ has a section too.

Now, using (3.2), we may replace W by any non-singular covering. (6.1) allows us to assume that f is flat with semi-stable fibres. Choose $\delta > 0$ such that $\mathscr{H}$ is contained in $\det(f_*\omega^k_{V/W})^\delta$. For $r' = \operatorname{rk}(f_*\omega^k_{V/W})$ we have an inclusion of $\det(f_*\omega^k_{V/W})$ into $\otimes^{r'} f_*\omega^k_{V/W}$. Hence we can find some s such that $\otimes^s f_*\omega^k_{V/W}$ contains $\mathscr{H}$. Now let $f^{(s)}\colon V^{(s)} \to W$ be as in (3.4). By (3.5) $\hat{S}^1(f^{(s)}_*\omega^k_{V^{(s)}/W})$ contains an ample sheaf.

Take $k' = k$ if $k > 1$ and $k' = 2$ if $k = 1$. Applying (5.4) we can find a covering $\tau\colon W' \to W$ and an ample sheaf $\mathscr{H}'$ such that $(\otimes^s \tau^* f_*\omega^{k'}_{V/W}) \otimes \mathscr{H}'^{-2s}$ is weakly positive. From (1.4, 3) we know that $\tau^* f_*\omega^{k'}_{V/W} \otimes \mathscr{H}'^{-2}$ is weakly positive. By definition $\hat{S}^\beta(\tau^* f_*\omega^{k'}_{V/W} \otimes \mathscr{H}'^{-2}) \otimes \mathscr{H}'^\beta$ is generated by global sections for $\beta \gg 0$. The nontrivial map from this sheaf to $\tau^* f_*\omega^{k'\cdot\beta}_{V/W} \otimes \mathscr{H}'^{-\beta}$ gives an inclusion of $\mathscr{H}'^\beta$ into $\tau^* f_*\omega^{k'\cdot\beta}_{V/W}$. Making β big enough we may assume that $\tau^*\mathscr{H}$ is contained in $\mathscr{H}'^\beta$ and we get (6.2).

Lemma 6.3 (K. Kodaira). *Let $\mathscr{L}$ be an invertible sheaf and $\mathscr{H}$ an invertible ample sheaf on the projective variety W. Then $\kappa(W, \mathscr{L})=$ $\dim(W)$ if and only if there exist an $b>0$ such that $\mathscr{H}$ is contained in $\mathscr{L}^b$.*

Proof. If $\mathscr{L}^b$ contains an ample sheaf $\kappa(W, \mathscr{L})=\dim(W)$. On the other hand we may assume $\mathscr{H}$ to have a section with the set of zeros D. $h^0(D, \mathscr{L}^\nu|_D)$ is bounded by a polynomial $P(\nu)$ of degree

$$\dim(D)=\dim(W)-1,$$

but $h^0(W, \mathscr{L}^\nu)$ is raising like $\nu^{\dim(W)}$. The exact sequence

$$0 \longrightarrow H^0(W, \mathscr{L}^\nu \otimes \mathscr{H}^{-1}) \longrightarrow H^0(W, \mathscr{L}^\nu) \longrightarrow H^0(D, \mathscr{L}^\nu|_D)$$

gives the inclusion wanted.

§ 7. The proofs of theorem II and corollary IV

Let us start with the following corollary of theorem III.

Corollary 7.1. *Let $f: V \to W$ be a fibre space, V and W non-singular projective varieties, $\mathscr{H}$ an ample invertible sheaf on W and $r>0$. Then there exists a divisor B and some $\eta>0$, such that* $\operatorname{codim}(f(B)) \geqslant 2$, *and such that the linear system*

$$H^0(V, \omega_{V/W}^{r \cdot \eta} \otimes \mathcal{O}_V(r \cdot \eta \cdot B) \otimes f^* \mathscr{H}^\eta)$$

defines a rational map Φ_1 from V to a variety X of dimension $\kappa(V_w)+$ $\dim(W)$. *Moreover there exist a rational map $p: X \to W$ such that $f=$ $p \cdot \Phi_1$.*

Proof. From theorem III we know that for $k>0$ there exists $\beta>0$ such that $\hat{S}^{2 \cdot \beta}(f_* \omega_{V/W}^k) \otimes \mathscr{H}^\beta$ is generated by global sections. We may choose k such that the sheaf $f_* \omega_{V/W}^k$ is nontrivial and such that $r|k$. Choose B such that, for $\omega=\omega_{V/W}(B)$, we have

$$\hat{S}^1(f_* \omega_{V/W}^\eta)=f_* \omega^\eta \text{ for all } \eta \leqslant k \cdot 2 \cdot \beta.$$

The natural map

$$\hat{S}^{2 \cdot \beta}(f_* \omega_{V/W}^k) \otimes \mathscr{H}^\beta \longrightarrow f_* \omega^{2 \cdot k \cdot \beta} \otimes \mathscr{H}^\beta$$

being nontrivial, we obtain an inclusion of $f^* \mathscr{H}^\beta$ into $\omega^{2 \cdot k \cdot \beta} \otimes f^* \mathscr{H}^{2 \cdot \beta}$. If we choose β big enough the sections of $\omega^{2 \cdot k \cdot \beta} \otimes f^* \mathscr{H}^{2 \cdot \beta}$ define a map Φ_1: $V \to X$ with connected general fibre $\Phi_1^{-1}(a)$ and

$$\kappa(\Phi_1^{-1}(a), \omega^{2 \cdot k \cdot \beta} \otimes f^* \mathscr{H}^{2 \cdot \beta}|_{\Phi_1^{-1}(a)})=0.$$

(see for example [15] 5.10).

Moreover we may assume $\mathscr{H}^{\beta}$ to be very ample, and hence $\Phi_1^{-1}(a)$ is contained in V_w. The easy addition formula (see for example [15] 6.12) gives

$$\kappa(V_w) \leqslant \dim(\Phi_1(V_w)) + \kappa(\Phi_1^{-1}(a)) = \dim(\Phi_1(V_w)).$$

On the other hand the restriction of the linear system to V_w is a subsystem of $H^0(V_w, \omega_{V_w}^{2\cdot k\cdot\beta})$. Hence

$$\kappa(V_w) = \dim(\Phi_1(V_w)) = \dim(X) - \dim(W).$$

Corollary 7.2. *Let $f: V \to W$ be a fibre space, fulfilling the assumptions of* (6.2). *Then there exists a divisor B and some $\eta > 0$ such that* codim $(f(B)) \geqslant 2$ *and such that the linear system $H^0(V, \omega_{V/W}^{\eta} \otimes \mathcal{O}_V(\eta \cdot B))$ defines a rational map Φ_1 from V to a variety X of dimension $\kappa(V_w) + \dim(W)$. Moreover there exists a rational map $p: X \to W$, such that $f = p \cdot \Phi_1$.*

Proof. From (6.2) we know, that there exists an ample invertible sheaf $\mathscr{H}$ on W and some divisor B', such that codim $(f(B')) \geq 2$ and such that $f^*\mathscr{H}$ is contained in $\omega_{V/W}^{\beta} \otimes \mathcal{O}_V(B')$ for some $\beta > 0$. Now we can argue like in the second half of the proof of (7.1), or we can apply (7.1), in order to see that the sub-system $H^0(V, \omega_{V/W}^{\eta} \otimes f^*\mathscr{H}^{\eta} \otimes \mathcal{O}_V(\eta \cdot B))$ of $H^0(V, \omega_{V/W}^{\beta\cdot\eta+\eta} \otimes \mathcal{O}_V(\eta \cdot B + \eta \cdot B'))$ has already enough sections.

In order to see that the "mistake" we made, allowing the divisor B to occur in the statements (7.1) and (7.2) is not too important, we need

Lemma 7.3. *Let $g: T \to W$ be a morphism of non-singular varieties. Then there exists a birational morphism of non-singular varieties $\tau: W' \to W$ and a desingularization T' of $T \times_W W'$, such that the induced morphism $g': T' \to W'$ has the following property: Let B' be any divisor of T' such that* codim $(g'(B')) \geqslant 2$. *Then B' lies in the exceptional locus of $\tau': T' \to T$.*

Proof. From [14] we have the existence of a blowing up W' such that the component Z of $T \times_W W'$, birational to T, is equidimensional over W'. Take T' to be any desingularization of Z.

Lemma 7.4. *Let $f: V \to W$ be a fibre space, fulfilling the assumptions made in theorem* II. *Let $\mathscr{M}$ be an invertible sheaf on W such that $\kappa(W, \mathscr{M}) \geqslant 0$. Then there exists a divisor B, such that* codim $(f(B)) \geqslant 2$ *and such that*

$$\kappa(V, \omega_{V/W} \otimes \mathcal{O}_V(B) \otimes f^*\mathscr{M}) \geqslant \kappa(V_w) + \operatorname{Max}\{\operatorname{Var}(f), \kappa(W, \mathscr{M})\}.$$

Before we prove this lemma, we show that theorem II follows from (7.3) and (7.4) and that corollary IV follows from (7.3) and (7.1):

(7.5) Let $f\colon V \to W$ be any fibre space of regular projective varieties. Applying (7.3), we find a fibre space $f'\colon V' \to W'$ and birational morphisms $\tau\colon W' \to W$ and $\tau'\colon V' \to V$ such that every divisor B' of V' with codim $(f'(B')) \geqslant 2$ is contained in the exceptional locus of τ'. Then $\tau'_*(\omega^k_{V'} \otimes \mathcal{O}_{V'}(k \cdot B')) = \omega^k_V$ for all $k \geqslant 0$.

If the fibre space f, and hence f' too, fulfill the assumptions of theorem II, we apply (7.4) to f' and $\mathcal{M} = \omega_{W'}$, and get

$$\begin{aligned}\kappa(V) = \kappa(V', \omega_{V'} \otimes \mathcal{O}_{V'}(B')) &\geqslant \kappa(V'_w) + \mathrm{Max}\,\{\mathrm{Var}\,(f'), \kappa(W')\} \\ &= \kappa(V_w) + \mathrm{Max}\,\{\mathrm{Var}\,(f), \kappa(W)\}.\end{aligned}$$

If instead the fibre space f is like in corollary IV, that means, if $\kappa(W) = \dim(W)$, and if $\mathcal{H}'$ is any invertible ample sheaf on W', we know from (6.3) that there exists some $r > 0$, such that $\mathcal{H}'$ is contained in $\omega^r_{W'}$. From (7.1) we get

$$\begin{aligned}\kappa(V) = \kappa(V', \omega_{V'} \otimes \mathcal{O}_{V'}(B')) &\geqslant \kappa(V', \omega^r_{V'/W'} \otimes f'^* \mathcal{H}' \otimes \mathcal{O}_{V'}(r \cdot B')) \\ &\geqslant \kappa(V'_w) + \dim(W') = \kappa(V_w) + \kappa(W).\end{aligned}$$

Proof of (7.4). By the definition of the variation, there exists a fibre space $f''\colon V'' \to W''$ of non-singular projective varieties, such that $\overline{C(W'')} \subseteq \overline{C(W)}$, Var $(f'') = \mathrm{Var}\,(f) = \dim(W'')$ and

$$V_w \sim V''_w \times_{\mathrm{Spec}\,(\overline{C(W'')})} \mathrm{Spec}\,(\overline{C(W)}).$$

The birational map of the general fibres being defined over a finite extension of $C(W)$, we can find a non-singular projective variety W', generically finite over W, and $\rho\colon W' \to W''$. Let V' be a desingularization of $V \times_W W'$ and $V'' \times_{W''} W'$ at the same time. We have a diagram

$$\begin{array}{ccccc} V & \xleftarrow{\tau'} & V' & \xrightarrow{\rho'} & V'' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f''} \\ W & \xleftarrow{\tau} & W' & \xrightarrow{\rho} & W'' \end{array}$$

Applying (7.3) to the morphism $\rho\colon W' \to W''$, we may assume that every divisor of W', which has its support in the non flat locus of ρ, is blown down under τ. If we apply (6.1) to the morphism $\rho\colon W' \to W''$, we can find an open subvariety U'' of W'' and a non-singular covering $U''_3 \to U''$ such that codim $(W'' - U'') \geqslant 2$ and such that $W' \times_{W''} U''_3$ has a desingularization which is flat and semistable over U''_3. Taking non-singular compactifications, we get morphisms $\rho_3\colon W'_3 \to W''_3$ and $\tau_3\colon W'_3 \to W$ such that every divisor of W'_3 which has its support in the closed subvariety, where ρ_3 is either not flat or where the fibres are not semistable, is blown down under τ_3. Of course we may assume, that from the beginning we have choosen W'' and W' in this way. By (3.6) we may even assume (leaving out a subvariety of U'' of codimension bigger than or equal to two) that we have:

There exists an open subvariety U'' of W'' such that ρ: $\rho^{-1}(U'')=U' \to U''$ is flat with semistable fibres, $Z'=U'\times_{W''}V''$ has only rational singularities, and codim $(\tau(W'-U'))\geqslant 2$.

If we write $Z''=f''^{-1}(U'')$ we have $\omega_{Z'/U'}=\rho'^*\omega_{Z''/U''}$ and Z' has only rational Gorenstein singularities. From (3.2) or just applying "flat base change" we get

$$f'_*\omega^k_{V'/W'}|_{U'}=pr_{1*}\omega^k_{Z'/U'}=\rho^*f''_*\omega^k_{Z''/U''}=\rho^*f''_*\omega^k_{V''/W''}|_{U'}.$$

The assumptions made in theorem II allow to apply (7.2) to the fibre space f'': $V''\to W''$.

Hence we can find $\eta>0$ and sections in $H^0(Z', \omega^\eta_{Z'/U'})$ such that the image of the corresponding rational map is a variety X over W'' of dimension $\kappa(V_w)+\mathrm{Var}\,(f)$ and the function field $\boldsymbol{C}(X)$ contained in $\boldsymbol{C}(V'')$.

Moreover we may assume that $\tau^*\mathcal{M}^\eta$ has enough sections, such that the image Y of the corresponding rational map is of dimension $\kappa(W, \mathcal{M})$. Of course, $\boldsymbol{C}(Y)$ is contained in $\boldsymbol{C}(W')$. The intersection of $\boldsymbol{C}(W')$ and $\boldsymbol{C}(V'')$ is $\boldsymbol{C}(W'')$ and the intersection of $\boldsymbol{C}(X)$ and $\boldsymbol{C}(Y)$ must be contained in $\boldsymbol{C}(W'')$. It can have at most the transcendental degree Min $\{\mathrm{Var}\,(f), \kappa(W, \mathcal{M})\}$ over $\boldsymbol{C}$.

Hence we can find sections in $H^0(Z', \omega^\eta_{Z'/U'}\otimes pr_1^*\tau^*\mathcal{M}^\eta)$ which map Z' to a variety of dimension bigger than or equal to

$$s=\kappa(V_w)+\mathrm{Max}\,\{\mathrm{Var}\,(f), \kappa(W, \mathcal{M})\}.$$

By (3.2) we have an inclusion

$$pr_{1*}\omega^\eta_{Z'/U'}\longrightarrow\tau^*f_*\omega^\eta_{V/W}|_{U'},$$

and hence $\tau'^*(\omega^\eta_{V/W}\otimes f^*\mathcal{M}^\eta)|_{Z'}$ has even more sections. By construction of U' and Z' we know that codim $(f(\tau'(V'-Z')))\geqslant 2$, and hence we can find the divisor B such that

$$\kappa(V, \omega_{V/W}\otimes\mathcal{O}_V(B)\otimes f^*\mathcal{M})=\kappa(V', \tau'^*(\omega_{V/W}\otimes\mathcal{O}_V(B)\otimes f^*\mathcal{M}))$$

is bigger than s.

§ 8. The proof of theorem I

Let f: $V\to W$ be a fibre space of projective varieties such that $\kappa(V_w)=$

$\dim(V_w)=n-m\leqslant 2$. In order to proof theorem I, we just have to consider the case where $\dim(W)=\mathrm{Var}(f)$, and we have to show that

$$\kappa(W, \det(f_*\omega^k_{V/W}))=\dim(W),$$

for some $k\gg 0$. In [18], 4.4 we obtained:

Theorem 8.1. *Let $f: V\to W$ be as above. Then, for $k, s\gg 0$, we have*

$$\kappa(W, \det(f_*\omega^{k\cdot s}_{V/W})^{r(k)}\otimes\det(f_*\omega^k_{V/W})^{-s\cdot r(s\cdot k)})=\mathrm{Var}(f),$$

where $r(k)$ denotes the rank of $f_*\omega^k_{V/W}$.

Sketch of the proof. In order to construct sections of an invertible sheaf on W, we may restrict this sheaf to an open subvariety of W such that the complement is of codimension bigger than or equal to two. Hence f may be assumed to be flat and projective. For $k, s\gg 0$, one knows that the natural map $S^s(f_*\omega^k_{V/W})\to f_*\omega^{k\cdot s}_{V/W}$ is surjective over W_0, and we even may assume its image to be locally free. Let

$$\mathscr{L}=\mathrm{Im}\,(\bigwedge^{r(k\cdot s)}(S^s(f_*\omega^k_{V/W}))\longrightarrow\det(f_*\omega^{k\cdot s}_{V/W})).$$

Take U to be an open subvariety of W such that there exists a trivialization $\psi: f_*\omega^k_{V/W}|_U\to\mathcal{O}_U^{\oplus r(k)}$. This trivialization defines a morphism $g_\psi: U\to \boldsymbol{P}=\boldsymbol{P}(\bigwedge^{r(k\cdot s)}(S^s(\boldsymbol{C}^{r(k)})))$.

The group $SL(\boldsymbol{C}^{r(k)})$ operates on $\boldsymbol{P}$. The main result, allowing to construct projective coarse moduli-schemes, and which is proven in [12] and [4], says: Let $u\in U$ be a point, such that $f^{-1}(u)$ is non-singular. If k and s are big enough, there exists a polynomial P of degree p, invariant under the $SL(\boldsymbol{C}^{r(k)})$-action, such that $P(g_\psi(u))\neq 0$. Moreover these polynomials separate points u, v, with $f^{-1}(u)$ and $f^{-1}(v)$ non-singular, but not isomorphic. These polynomials define sections $P_\psi\in H^0(U, \mathscr{L}^p|_U)$. The invariance under the $SL(\boldsymbol{C}^{r(k)})$ just means that, if we take ψ' to be another trivialization, we have $P_\psi=P_{\psi'}\cdot\det(\psi'\cdot\psi^{-1})^{-a}$, and an easy calculation shows that $a=p\cdot s\cdot r(k\cdot s)\cdot r(k)^{-1}$.

Hence P defines a global section of a power of

$$\mathscr{L}^{r(k)}\otimes\det(f_*\omega^k_{V/W})^{-s\cdot r(k\cdot s)}.$$

Corollary 8.2. *$f: V\to W$ being as above, we have $\kappa(W, \det(f_*\omega^k_{V/W}))=\dim(W)$ for some $k\gg 0$.*

Proof. From (8.1) and (6.3) we know that, for $a, s, k\gg 0$, there exists an ample subsheaf $\mathscr{H}$ of

$$\det(f_*\omega^{k\cdot s}_{V/W})^{a\cdot r(k)}\otimes\det(f_*\omega^k_{V/W})^{-a\cdot s\cdot r(k\cdot s)}.$$

From theorem III and (1.4, 6) we know that $\det(f_*\omega^k_{V/W})$ is weakly positive. Hence $\det(f_*\omega^k_{V/W})^{a\cdot s\cdot r(k\cdot s)\cdot 2\cdot\beta}\otimes\mathscr{H}^\beta$ has nontrivial sections for some $\beta>0$. Putting everything together, we see that $\mathscr{H}^\beta$ is contained in $\det(f_*\omega^{k\cdot s}_{V/W})^{a\cdot r(k)\cdot 2\cdot\beta}$.

§ 9. Classification theory

As explained in [15] (see also [2] and [18]) every answer to $C_{n,m}$ has immediate applications to classification theory of higher dimensional varieties. By the results described in [9] and in this article, we know $C_{n,m}$ whenever $n\leqslant 4$. Hence we obtain (using [18] Satz V, for example):

Theorem 9.1. *Let V be a non-singular projective variety,* $\dim(V)\leqslant 4$ and $\kappa(V)=-\infty$. *Let $f\colon V\to W$ be the Stein-factorization of the Albanese map, W' a desingularization of W and V' a desingularization of $V\times_W W'$. Then the induced fibre space $f'\colon V'\to W'$ has a general fibre V'_w such that $\kappa(V'_w)=-\infty$, $\kappa(W')\geqslant 0$ and $q(W')=q(V)$.*

We also can apply $C_{n,m}$ to the Albanese map α of a variety V with $\kappa(V)=0$ and $\dim(V)\leqslant 4$, in order to show that α is a fibre space. However, Y. Kawamata obtained in [6] (using a special case of corollary IV) already:

Theorem 9.2 ([6], Theorem I). *Let V be a non-singular projective variety, $\kappa(V)=0$. Then the Albanese map $\alpha\colon V\to A(V)$ is a fibre space.*

Knowing $C_{n,1}$, $C_{n,n-1}$ and $C_{n,n-2}$, we get:

Corollary 9.3. *Let V be a non-singular projective variety, $\kappa(V)=0$. Assume that $q(V)=1$ or $q(V)\geqslant\dim(V)-2$. Then the general fibre F of the Albanese map is of Kodaira dimension zero.*

Knowing $C^+_{n,n-1}$ and $C^+_{n,n-2}$ to be true if $\kappa(V_w)=0$, we can even say:

Corollary 9.4. *Let V be a non-singular projective variety, $\kappa(V)=0$ and $q(V)\geqslant\dim(V)-2$. Then we can find V', birationally equivalent to V such that the Albanese map $\alpha'\colon V'\to A(V)$ is an étale fibre bundle, whose fibre F is a curve or a minimal surface of Kodaira dimension zero. Especially there exists $k>0$ such that $\omega^k_{V'}=\mathcal{O}_{V'}$.*

In fact, the general fibre of α has a minimal model, and we may apply Satz V of [18]. For the convenience of the reader we give a simplified proof.

Lemma 9.5. *Let V be a non-singular projective variety, $\kappa(V)=0$, A*

an abelian variety and f: $V \to A$ *a fibre space. Assume one of the following two conditions*:

i) Var $(f)=0$ *and the non-singular fibres of* f *have a minimal model* F.

ii) *There exists an open subvariety* U *of* A *and an étale covering* U' *of* U, *such that* $U' \times_A V \cong U' \times F$ *for some variety* F.

Then there exists a variety V', *birational to* V, *such that the induced map* f': $V' \to A$ *is an étale fibre bundle with fibre* F.

Remark 9.6. 1) From the condition ii) we know that Var $(f)=0$. 2) From (9.5) we get $\kappa(F)=0$. However, the definition of the variation and (3.2) give $C_{n,m}$ for all fibre spaces f: $V \to W$ with Var $(f)=0$.

Remark 9.7. In the proof of Lemma 2.9 in [18] one step is left out. One has to replace "Ist I_0 eine Zusammenhangskomponente von I, so$\cdots$" in the third sentence by: "Die Steinfaktorisierung eines glatten Morphismus ist étale. Indem wir Z durch eine étale Überlagerung ersetzen, dürfen wir annehmen, daß eine Zusammenhangskomponente I_0 von I existiert, so daß $I_0 \to Z$ zusammenhängende Fasern hat. Dann$\cdots$"

Proof of (9.5). Under the assumption ii) let W' be a non-singular compactification of U'. In the case i) we know by definition of Var (f) that there exists a non-singular variety W' and a generically finite morphism τ: $W' \to A$ such that $F \times W'$ is birational to $V \times_A W'$ for a minimal model F. In both cases we may assume that $C(W')$ is Galois over $C(A)$ and that the Galois group G acts biregularly on W'. Every $\sigma \in G$ defines an isomorphism $id \times \sigma$ on $F \times W'$ and an isomorphism $id \times_A \sigma$ on $V \times_A W'$. This second isomorphism induces a birational map of $F \times W'$ which we denote by σ'.

Claim 9.8. σ' is an isomorphism.

Proof. We only have to show that $\sigma''=(id \times \sigma)^{-1} \cdot \sigma'$ is an isomorphism. In the case ii) this is true over the open subvariety U' constructed above, and in the case i) we can find such U', since F is minimal. From the "easy addition formula" (see [15] 6.12) we know that $\kappa(F) \geqslant 0$ and hence the connected components of Aut (F) are isomorphic to abelian varieties. σ'' defines a rational map s_σ from W' into Aut $(F) \times W'$, such that $pr_2 \cdot s_\sigma = id_{W'}$. We know that we can find a blowing up α: $W'' \to W'$ such that $s_\sigma \cdot \alpha$ is a morphism. However, the fibres of α are unions of projective spaces, and any morphism from a projective space to an abelian variety is constant. Hence s_σ is regular itself.

Let B be a prime-divisor of A and B' one component of the proper transform of B in W'. Let $e(B')$ and $e(F\times B')$ denote the order of ramification over A and $(F\times W')/G$ respectively. We write $d(B')=e(B')-e(F\times B')$.

Claim 9.9. $d(B')=0$ for all prime-divisors B' of W', such that $\tau(B')$ is a divisor.

Proof. If $d(B')>0$ for a prime-divisor B', then $d(B'_i)>0$ for all components B'_i of the proper transform of $B=\tau(B')$ in W'. The definition of $d(\ \ )$ makes sense, even if W' is not Galois over A. In order to find a contradiction of

(*) $d(B'_i)>0$ *for all components of the proper transform in* W' *of a prime-divisor* B *of* A,

we may replace W' by any non-singular variety, generically finite over W'.

Using (2.5) we may therefore assume that there exists a regular blowing up W of A, such that the non-singular variety W' is finite over W. Let T' be a desingularization of $V\times_A W'$, dominating $F\times W'$. Blowing up V, if necessary, we may assume that the discriminant $\Delta(T'/V)$ is a normal crossing divisor, and that we have the following diagram

$$\begin{array}{ccccccc} F\times W' & \xleftarrow{\ \gamma\ } & T' & \xrightarrow{\ \eta'\ } & V & \xrightarrow{\ id\ } & V \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ W' & \xleftarrow{\ id\ } & W' & \xrightarrow{\ \eta\ } & W & \xrightarrow{\ \rho\ } & A\,. \end{array}$$

Applying (3.3) and using the notation introduced there, we have for $a\gg 0$ and $k>0$ an inclusion

$$g'_*\Big(\omega_{T'/W'}^{k\cdot a+1}\otimes\eta'^*\mathcal{O}_V\Big(k\cdot\sum_{D\in I}\delta(D)\cdot D\Big)\Big)\otimes\eta^*\omega_W^{k\cdot a+1}\longrightarrow\eta^*g_*\omega_V^{k\cdot a+1}.$$

Let D' be the proper transform of $\bigcup_i F\times B'_i$ in T' and E any exceptional divisor of γ. By definition of $d(\ \ )$ and $\delta(\ \ )$ we get from (*) that D' is smaller than or equal to $\sum_{D\in I}\delta(D)\cdot D$. Moreover we may choose a and k such that $\omega_F^{k\cdot a+1}$ has a section, and such that $\mathcal{O}_{T'}(E)$ is contained in $\omega_{T'/W'}^{k\cdot a+1}$. If F is a positive divisor with support in the exceptional locus of ρ, $\mathcal{O}_W(F)$ is contained in ω_W^{β} for some $\beta\gg 0$. Putting everything together we find for any $\nu>0$ some a and k such that $\eta^*\rho^*\mathcal{O}_A(B)^{\nu}$ is contained in

$$g'_*\Big(\omega_{T'/W'}^{k\cdot a+1}\otimes\eta'^*\mathcal{O}_V\Big(k\cdot\sum_{D\in I}\delta(D)\cdot D\Big)\Big)\otimes\eta^*\omega_W^{k\cdot a+1}$$

and hence in $\eta^*g_*\omega_V^{k\cdot a+1}$. However B is a positive divisor on an abelian variety and $\mathcal{O}_A(B)^\nu$ has lots of sections for $\nu \gg 0$. This contradicts the assumption $\kappa(V)=0$.

From now on we assume again that W' is Galois over A and that we have choosen W' such that the degree of W' over A is as small as possible. Let H be the subgroup of G, generated by the ramification groups $I(B')$ of prime-divisors B' of W', such that $\tau(B')$ is an divisor. (9.5) follows from:

Claim 9.10. $H=id_{W'}$.

In fact, W' must be étale over an open subvariety U_0 of A with codim $(A-U_0)\geqslant 2$. Replacing W' by the normalization of A in $\mathbf{C}(W')$, we may assume W' to be étale over A and G to operate fixpoint free on $F\times W'$. If we take $V'=(F\times W')/G$ we get (9.5).

Proof of (9.10). Let $A(W')$ be the group of sections of Aut $(F)\times W'$ over W', Aut $(F)_0$ the connected component of zero in Aut (F) and $A_0(W')$ the sections of Aut $(F)_0\times W'$ over W'. The group H operates on $A(W')$ and $A_0(W')$: Let s^η denote the conjugation of the section s by $\eta \in H$. The sections s_σ out of $A(W')$, we introduced in the proof of (9.8), fulfill: $s_{\sigma\cdot\eta}=s_\sigma^\eta\cdot s_\eta$ for σ, $\eta \in H$. This means that (s_σ) defines an 1-cocycle and hence an element ξ of $H^1(H, A(W'))$. If $\sigma \in I(B')$, for some prime-divisor B', the automorphism σ' operates trivially on $F\times B'$ as we have seen in (9.9). Especially $s_\sigma \in A_0(W')$. Hence the whole cocycle has values in $A_0(W')$. Since Aut $(F)_0$ is an abelian variety, ξ must be of finite order, lets say of order k. If $A_0^k(W')$ denotes the points of order k in $A_0(W')$, ξ is given by an 1-cocycle (s'_σ) with values in $A_0^k(W')$. The two cocycles differ by a coboundary, that means, there exists $a \in A_0(W')$ such that $s'_\sigma= a^\sigma\cdot s_\sigma\cdot a^{-1}$. Changing the trivialization by a, we may assume that $s'_\sigma=s_\sigma$. If $\sigma \in I(B')$, again we know that $s_\sigma|_{B'}$ is the unit of the finite group $A_0^k(W')$ and hence it is trivial everywhere.

Since H is generated by the ramification groups, s_σ is trivial for all $\sigma \in H$ and hence $\sigma'=(id\times\sigma)$. In this case we have $(F\times W')/H=F\times (W'/H)$, and $V\times_A (W'/H)$ is birational to $F\times(W'/H)$. Moreover in the situation considered in (9.5) ii) $V\times_A(U'/H)$ is isomorphic to $F\times(U'/H)$. This contradicts the minimality of the degree of W' over A.

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