BICANONICAL AND ADJOINT LINEAR SYSTEMS ON SURFACES OF GENERAL TYPE

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Abstract. This note contains a new proof of a theorem of Gang Xiao saying that the bicanonical map of a surface S of general type is generically finite if and only if $p_2(S) > 2$. Such properties are also studied for adjoint linear systems $|K_S + L|$, where L is any divisor with $h^0(S, \mathcal{O}_S(L)) \geq 2$.

Introduction

Let S be a complex minimal surface of general type. Since

$$K_S^2 + 1 - q(S) + p_g(S) \geq 2$$

the Riemann-Roch Theorem implies that $p_2(S) \geq 2$. If $p_2(S) = 2$, the bicanonical map is composite with a pencil. It is the aim of this note, to give a short prove of the Theorem of G. Xiao, stating the converse.

Theorem 0.1 (Theorem 1 of [13]). *Let S be a minimal projective surface of general type. Then the bicanonical map of S is generically finite if and only if $p_2(S) > 2$.*

The proof of G. Xiao depends on his study of genus 2 fibration over curves and on Horikawa's classification of the possible degenerations. As pointed out by F. Catanese, there might be a simpler proof relying on the base point freeness of $|2K_S|$ for surfaces with $p_g(S) = 0$ and $2 \leq K_S^2 \leq 4$.

We choose a different approach, and we will deduce the Theorem from vanishing theorems for $\mathbb{Q}$-divisors, using in addition just some well known properties of surfaces of general type.

In the last section, we show that adjoint linear systems $|K_S + L|$ on surfaces of general type can only be composite with a pencil of curves, if L is a divisor with $h^0(S, \mathcal{O}_S(L)) \leq 2$. We discuss some examples, showing that this bound is sharp.

For a linear system $|L|$ on a surface S the induced rational map is denoted by φ_L. The linear system is composite with a pencil of curves,

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if $\dim \varphi_L(S) = 1$. The symbol $\equiv$ stands for the numerical equivalence of divisors, whereas $\sim$ denotes the linear equivalence. K_S denotes the canonical divisor, and if $f : S \to B$ is a surjective morphism, $K_{S/B} = K_S - f^*K_B$. The base field is $\mathbb{C}$.

1. **Proof of Theorem 0.1**

Recall the Kawamata-Viehweg vanishing theorem (see [9] or [12]).

Theorem 1.1 (see [7], p. 49)**.** *Let X be a smooth projective variety and L a divisor on X. Assume that D is an effective $\mathbb{Q}$-divisor with normal crossing supports such that one of the following holds true:*

(i) *$L - D$ is nef and big.*
(ii) *$L - D$ is nef and $\kappa(L - \llcorner D \lrcorner) = \dim X$.*

Then $H^i(X, \mathcal{O}_S(K_X + L - \llcorner D \lrcorner)) = 0$ for all $i > 0$.

Remark 1.2. As well known, on surfaces, one may apply the vanishing theorems without the assumption "normal crossings". In fact, if $\tau : X' \to X$ is a blowing up, with τ^*D a normal crossing divisor, then

$$R^i\tau_*\mathcal{O}_{X'}(K_{X'} + \tau^*L - \llcorner D' \lrcorner) = 0, \quad \text{for } i > 0,$$

and for $i = 0$ it coincides with $\mathcal{O}_X(K_X + L - \llcorner D \lrcorner)$ in codimension one. If X is a surface, for $i > 0$

$$0 = H^i(X', \mathcal{O}_{X'}(K_{X'} + \tau^*L - \llcorner D' \lrcorner))$$
$$= H^i(X, \tau_*\mathcal{O}_{X'}(K_{X'} + \tau^*L - \llcorner D' \lrcorner)) = H^i(X, \mathcal{O}_X(K_X + L - \llcorner D \lrcorner)).$$

We will also use the following simple observation, due to Xiao (see [13], Lemme 8).

Lemma 1.3. *Let S be a minimal surface of general type with $q(S) = 0$ and $K_S^2 \le 2$. Let θ be a non-trivial invertible torsion sheaf on S. Then $H^1(S, \theta) = 0$.*

Proof. There exists an étale cover $\tau : T \to S$ with $\tau^*\theta = \mathcal{O}_T$, hence θ is a direct factor of $\tau_*\mathcal{O}_T$. Since $K_S^2 \le 2 \le 2\chi(\mathcal{O}_S)$ Corollary 5.8 of [2] implies that the fundamental group of S is finite, hence the one of T as well. Then both $H^1(T, \mathcal{O}_T)$ and $H^1(S, \theta)$ are zero. $\square$

As a first step, let us reduce the proof of Theorem 0.1 to the case $p_2(S) = 3$.

Proposition 1.4. *Let S be a minimal smooth surface of general type. Then*

(1) *the bicanonical map of S is generically finite if $p_2(S) \ge 4$;*
(2) *the linear system $|2K_S|$ is not composite with an irrational pencil of curves for $p_2(S) = 3$.*

Proof. Suppose for some S with $p_2(S) \geq 2$ the linear system $|2K_S|$ is composite with a pencil, or for $p_2(S) = 3$ with an irrational pencil. Let $\pi : S' \longrightarrow S$ be any birational modification such that $|2\pi^*(K_S)|$ defines a morphism ϕ'_2 and let B'_2 be its image. Consider the Stein factorization

$$\phi'_2 : S' \xrightarrow{f} B_2 \longrightarrow B'_2.$$

For some fibres C_i of f and for a general fibre C. We may write

$$\pi^*(2K_S) \sim \sum_{i=1}^{a} C_i + Z_2 \equiv a \cdot C + Z_2,$$

where Z_2 is the fixed part. By assumption on the the smooth curve B_2 the sheaf $f_*(\mathcal{O}_{S'}(2K_{S'}))$ is invertible of degree a and the space of its global sections is of dimension ≥ 4, or of dimension ≥ 3 if $B_2 \neq \mathbb{P}^1$. In both cases one finds $a \geq 3$.

Set $G = \pi^*(K_S) - \frac{1}{a}Z_2$. We have $K_{S'} + \ulcorner G \urcorner \leq K_{S'} + \pi^*(K_S)$ and

$$G - C \equiv \frac{a-2}{a}\pi^*(K_S)$$

is nef and big. Thus 1.1 implies that

$$|K_{S'} + \ulcorner G \urcorner||_C = |K_C + D|,$$

for some divisor $D = \ulcorner G \urcorner|_C$ of positive degree on the curve C. The genus of C can not be zero or one, hence $h^0(C, K_C + D) \geq 2$. This implies that the morphism given by $|K_{S'} + \pi^*(K_S)|$ can not factor through f, a contradiction. □

Proposition 1.5. *Let S be a smooth minimal surface of general type with $p_2(S) = 3$. Assume that $|2K_S|$ is composite with a pencil of curves. Then*

(i) $K_S^2 = 2$ *and* $p_g(S) = q(S) \leq 1$.
(ii) $|2K_S|$ *is composite with a rational pencil of curves of genus* 2.
(iii) $|2K_S|$ *defines a morphism on S, i.e. the movable part of $|2K_S|$ is base point free.*
(iv) *Let E be a component of the fixed part of $|2K_S|$. Then $E \cdot K_S = 0$ and E is a (-2) curve.*

Proof. Since $p_2(S) = 3$ one has $p_g(S) \leq 2$. The Riemann-Roch theorem and the positivity of the Euler-Poincaré characteristic imply that

$$0 < K_S^2 = 3 - 1 + q(S) - p_g(S) \leq 2.$$

By [3], Theorems 11 and 12, $q(S) = 0$ if either $K_S^2 = 1$ or if $K_S^2 = p_g(S) = 2$. Hence in order to prove (i), one just has to exclude the case $K_S^2 = 1$, $p_g(S) = 1$ and $q(S) = 0$.

Since $p_2(S) = 3$ Proposition 1.4 implies that $|2K_S|$ is composite with a rational pencil of curves. Let $\pi : S' \to S$ be again a minimal birational modification such that $|2K_{S'}|$ defines a morphism $f : S' \to \mathbb{P}^1$. The sheaf $f_*\mathcal{O}_S(2K_S)$ is invertible of degree two, hence we may write

$$2K_{S'} \sim 2C' + Z'_2$$

for a general fibre C' of f. Set $C = \pi_*(C')$ and $Z_2 = \pi_*(Z'_2)$, then $2K_S \sim 2C + Z_2$.

If $K_S^2 = 1$ one has $C^2 \le K_S \cdot C \le 1$. Since the genus of C is at least two, $K_S \cdot C + C^2 \ge 2$, which implies $K_S \cdot C = C^2 = 1$ and $K_S^2 \cdot C^2 = (K_S \cdot C)^2$. By the Index Theorem $K_S \equiv C$. As shown in [3] or [4] the condition $K_S^2 = p_g(S) = 1$ implies that on S numerical equivalence coincides with linear equivalence. Hence $K_S \sim C$, a contradiction since $p_g(S) \ne h^0(S, \mathcal{O}_S(C)) = 2$.

Up to now, we obtained (i). For (iii) suppose that π can not be chosen to be an isomorphism, hence $C^2 > 0$. Then $2 = K_S^2 \ge K_S \cdot C \ge C^2$. On the other hand, the index theorem gives

$$K_S^2 \cdot C^2 \le (K_S \cdot C)^2.$$

Since $K_S \cdot C + C^2$ is even, one finds $K_S^2 = K_S \cdot C = C^2 = 2$, hence $K_S \equiv C$, and $Z_2 = 0$.

Assume $p_g(S) = 1$. Let $D \in |K_S|$ be the unique effective divisor. Then there are two fibers C'_1 and C'_2 of f such that, for $C_i = \pi(C'_i)$ one has $2D = C_1 + C_2$. If $C_1 \ne C_2$, then the C_i are both 2-divisible for $i = 1,\ 2$ and $D \equiv 2P$, where P is a divisor. This implies $D^2 \ge 4$, a contradiction. If $C_1 = C_2$, then $D = C_1$ and thus $h^0(S, \mathcal{O}_S(D)) = 2$, again a contradiction.

Assume $p_g(S) = 0$, hence $q(S) = 0$. Then the sheaf

$$\theta = \mathcal{O}_S(K_S - C)$$

is a non-trivial invertible torsion sheaf on S. The Riemann-Roch Theorem implies $h^1(S, \theta) = 1$, contradicting Lemma 1.3.

So (iii) holds true and we may choose $S' = S$. Since for a general fibre C of f one has $g(C) \ge 2$ and $K_S \cdot C \le K_S^2 = 2$, one finds $g(C) = 2$, and $Z_2 \cdot K_S = 0$. □

Proof of Theorem 0.1. By 1.4 and 1.5 it remains to show, that there can not exist a surface with:

1.6. *S is a minimal surface of general type with $p_2(S) = 3$, with $K_S^2 = 2$ and with $p_g(S) = q(S) \le 1$. The bicanonical map is a genus two fibration $f : S \longrightarrow \mathbb{P}^1$.*

Writing again Z_2 for the fixed part of $|2K_S|$ and C for a general fibre of f, one has $2K_S \sim 2C + Z_2$. Let $Z_v \le Z_2$ be the largest effective divisor contained in fibres of f, and $Z_h = Z_2 - Z_v$ the horizontal part of Z_2. In particular $C \cdot K_S = C \cdot Z_h = 4$. We will study step by step the divisors Z_v and Z_h.

Claim 1.7. The maximal multiplicity a in Z_2 of an irreducible component is two.

Proof. Suppose $a > 2$, and denote by Γ the total sum of reduced components of multiplicity a in Z_2. We may write

$$\Gamma = \Gamma_1 + \cdots + \Gamma_s,$$

where the Γ_i are connected pairwise disjoint. 1.5, (iv), implies that each Γ_i is a connected tree of rational curves, thus 1-connected. We may replace $2C$ by the sum of two different general fibres of f, say C_1 and C_2. Then

$$K_S - \frac{1}{a}C_1 - \frac{1}{a}C_2 - \frac{1}{a}Z_2$$

is nef and big, and 1.1 implies that

$$H^1(2K_S - \Gamma_1 - \cdots - \Gamma_s) = H^1(2K_S + \ulcorner -\tfrac{1}{a}C_1 - \tfrac{1}{a}C_2 - \tfrac{1}{a}Z_2 \urcorner) = 0.$$

Thus we have a surjective map

$$H^0(S, 2K_S) \longrightarrow H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}) \oplus \cdots \oplus H^0(\Gamma_s, \mathcal{O}_{\Gamma_s}) = \bigoplus^s \mathbb{C},$$

contradicting $\Gamma \le Z_2$. □

Claim 1.8. The horizontal part Z_h of Z_2 is either reduced, or $Z_h = 2H$ for an irreducible (-2) curve H.

Proof. If not, there is an irreducible curve H_1 with $Z_h - 2H_1 \neq 0$. By 1.7 the multiplicities occurring in Z_2 are at most 2, and $Z_h \cdot C = 4$ implies that either $Z_h - 2H_1 = 2H_2$ for a reduced (-2)-curve H_2, or $Z_h - 2H_1$ is reduced. Let us write $H_2 = 0$ in the second case, such that in both cases

$$\tfrac{1}{2}Z_h - \llcorner \tfrac{1}{2}Z_h \lrcorner + H_2 \neq 0.$$

Consider the effective $\mathbb{Q}$-divisor $G = \frac{1}{2}Z_2 - H_2$. Obviously

$$K_S - G \equiv C + H_2$$

is nef. On the other hand,

$$2(K_S - \llcorner G \lrcorner) \geq 2C + Z_h - 2\ulcorner \tfrac{1}{2}Z_h \urcorner + 2H_2$$

is big. By the vanishing theorem 1.1, we have

$$H^1(S, 2K_S - \llcorner G \lrcorner) = 0.$$

The divisor $\llcorner G \lrcorner \geq H_1$ is again the sum over reduced connected trees Γ_i of (-2)-curves, say

$$\ulcorner G \urcorner = \Gamma_1 + \cdots + \Gamma_s.$$

Thus we have a surjective map

$$H^0(S, 2K_S) \longrightarrow H^0(\Gamma_1, \mathcal{O}_{\Gamma_1}) \oplus \cdots \oplus H^0(\Gamma_s, \mathcal{O}_{\Gamma_s}) = \bigoplus^s \mathbb{C},$$

contradicting $0 < 2\llcorner G \lrcorner \leq 2G \leq Z_2$. □

Claim 1.9. Z_h is either the sum of 4 disjoint sections of f or twice an irreducible curve H. Moreover $Z_v = 0$ in both cases.

Proof. If $Z_h = 2H$ for an irreducible curve H, one has $Z_h^2 = -8$. Otherwise 1.8 only leaves the possibility $Z_h = H_1 + \cdots + H_t$, for $t \le 4$. In this case, $Z_h^2 \ge -2t \ge -8$, and $Z_h^2 = -8$ if and only if $t = 4$ and $H_i \cdot H_j = 0$ for $i \ne j$. The inequality

$$0 = 2K_S \cdot Z_h = 8 + Z_v \cdot Z_h + Z_h^2, \tag{1.9.1}$$

implies $Z_h^2 \le -8$, and we obtain the first part of 1.9.

In both cases (1.9.1) is an equality, hence $Z_v \cdot Z_h = 0$. Finally the equality

$$0 = 2K_S \cdot Z_v = 2C \cdot Z_v + Z_v^2 + Z_v \cdot Z_h$$

implies $Z_v^2 = 0$ and by the Index theorem $Z_v \equiv 0$. Since $Z_v \ge 0$ one finds $Z_v = 0$. □

Claim 1.10. In 1.9 the case $Z_h = 2H$ does not occur, and

$$Z_h = H_1 + \cdots + H_4$$

implies $p_g(S) = q(S) = 0$.

Proof. Assume that $p_g(S) = 1$, and let D denote the effective canonical divisor. Then $2D = C_1 + C_2 + Z_h$ for fibres C_i of f. First of all this implies that the multiplicity of Z_h is divisible by 2, hence $Z_h = 2H$, and $C_1 + C_2$ must be divisible by 2, as well. Since for any divisor B the intersection number $B^2 + B \cdot K_S$ must be even, and since $C_i \cdot K_S = 2$ the fibres C_i can not be divisible by two. Hence $C_1 = C_2$ and $D = C_1 + H$, a contradiction since $p_g(S) < h^0(S, \mathcal{O}_S(D)) = 2$.

If $p_g(S) = 0$, then by 1.5 (i) $q(S) = 0$. In case $Z_h = 2H$ one finds $K_S \equiv C + H$ and $\theta = \mathcal{O}_S(K_S - C - H)$ is a 2-torsion sheaf. The Riemann-Roch Theorem implies that $h^1(S, \theta) = 1$, contradicting 1.3. □

It remains to exclude the existence of a surface with:

1.11. *S is a minimal surface of general type, $f : S \to \mathbb{P}^1$ the bicanonical map and for a fibre C of f and for pairwise disjoint (-2) curves $H_1, \ldots, H_4$*

$$2K_{S/\mathbb{P}^1} = 6C + H_1 + \cdots + H_4.$$

Let us write $H = H_1 + \cdots + H_4$. On some open dense subset $U \subset \mathbb{P}^1$ there is a natural involution ι on $f^{-1}(U)$ with quotient $f^{-1}(U) \to \mathbb{P}^1 \times U$. Since S is minimal ι extends to an involution on S, denoted again by ι. The equality

$$0 = 2K_S \cdot \iota(H_i) = 2C \cdot \iota(H_i) + (H_1 + H_2 + H_3 + H_4) \cdot \iota(H_i)$$

implies that $\iota(H_i) \in \{H_1, H_2, H_3, H_4\}$, hence $\iota(H) = H$. For U small enough, each effective bicanonical divisor of $f^{-1}(U)$ is the pullback of a divisor on $\mathbb{P}^1 \times U$, hence none of the H_i can be fixed under ι. Renumbering we may assume that $\iota(H_1) = H_2$ and $\iota(H_3) = H_4$.

Let E be any (-2)-curve on S, not equal to one of the H_i. The equality

$$0 = 2K_S \cdot E = 2C \cdot E + (H_1 + H_2 + H_3 + H_4) \cdot E$$

implies that $H_i \cdot E = 0$ for all i. Hence E is a component of a fibre not meeting the H_i.

On the other hand let E be any component of a fibre of f. If E does not meet H, then $E \cdot K_S = 0$, hence E is a (-2)-curve.

The morphism $\delta : S \to S'$ to the relative minimal model contracts exactly the (-2) curves of the fibres. Hence all fibres of $f' : S' \to \mathbb{P}^1$ are reduced and all of their components E' meet $H' = \delta(H)$. Moreover the intersection number $E \cdot K_S = E \cdot H$ on S is even. So the reducible fibres of f' have at most two components E'_1 and E'_2, both meeting H' in two points. The components E'_1 and E'_2 need not be Cartier divisors. However $E'_1 + E'_2$ is Cartier, as well as the images H'_i of the H_i.

We write ι' for the automorphism of S' induced by ι. Since $p_g(S) = q(S) = 0$ the direct image $f_*\mathcal{O}_S(K_{S/\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Consider the restriction map

$$\eta : {f'}_*\mathcal{O}_{S'}(K_{S'/\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \longrightarrow \mathcal{O}_{H'_1}(2) = \mathcal{O}_{H_i}(K_{S'/\mathbb{P}^1} \cdot H'_1).$$

Since $\mathcal{O}_C(K_C)$ is generated by global sections η is non-zero, hence its kernel is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(\epsilon)$, for $\epsilon = 0$ or 1. Let σ' be a general section of $\mathrm{Ker}(\eta)$, and let σ be the induced section of $\mathcal{O}_{S'}(K_{S'/\mathbb{P}^1})$. By construction H'_1 lies in the zero-locus B of σ. For some open dense $U \subset \mathbb{P}^1$ the divisor $B|_{f'^{-1}(U)}$ is invariant under ι'. Then the section σ is zero on $H'_1 + H'_2$. Altogether we found an effective Cartier divisor D' with

$$\epsilon \cdot C + H'_1 + H'_2 + D' \sim K_{S'/\mathbb{P}^1}.$$

By construction D' does not contain a whole fibre. So it is concentrated in the reducible fibres of f'. Let ${f'}^{-1}(p) = E'_1 + E'_2$ be one of such fibres, and let $\alpha_1 \cdot E'_1 + \alpha_2 \cdot E'_2$ be the part of D' concentrated in ${f'}^{-1}(p)$. Then one of the α_i must be zero, say α_1, hence $\alpha_2 > 0$.

The divisor $\iota'^*(\alpha_2 \cdot E'_2)$ is the part of $\iota'^*(D')$ lying in ${f'}^{-1}(p)$. If $\iota'^*(E'_2) = E'_1$

$$\alpha_2 \cdot E'_2 - \iota'^*(\alpha_2 \cdot E'_2) = \alpha_2 \cdot E'_2 - \alpha_2 \cdot E'_1$$

is the part concentrated in ${f'}^{-1}(p)$ of a divisor, linear equivalent to zero. Then the same holds true for

$$\alpha_2 \cdot \delta^*(E'_2) - \alpha_2 \cdot \delta^*(E'_1).$$

Obviously this is not possible, hence E'_i is invariant under ι'.

We may assume that $E'_1 \cap H'_1 \neq \emptyset$. The component E'_1 meets exactly one of the other H'_i, and being invariant under ι', this can only be H'_2.

Write $D = \delta^*(D')$ and E_i for the proper transform of E'_i. If D' contains E'_2, it can not contain E'_1, hence D does not contain E_1. Since

$$\epsilon \cdot C + H_1 + H_2 + D \sim K_{S/\mathbb{P}^1}$$

one finds $1 = E_1 \cdot K_{S/\mathbb{P}^1} \geq E_1 \cdot (H_1 + H_2) = 2$, obviously a contradiction. So D' only contains components of reducible fibres meeting H'_1 and H'_2 but neither H'_3 nor H'_4. So $D \cdot H_3 = 0$ and

$$H_3 \cdot (\epsilon \cdot C + H_1 + H_2 + D) = \epsilon < H_3 \cdot K_{S/\mathbb{P}^1} = 2,$$

a contradiction. □

2. **Adjoint linear systems**

Let S be a surface of general type, not necessary minimal, and let L be a divisor on S. There are few criteria known, which imply that φ_{K_S+L} is generically finite, though the linear system $|K_S+L|$ quite well understood (see for instance [11] and [5]).

By [14], for a surface S of general type with $q(S) \geq 3$ the map φ_{K_S} is generically finite, hence the same holds true for φ_{K_S+L} whenever $L \geq 0$. We will prove here

Proposition 2.1. *Let S be a smooth projective surface of general type and let L be an effective divisor on S with $h^0(S, \mathcal{O}_S(L)) > 2$. Then φ_{K_S+L} is generically finite.*

If $h^0(S, \mathcal{O}_S(L)) = 2$ obviously $|L|$ is composite with a pencil. The method used to prove 2.1 will also show:

Addendum 2.2. *Assume in 2.1 that $h^0(S, \mathcal{O}_S(L)) = 2$. Then φ_{K_S+L} is generically finite, except possibly in one of the following cases:*

- (a) *$p_g(S) = 0$ and $|L|$ is composite with a rational pencil of hyperelliptic curves.*
- (b) *$0 < q(S) \leq 2$ and $|L|$ is composite with a rational pencil of curves of genus $g = q(S) + 1$.*

The next two examples shows that the exceptional cases 2.2, (a) and (b), really occur.

Example 2.3. In [13], p. 46 - 49, one finds an example of a surface S of general type with $p_g(S) = q(S) = 0$ and $K_S^2 = 2$, having a pencil $f : S \to \mathbb{P}^1$ of curves of genus 2. If C denotes a general fibre, then

$$H^0(S, \mathcal{O}_S(K_S + C)) = H^0(C, \mathcal{O}_C(K_C)) = \mathbb{C}^{\oplus 2},$$

and $|K_S + C|$ is composite with a rational pencil of genus 2 curves.

Example 2.4. Let C be a smooth curve of genus 2, and let θ be an invertible 2-torsion sheaf on C, with $\theta \neq \mathcal{O}_C$. For $T = \mathbb{P}^1 \times C$ let $p_1 : T \to \mathbb{P}^1$ and $p_2 : T \to C$ be the projections. For $a \geq 3$ consider

$$\delta = p_1^*(O(a)) \otimes p_2^*(\theta).$$

Since $\delta^2 \cong \mathcal{O}_T(D)$ for a non-singular divisor D, one obtains a smooth double cover $\pi : S \to T$ with

$$\pi_*\mathcal{O}_S(K_S) = \mathcal{O}_T(K_T) \oplus \mathcal{O}_T(K_T) \otimes \delta.$$

It is easy to see that S is a minimal surface of general type, and that $|K_S|$ is composite with a pencil of curves of genus 3. In fact φ_{K_S} coincides with $f = p_1 \circ \pi$. For a general fiber C of f, choose $L = C$. Then $h^0(S, \mathcal{O}_S(L)) = 2$, but $|K_S + L|$ is composite with the same pencil as $|K_S|$.

Note that f is an isotrivial family of curves of genus 3, that

$$f_*\mathcal{O}_S(K_S) = \mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2},$$

and that $q(S) = 2$.

In Examples 2.3 and 2.4 the divisor L is nef, but not big.

Question 2.5. Does there exists a minimal surfaces S of general type and a nef and big divisor L on S with $h^0(S, \mathcal{O}_S(L)) = 2$, for which $|K_S + L|$ is composite with a pencil of curves?

Such examples exist on surfaces S of smaller Kodaira dimension, or on surfaces S of general type for $h^0(S, \mathcal{O}_S(L)) = 1$:

Example 2.6. Let $f : S \to \mathbb{P}^1$ be a family of elliptic curves admitting a section G, and with S non-singular and projective. For a general fibre C of f choose $L_m = mF + G$. Then L_m is nef and big, whenever $m > \text{Max}\{0, -\frac{G^2}{2}\}$, and $h^0(S, \mathcal{O}_S(L_m)) = m + 1$. However $|K_S + L_m|$ is always composite with a pencil.

Example 2.7. Let S be a minimal surface of general type with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$. Denote by L a divisor numerically equivalent to K_S. Then $h^0(S, \mathcal{O}_S(L)) \leq 1$ and $h^0(S, \mathcal{O}_S(K_S + L)) = 2$. Thus $|K_S + L|$ is automatically composite with a rational pencil of curves. One may refer to [10] for a classification of such pairs (S, L).

Proof of 2.1 (and of 2.2). Replacing S by a blowing up, we may assume that the moving part of L has no fixed points, hence that φ_L is a morphism.

Let us first consider the case that $|L|$ is composite with a pencil of curves. Take the Stein factorization

$$g : S \xrightarrow{\ f\ } B \xrightarrow{\ \rho\ } \mathbb{P}(H^0(S, \mathcal{O}_S(L))), \tag{2.7.1}$$

so f is a pencil of curves of genus $g \geq 2$. As in the proof of 1.4 one easily sees that $h^0(S, \mathcal{O}(L)) > 2$ implies that $L \geq C_1 + C_2$ for two fibres C_i of f. The same holds true for $h^0(S, \mathcal{O}_S(L)) = 2$, if ρ is not an isomorphism. In both cases we may as well assume that $L = C_1 + C_2$.

As explained in [7], 7.18, Kollárs vanishing theorem implies that the locally free sheaf $f_*\mathcal{O}_S(K_{S/B})$ is numerically effective, and that $\mathcal{E} = f_*\mathcal{O}_S(K_S + C_1 + C_2)$ is generated by global sections. Hence the tautological sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the projective bundle $\mathbb{P}(\mathcal{E})$ is globally generated.

If the genus $g(B) > 0$, as a tensor product of a numerically effective vector bundle with an invertible sheaf of positive degree, $\mathcal{E}$ is ample.

If $B \cong \mathbb{P}^1$ the sheaf $\mathcal{E} = f_* \mathcal{O}_S(K_{S/B})$ is a direct sum of line bundles of non-negative degree, say $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_g$. If $q(S) = 0$, by the Leray spectral sequence $H^1(\mathbb{P}^1, f_* \mathcal{O}_S(K_S)) = 0$, hence $\nu_1 > 0$. If $q(S) \neq 0$, one has $p_g(S) > 0$, hence $\nu_g \geq 2$.

Altogether, in both cases the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is globally generated and big. φ_{K_S+L} factors like

$$S \xrightarrow{\varphi} \mathbb{P}(\mathcal{E}) \xrightarrow{\varphi'} \mathbb{P}^M, \tag{2.7.2}$$

where φ is the relative canonical map and φ' the rational map induced by global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Since the genus of the fibres of f is at least two, φ is generically finite. $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, as well as its restriction to the closure of the image of φ, are globally generated and big, hence φ_{K_S+L} is generically finite.

Before finishing the proof of 2.1 let us look to the case

$$h^0(S, \mathcal{O}_S(L)) = 2, \text{ and } B \xrightarrow{\cong} \mathbb{P}^1$$

in (2.7.1). Here we may assume that $L = C$ for a general fibre of $f : S \to \mathbb{P}^1$. Write again $f_* \mathcal{O}_S(K_{S/B})$ as a direct sum of line bundles of non-negative degrees $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_g$. If φ_{K_S+L} is composite with a pencil, [14] implies that $q(S) < 3$. Note that $\nu_i = 0$ for $i = 1, \ldots, q(S)$.

If $p_g(S) > 0$, one also knows that $\nu_g \geq 2$. Hence if $g > q(S) + 1$, the sheaf $f_* \mathcal{O}_S(K_S + C)$ contains a subbundle $\mathcal{E}$ of rank ≥ 2 which is globally generated and non trivial, i.e. not the direct sum of copies of $\mathcal{O}_{\mathbb{P}^1}$. For this bundle consider again the maps (2.7.2). The first one, φ, is fibrewise given by ≥ 2 independent sections of the canonical linear system, hence it is generically finite. Since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and its restriction to the image of φ are again generated by global sections and big, $\varphi' \circ \varphi$ is generically finite and one obtains 2.2, for $p_g(S) > 0$.

If $p_g(S) = 0$, hence $q(S) = 0$, then $\nu_1 = \cdots = \nu_g = 1$, and $\mathcal{E} = f_* \mathcal{O}_S(K_S + C)$ is trivial. Then $\mathbb{P}(\mathcal{E}) = \mathbb{P}^1 \times \mathbb{P}^{g-1}$ and in (2.7.2) φ is generically finite, whereas φ' is the projection to the second factor. The restriction of φ_{K_S+L} to a smooth fibre F coincides with $|K_F|$. So for F non hyperelliptic, the assumption that $|K_S + L|$ is composite with a pencil, implies that all smooth fibres F are isomorphic and that $(\varphi_L, \varphi_{K_S+L})$ is a birational map $S \to \mathbb{P}^1 \times F$, a contradiction.

To finish the proof of 2.1 it remains to consider the case that φ_L is generically finite. If $p_g(S) > 0$, the linear system $|L|$ is a subsystem of $|K_S + L|$, hence the latter can not be composite with a pencil of curves.

For $p_g(S) = q(S) = 0$, blowing up S if necessary, we assume that both, φ_{K_S+L} and φ_L are morphism, hence that the movable parts M of $K_S + L$ and L^0 of L have no fixed points. Replacing L by L^0 we may assume L to be big and globally generated.

Take the Stein factorization

$$\varphi_{K_S+L} : S \xrightarrow{h} B \longrightarrow \mathbb{P}(H^0(S, \mathcal{O}_S((K_S + L) - 1))).$$

If φ_{K_S+L} is not generically finite, h is a fibration onto a smooth curve B with general fibre C. One may write $M \sim \sum_{i=1}^{a} C_i$ for fibres C_i of h and for $a \geq h^0(S, \mathcal{O}_S(K_S + L)) - 1$. Noting that

$$h^0(S, \mathcal{O}_S(K_S + L)) = \frac{1}{2}L \cdot (K_S + L) + \chi(\mathcal{O}_S) = \frac{1}{2}L \cdot (K_S + L) + 1$$

one obtains the inequality

$$L \cdot (K_S + L) \geq L \cdot M \geq (\frac{1}{2}L \cdot (K_S + L))(L \cdot C),$$

hence $1 \leq L \cdot C \leq 2$.

Consider next the natural map

$$H^0(S, \mathcal{O}_S(L)) \xrightarrow{\alpha} W \subset H^0(C, \mathcal{O}_C(L|_C)),$$

with W the image of α. Because $|L|$ is not composite with a pencil,

$$h^0(C, \mathcal{O}_C(L|_C)) \geq \dim_{\mathbb{C}} W \geq 2.$$

Noting that the genus $g(C) \geq 2$, one has $h^0(C, \mathcal{O}_C(\Gamma)) \leq j$ whenever Γ is a divisor with

$$1 \leq \deg(\Gamma) \leq j.$$

Hence

$$h^0(C, \mathcal{O}_C(L|_C)) = \dim_{\mathbb{C}} W = L \cdot C = 2.$$

This implies that $h^0(S, \mathcal{O}_S(L - C)) \geq 1$ and $L - C \geq 0$. Since

$$|K_S + C||_C = |K_C|,$$

one finds $\dim \varphi_{K_S+L}(C) = 1$, contradicting the choice of C as a fibre of h. □

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