SHORT EQUATIONS FOR THE GENUS 2 COVERS OF DEGREE 3
OF AN ELLIPTIC CURVE

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ABSTRACT. E. Kani [4] has shown that the Hurwitz functor $H_{E/K,3}$, which parameterizes the (normalized) genus 2 covers of degree 3 of one elliptic curve $E$ over a field $K$, is representable. In this paper the moduli scheme $H_{E/k,3}$ and the universal family are explicitly calculated over an algebraically closed field $k$ and described by short equations.

INTRODUCTION

T. Shaska [7] has given a long equation, which describes the genus 2 curves with covers of degree 3 onto elliptic curves over an algebraically closed field of characteristic 0. One can obtain simpler equations for the genus 2 covers of degree 3 of one elliptic curve. We use the theoretical framework, which has been introduced by E. Kani [4]. Let $K$ be a field with $\text{char}(K) \neq 2, 3$. We consider a $K$-scheme $S$, an elliptic curve $E/K$ with zero point 0 and a relative genus 2 curve $C/S$. A normalized genus 2 cover $f : C \to E \times S = E_S$ of degree 3 is a morphism of $S$-schemes of degree 3 such that $f_*(W_{C/S}) = 2 \cdot [0_{E/S}] + E_S[2]$, where $W_{C/S}$ is the divisor of Weierstrass points and $[0_{E/S}]$ is the zero section of $E \times S \to S$. Two genus 2 covers $f_1 : C_1 \to E_S$ are isomorphic if there is an isomorphism $\varphi : C_1 \to C_2$ such that $f_1 = f_2 \circ \varphi$. One denotes by $H_{E/K,3}(S)$ the set of isomorphism classes of normalized genus 2 covers of degree 3 onto $E_S$. The assignment $S \to H_{E/K,3}(S)$ yields a Hurwitz functor $H_{E/K,3} : \text{Sch}_{/K} \to \text{Sets}$, which is represented by a smooth and geometrically connected modular curve $H_{E/K,3}$ ([4], Theorem 1.1).

Let $k$ be an algebraically closed field with $\text{char}(k) \neq 2, 3$. Here $H_{E/k,3}$ and the universal family $C \to H_{E/k,3}$ will be explicitly calculated. We construct and parameterize genus 2 covers of elliptic curves by suitable coverings $u : \mathbb{P}^1 \to \mathbb{P}^1$ (Frey-Kani coverings) in Section 1. Section 2 treats the positions of the ramification points of the Frey-Kani coverings. The pattern of these positions allows us to determine $H_{E/k,3}$.

Gerhard Frey gave some hints for the presentation of this paper. Martin Möller helped to improve this paper. I would like to thank them, and Eckart Viehweg for all his time and effort spent in guiding me for my Diplomarbeit (“master thesis”), from which this paper is originated.

1. Construction of genus 2 covers

Let us fix some elliptic curve $E$, which is given by the 4 different points $0, 1, \lambda, \infty \in \mathbb{P}^1$, over an algebraically closed field $k$ with $\text{char}(k) \neq 2, 3$. We consider a normalized genus 2 cover $f : C \to E$ of degree 3. There exist covers $h : C \to \mathbb{P}^1$ and $i : E \to \mathbb{P}^1$ of degree 2
and there is a cover \( v : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree 3, which is called "Frey-Kani covering", (see [2], [5] and [6]) such that this diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{f} & E \\
\downarrow{h} & & \downarrow{i} \\
\mathbb{P}^1 & \xrightarrow{v} & \mathbb{P}^1
\end{array}
\]

Let the zero point of \( E \) lie over \( \infty \), and 0, 1 and \( \infty \) be ramification points of a cover \( u : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree 3. Assume that \( u(0) = 0, u(1) = 1 \) and \( u(\infty) = \lambda \). Later we will see that almost all covers \( u \) are Frey-Kani coverings. Let us first study \( u \). Then the results of the following calculation will be used for a calculation and parametrization of the normalized genus 2 covers \( f : C \to E \) of degree 3. We have for some \( p_1, c \in k \):

\[
u(x_0 : x_1) = (g(x_0, x_1) : c(x_1 - p_1x_0)x_1^2)
\]

with

\[g(x_0, x_1) = x_0^3 + g_2 x_1^2 x_0 + g_1 x_1 x_0^2 + g_0 x_0^3\]

Therefore we get by \( u(0 : 1) = (1 : \lambda) \), which implies that \( \lambda = c \), and by \( u(1 : 1) = (1 : 1) \):

\[
p_1 = 1 - \frac{g(1, 1)}{\lambda}
\]

Now we want to consider the situation over the branch points 1 and \( \lambda \) of \( u \). Therefore we define \( \hat{u} := (x_0 : x_1 - x_0) \circ u \) and \( \check{u} := (x_0 : x_1 - \lambda x_0) \circ u \). Then one has \( \hat{u}(0) = -1 \), \( \hat{u}(1) = 0 \), and \( \hat{u}(\infty) = \lambda - 1 \). We obtain for some \( p_2, d \in k \):

\[
\hat{u}(x_0 : x_1) = (g(x_0, x_1) : d(x_1 - p_2 x_0)(x_1 - x_0)^2)
\]

Thus, we get by \( \hat{u}(0 : 1) = (1 : \lambda - 1) \), which implies \( \lambda - 1 = d \), and by \( \hat{u}(1 : 0) = (1 : -1) \):

\[
p_2 = \frac{g_0}{\lambda - 1}
\]

One has \( \hat{u}(0) = -\lambda \), \( \hat{u}(1) = 1 - \lambda \), and \( \hat{u}(\infty) = 0 \). We obtain for some \( e, p_3 \in k \):

\[
\check{u}(x_0 : x_1) = (g(x_0, x_1) : e(p_3 x_1 - x_0)x_0^2)
\]

Therefore we get by \( \check{u}(1 : 0) = (1 : -\lambda) \), which implies that \( \lambda g_0 = e \), and by \( \check{u}(1 : 1) = (1 : 1 - \lambda) \):

\[
p_3 = 1 + \frac{(1 - \lambda)g(1, 1)}{\lambda g_0}
\]

By the definitions of \( \hat{u} \) and \( \check{u} \), and the preceding results, we get:

\[
(\lambda - 1)(x_1 - \frac{g_0}{\lambda - 1} x_0)(x_1 - x_0)^2 = \lambda(x_1 - (1 - \frac{g(1, 1)}{\lambda})x_0)x_0^2 - g(x_0, x_1),
\]

\[
\lambda g_0((1 + \frac{(1 - \lambda)g(1, 1)}{\lambda g_0})x_1 - x_0)x_0^2 = \lambda(x_1 - (1 - \frac{g(1, 1)}{\lambda})x_0)x_0^2 - \lambda g(x_0, x_1)
\]

These equations of polynomials imply the following equations of coefficients of \( x_1^2 x_0 \):

\[
-2(\lambda - 1) - g_0 = g(1, 1) - \lambda - g_2,
\]

\[
g(1, 1) - \lambda - \lambda g_2 = 0
\]
By (1.0.3) and (1.0.4), one has the equations \( g(1, 1) = \lambda - \lambda p_1 \) and \( g_0 = \lambda p_2 - p_2 \). We substitute for \( g(1, 1) \) and \( g_0 \) in (1.0.6) and (1.0.7) and obtain:

\[
(1.0.8) \quad p_1 = -g_2 = -2(\lambda - 1) + p_2 - \lambda p_2 + p_1 \lambda
\]

\[\Rightarrow (\lambda - 1)(-2 - p_2 + p_1) = 0\]

One can divide by \( \lambda - 1 \), because we have \( \lambda \neq 1 \). Thus, we have:

\[
(1.0.9) \quad -2 - p_2 + p_1 = 0 \iff p_2 = p_1 - 2
\]

One obtains by (1.0.3), (1.0.4) and (1.0.5):

\[
(1.0.10) \quad p_3 = 1 + \frac{(1-\lambda)g(1, 1)}{\lambda g_0} = 1 - \frac{1 - p_1}{p_1 - 2} = \frac{2p_1 - 3}{p_1 - 2}
\]

The equations \( p_2 = p_1 - 2 \) and \( p_2 = \frac{g_0}{1-\lambda} \iff p_2(\lambda - 1) = g_0 \) imply:

\[
(1.0.11) \quad (\lambda - 1)(p_1 - 2) = g_0
\]

Using the equations (1.0.3), (1.0.8) and (1.0.11) we get:

\[
(1.0.12) \quad g_1 = g(1, 1) - 1 - g_2 - g_0 = \lambda - p_1 \lambda - 1 + p_1 + (1 - \lambda)(p_1 - 2) = -2p_1 \lambda + 2p_1 + 3\lambda - 3
\]

Thus, by (1.0.8), (1.0.11), and (1.0.12), we obtain

\[
(1.0.13) \quad g(1, x) = x^3 - p_1 x^2 + (-2\lambda p_1 + 2p_1 + 3\lambda - 3)x - (1 - \lambda)(p_1 - 2).
\]

Hence the cover \( u \) is completely determined by (1.0.2), \( \lambda = c \), and (1.0.13). Now we can apply the results of this calculation and construct normalized genus 2 covers of degree 3. Later we will use that we get by (1.0.13):

\[
(1.0.14) \quad \lambda(x - p_1)x^2 - \lambda g(1, x) = -\lambda(1 - \lambda)((2p_1 - 3)x - p_1 + 2),
\]

\[
(1.0.15) \quad \lambda(x - p_1)x^2 - g(1, x) = (\lambda - 1)(x - p_1 + 2)(x - 1)^2
\]

Now we consider smooth curves \( C \) of genus 2, which are given by

\[
(1.0.16) \quad y^2 = (x - p_1)(x - p_1 + 2)((2p_1 - 3)x - p_1 + 2)g(1, x)
\]

for some \( p_1 \).

**Proposition 1.1.** The curve \( C \) given in (1.0.16) is smooth if \( u \) is not ramified over \( \infty \).

Let \( h : C \to \mathbb{P}^1 \) and \( i : E \to \mathbb{P}^1 \) be the natural projections, which are given by \( (x, y) \to x \). Then the normalized covers \( f_\pm : C \to E \) of degree 3 with Frey-Kani covering \( u \) for \( h \) and \( i \) are given by

\[
f_\pm(x, y) = (g(1, x) : \lambda(x - p_1)x^2 : \pm \frac{(\lambda - 1)yx(x - 1)}{g(1, x)}).
\]

**Proof.** Using that \( u \) is unramified over \( \infty \) we see that \( g(1, x) \) has 3 different zeros. By (1.0.9) and (1.0.10), we conclude that \( u \) maps the 3 other points in \( \mathbb{P}^1 \), which give the Weierstrass points of \( C \), to 3 different points. Therefore the Weierstrass points of \( C \) are given by 6 different points of \( \mathbb{P}^1 \), and \( C \) is smooth. Let \( X_1 = \lambda(x - p_1)x^2 \) and \( X_0 = g(1, x) \). Note that \( E = \{y^2x_0 - x_1(x_1 - x_0)(x_1 - \lambda x_0) = 0\} \subset \mathbb{P}^2 \). There is a cover \( f : C \to E \) with Frey-Kani covering \( u \) for the chosen degree 2 covers \( h \) and \( i \) if and only if there is an \( Y \in \k(C) \), which satisfies the equation \( Y^2X_0 = X_1(X_1 - X_0)(X_1 - \lambda X_0) \) in \( k(C) \). By (1.0.14) and (1.0.15), one can easily check that
Proposition 1.2. Using Proposition 1.1 we have a bijection \( \chi : \mathcal{U} \to \mathcal{H}_{E/k,3}(k) \). This map is given by \( u \to [f_\pm : C \to E] \).

**Proof.** By the hyperelliptic involution on \( C \), we conclude that the covers \( f_\pm : C \to E \) lie in the same isomorphism class, and the map \( \chi : \mathcal{U} \to \mathcal{H}_{E/k,3}(k) \) is well defined.

Let \( f : C \to E \) be a normalized genus 2 cover of degree 3. The Frey-Kani covering of \( f \) is not ramified over \( \infty \) and has ramification points over 0, 1, and \( \lambda \) (see [1]). One can put these ramification points 0, 1, and \( \infty \) such that the Frey-Kani covering is some \( u \in \mathcal{U} \).

In [1], page 92-93 the curve \( C \) is described by the Frey-Kani covering. This description of \( C \), (1.0.14) and (1.0.15) imply that \( C \) is given by (1.0.16) for some \( p_1 \). One can assume that \( h \) and \( i \) are given by \( (x,y) \to x \). Hence by Proposition 1.1, we conclude that \( \chi \) is surjective.

Let \( \chi(u_1) = [f_1 : C_1 \to E] = [f_2 : C_2 \to E] = \chi(u_2) \) and \( h_i : C_i \to \mathbb{P}^1 \) be the natural degree 2 covers (for \( i = 1,2 \)). Then there is an isomorphism \( i : C_1 \to C_2 \) such that \( f_1 = f_2 \circ i \) and an \( a \in \text{Aut}(\mathbb{P}^1) \) such that \( a \circ h_1 = h_2 \circ i \) (see [3], page 304, IV, Exercise 2.2.(a)). Therefore we conclude \( u_1 = u_2 \circ a \). Let \( x \in \{0,1,\infty\} \). We get by our assumptions that \( u_1(x) = u_2(x) \). These points are ramification points of \( u_1 \) and \( u_2 \). Thus, \( u_1 = u_2 \circ a \) implies that \( a(x) = x \ (\forall x \in \{0,1,\infty\}) \). Therefore we have \( a = \text{id} \) and \( \chi \) is injective. \( \square \)

2. The reckoning of the moduli space

By \( u(\infty) = \lambda \neq 0 = u(p_1) \), we conclude that \( p_1 \) and \( \infty \) can not coincide. Thus, (1.0.2), \( c = \lambda \), and (1.0.13) imply that \( g(1,x) \) is determined by \( p_1 \) and \( \lambda \), and that:

**Lemma 2.1.** We have an injective map \( \iota : \mathcal{U} \to \mathbb{A}^1 \), which maps the morphism, which is given by \( x \to \frac{\lambda(x-p_1)x^2}{g(1,x)} \), to \( p_1 \).

**Remark 2.2.** The morphism, which is given by \( x \to \frac{\lambda(x-p_1)x^2}{g(1,x)} \) for some \( p_1 \), has the degree 3 if and only if we have \( g(1,p_1) \neq 0 \) and \( g(1,0) \neq 0 \). By (1.0.13), one can easily see that this is true if and only if \( p_1 \in k \setminus \{1,2\} \). Thus, by (1.0.14) and (1.0.15), we have \( p_1 \in \iota(\mathcal{U}) \) if and only if we have \( p_1 \in k \setminus \{1,2\} \) and \( g(1,x) \) has 3 different zeros.

Let \( p_1 \neq 1,2,\infty \), and \( u \) be the morphism, which is given by \( x \to \frac{\lambda(x-p_1)x^2}{g(1,x)} \). The preceding remark, (1.0.14) and (1.0.15) imply that \( u \) has the degree 3 and satisfies our choice of coordinates of ramification points, and that we have \( p_1 \in \iota(\mathcal{U}) \) if and only if \( u \) does not have a ramification point over \( \infty \). By the Hurwitz formula, \( u \) has a ramification divisor of degree 4. But we do not know the position of a fourth branch point. Let \( u(\mu) = u(\delta) = \zeta \)
and δ be a ramification point of u. We have \((x_0 : x_1 - \zeta x_0) \circ u = (g : F(x_1 - \mu x_0)(x_1 - \delta x_0)^2)\) for some \(F \in k\) resp.,

\[
F(x_1 - \mu x_0)(x_1 - \delta x_0)^2 = (\lambda - \zeta)x_1^3 + (g(1, 1) - \lambda - \zeta g_2)x_1^2x_0 - \zeta g_1 x_1 x_0^2 - \zeta g_0 x_0^3.
\]

This equation of polynomials implies the following equations of coefficients:

\[
F = \lambda - \zeta
\]

\[
-F\mu - 2F\delta = g(1, 1) - \lambda - \zeta g_2
\]

\[
F\delta^2 + 2F\delta\mu = -\zeta g_1
\]

\[
-\mu\delta^2 F = -\zeta g_0
\]

Using \(F = \lambda - \zeta\), \(p_1 = -g_2\), \(-p_1\mu = g(1, 1) - \lambda\), (1.0.8), and (1.0.3) we substitute in the second equation:

\[
(-\lambda + \zeta)\mu + 2(-\lambda + \zeta)\delta = -p_1\lambda + \zeta p_1
\]

Let \(\lambda \neq \zeta\). Then we have:

(2.2.1)

\[
\mu = p_1 - 2\delta
\]

By \(\mu = p_4 - 2\delta\) and and the same substitution as above, we obtain:

(2.2.2)

\[
(\lambda - \zeta)(2\delta p_1 - 3\delta^2) = -\zeta(1 - \lambda)(2p_1 - 3),
\]

(2.2.3)

\[
(2\delta^3 - p_1\delta^2)(\lambda - \zeta) = \zeta(1 - \lambda)(p_1 - 2)
\]

By (2.2.3), we get the equation \(\zeta = \frac{(2\delta^3 - p_1\delta^2)(\lambda - \zeta)}{(1 - \lambda)(p_1 - 2)}\). Thus, one has by (1.0.10) and (2.2.2):

\[
(\lambda - \zeta)(2\delta p_1 - 3\delta^2) = (2\delta^3 - p_1\delta^2)(\lambda - \zeta)(-p_3)
\]

The solution \(\zeta = \lambda\) gives the ramification point \(\infty\) and \(\delta = 0\) gives the ramification point 0. Therefore we can divide by \(\lambda - \zeta\) and \(\delta\), and get

\[
2p_1 - 3\delta = (2\delta^2 - p_1\delta)(-p_3) \iff 0 = \delta^2 - \frac{p_1 p_3 + 3}{2p_3} \delta + \frac{p_1}{p_3} = (\delta - 1)(\delta - \frac{p_1}{p_3}),
\]

because we get by (1.0.10):

\[
\frac{p_1 p_3 + 3}{2p_3} = \frac{p_1(2p_1 - 3) + 3(p_1 - 2)}{2(2p_1 - 3)} = \frac{2p_1^2 - 4p_1 + 4p_1 - 6}{2(2p_1 - 3)} = \frac{p_1}{p_3} + 1
\]

Thus, a ramification point \(\delta\) of \(u\) is given by

\[
\delta = \frac{p_1}{p_3} = \frac{p_1(p_1 - 2)}{2p_1 - 3}
\]

We substitute \(\delta = \frac{p_1(p_1 - 2)}{2p_1 - 3}\) in (2.2.2) and get:

\[
(\lambda - \zeta)(2\frac{p_1(p_1 - 2)}{2p_1 - 3} p_1 - 3\frac{p_1(p_1 - 2)}{2p_1 - 3})^2 = -\zeta(1 - \lambda)(2p_1 - 3)
\]

\[
\iff (\lambda - \zeta)p_1^2(p_1 - 2) = -\zeta((1 - \lambda)(2p_1 - 3))^3 \iff \lambda p_1^2(p_1 - 2) = \zeta((\lambda - 1)(2p_1 - 3)^3 + p_1^3(p_1 - 2))
\]

(2.2.4)

\[
\iff \zeta = \frac{\lambda p_1^3(p_1 - 2)}{(\lambda - 1)(2p_1 - 3)^3 + p_1^3(p_1 - 2)}
\]

Recall that \(E\) is determined by \(\lambda\), and that \(E\) resp., \(\lambda\) is fixed. Hence by 2.2.4, the set of values of \(p_1\), which induce a 4th ramification point over \(\infty\), is given by \(Z = \{(\lambda - 1)(2p_1 - 3)^3 + p_1^3(p_1 - 2) = 0\} \subset \mathbb{P}^1\).

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Remark 2.3. Four different points lie in $\mathbb{Z}$.

Proof. We consider the morphism $\tilde{\zeta} : \mathbb{P}^1 \to \mathbb{P}^1$, which is given by

$$\tilde{\zeta}(p_1) = \frac{\lambda p_1^3(p_1 - 2)}{(\lambda - 1)(2p_1 - 3)^3 + p_1^3(p_1 - 2)} = (1 + \frac{\lambda - 1}{\lambda} \frac{(2p_1 - 3)^3}{p_1^3(p_1 - 2)})^{-1}.$$ 

Therefore the ramification points of $\tilde{\zeta}$ are the ramification points of the morphism $\sigma$, which is given by $p_1 \to \frac{(2p_1 - 3)^3}{p_1^3(p_1 - 2)}$. The derivative of $\sigma$ is

$$\sigma'(p_1) = \frac{6(2p_1 - 3)^2 p_1^3(p_1 - 2) - (2p_1 - 3)^3(3p_1^2(p_1 - 2) + p_1^3)}{p_1^6(p_1 - 2)^2} = -\frac{2(2p_1 - 3)^2 p_1^2(p_1 - 3)^2}{p_1^6(p_1 - 2)^2}.$$ 

Thus, the ramification points of $\tilde{\zeta}$ are 0, $\frac{3}{2}$ and 3. The statement follows by the fact that all $x \in \{0, \frac{3}{2}, 3\}$ fulfill $(\lambda - 1)(2 \cdot x - 3)^3 + x^3(x - 2) \neq 0$. □

By Lemma 2.1 and Remark 2.2, we have a bijection between $\mathcal{U}$ and the set of closed points of $\mathbb{P}^1 \setminus (\{1, 2, \infty\} \cup \mathbb{Z})$, where $\mathbb{P}^1 \setminus (\{1, 2, \infty\} \cup \mathbb{Z})$ is an (Zariski) open subset of $\mathbb{P}^1$. Recall that $\mathcal{H}_{E/k,3}$ is represented by a smooth modular curve $H_{E/k,3}$ (see [4], Theorem 1.1). Therefore, we conclude by Proposition 1.1 and Proposition 1.2:

Theorem 2.4. We have $H_{E/k,3} \cong \mathbb{P}^1 \setminus (\{1, 2, \infty\} \cup \mathbb{Z})$. The fiber $C_{p_1}$ of the universal family $C \to \mathbb{P}^1 \setminus (\{1, 2, \infty\} \cup \mathbb{Z})$ is given by (1.0.16) for all $p_1 \in \mathbb{P}^1 \setminus (\{1, 2, \infty\} \cup \mathbb{Z}$.

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