Raynaud vector bundles

Georg Hein

October 6, 2008

Abstract

We construct vector bundles R_μ^{rk} on a smooth projective curve X having the property that for all sheaves E of slope μ and rank rk on X we have an equivalence: E is a semistable vector bundle $\iff \mathrm{Hom}(R_\mu^{\mathrm{rk}}, E) = 0$.

As a byproduct of our construction we obtain effective bounds on r such that the linear system $|R \cdot \Theta|$ has base points on $\mathrm{U}_X(r, r(g-1))$.

1 Introduction

Notation: Throughout this paper X is a smooth projective curve of genus g over some algebraically closed field k.

Raynaud constructed in his article [16] vector bundles $\{P_m\}_{m \geq 1}$ with the property that $\mu(P_m) = \frac{g}{m}$ and $h^0(P_m \otimes L) \neq 0$ for all line bundles L of degree zero. We showed in our article [10] that the converse also holds, that is: $h^0(E \otimes L) \neq 0$ for all line bundles of degree zero if and only if we have non zero morphisms $P_{\mathrm{rk}(E)g+1} \to E$. Furthermore, Raynaud showed that a vector bundle E of rank two and slope $g-1$ is semistable if and only if, there exists a line bundle L of degree zero with $h^0(E \otimes L) = 0$. Thus, we deduce:

Theorem 1.1 *For a coherent sheaf E on X of rank two and slope $g-1$ we have an equivalence*

$$E \text{ is a semistable vector bundle} \iff \mathrm{Hom}(P_{2g+1}, E) = 0\,.$$

This way we obtain another condition equivalent to semistability. This condition is convenient, because we need only check the behavior of E with respect to one bundle in order to determine whether it is semistable. This motivates the following definition:

Definition: A vector bundle R_μ^{rk} is called a Raynaud bundle, if we have an equivalence

$$E \text{ is a semistable vector bundle} \iff \mathrm{Hom}(R_\mu^{\mathrm{rk}}, E) = 0$$

for all coherent sheaves E of rank rk and slope μ.

Raynaud's bundle P_{2g+1} is a first example of a Raynaud bundle. Theorem 1.1 could also be formulated as: P_{2g+1} is a Raynaud bundle R^2_{g-1}. We derive from this Theorem the existence of Raynaud bundles R^2_μ for all integer slopes μ. The main result of this paper is:

Theorem 1.2 *For all pairs (rk, μ) there exists a Raynaud bundle R_μ^{rk}.*

This theorem is the equivalence (i) $\iff$ (v) of Theorem 2.12. We remark that such a Raynaud bundle is not unique. Indeed, twisting a Raynaud bundle with a line bundle of degree zero gives another, as well as taking the direct sum of two such bundles. In Section 2 we construct a Raynaud bundle R_μ^{rk}. Implicitly this construction appears in Proposition 2 of [9]. However, there its construction is embedded in the theory of the derived category. Here we work out this construction more concretely, give the numerical invariants (Proposition 2.3), show the relation to base points of the Θ-divisor, and sum up the main properties in Theorem 2.12.
When writing $H^*(G) = 0$, we mean that $h^i(G) = 0$ for all integers i. Since X is of dimension one, this is equivalent to $h^0(G) = 0 = h^1(G)$ for a coherent sheaf G. It is well known, that $H^*(E \otimes F)$ implies the semistability of E and F. The converse also holds (see for example theorem 7.4 in [15]). The strange duality which was raised in [2] by Beauville, and established by A. Marian and D. Oprea in [11] (it was earlier shown for general curves in [4] and later generalized to all curves in [5] by P. Belkale) allows the following geometric description of base points of the generalized Θ-divisor on $\mathrm{U}_X(r, r(g-1))$: A vector bundle $E \in \mathrm{U}_X(r, r(g-1))$ is a base point of the linear system $|k \cdot \Theta|$ if and only if $H^*(E \otimes F) \neq 0$ for all vector bundles F with trivial determinant of rank k. Using this approach, we can construct base points of $|k \cdot \Theta|$ for all integers k.
This the purpose of section 3. To do so, we start by fine tuning the construction of Raynaud bundles for the case when $\mu = g - 1$. This allows the construction of Raynaud bundles of smaller rank than those obtained in Section 2. This way we obtain upper bounds for r for the base point freeness of $|R \cdot \Theta|$ on the moduli spaces $\mathrm{U}_X(r, r(g-1))$ see Proposition 3.6 and Corollary 3.8. They imply upper bounds for the base point freeness of the Θ-divisor on $\mathrm{SU}_X(r)$ (see Proposition 3.7). However the bounds for base points of $|\Theta|$ on $\mathrm{SU}_X(r)$ are not optimal see Arcara's result in [1] and Pauly's recent result in [12] (see also older results of Popa in [13]). O. Schneider used Raynaud's original bundles to produce base points of $|\Theta|$ on $\mathrm{SU}_X(r)$ as extensions of Raynaud's bundle by line bundles in [17].
Corollary 3.8 gives a polynomial p_R of degree E such that the linear system $|R \cdot \Theta|$ is not base point free on $\mathrm{U}_X(r, r(g-1)$ for all $r \geq p_R(g)$. This way we give a partial answer to a problem posed by Popa (see [15, problem 6.10]).
For $X = \mathbb{P}^1$ there exist semistable bundles E only for integer slopes. We see that the line bundle $\mathcal{O}_{\mathbb{P}^1}(\mu + 1)$ is a Raynaud bundle R_μ^{rk} in this case. For an elliptic curve X the existence of Raynaud bundles is well known too (see Lemma 5 in [8]). For example: every stable bundle F of rank $\mathrm{rk} + 1$ and degree one is a R_0^{rk}. Therefore, we may assume $g \geq 2$.

Acknowledgment: The author wants to thank the referee for the useful comments. This work has been supported by the SFB/TR 45 "Periods, moduli spaces and arithmetic of algebraic varieties".

2 The Raynaud bundle R_μ^{rk}

2.1 Construction of $S_{\mu,R,m}$ for $\mu \in [-g-1, -g)$

Let X be a smooth projective curve of genus g over k. We fix be a line bundle L_1 on X of degree one. By L_{-1} we denote its dual. Let $\mu = \frac{d}{r} \in [-g-1, -g)$ be a rational number, where d and r are coprime integers with $r \geq 1$. Furthermore, we fix a positive integer R.

A semistable vector bundle E of slope μ is a base point of the linear system $|R\cdot\Theta|$, if for all vector bundles F with $\mathrm{rk}(F) = r\cdot R$ and $\det(F) = L_1^{\otimes rR(g-1)-dR}$ we have $H^*(X, E\otimes F) \neq 0$. (The reader not familiar with base point of the generalized Θ-divisor may take this as a definition. Indeed, this description of base points follows from Marian's and Oprea's result [11]. See also Beauville's or Popa's surveys [2], [3], and [15] for the definition of base points.)
We consider the two sheaves

$$M_1 = M_1(\mu, R) := L_{-1}^{\otimes rR(g-1-\mu)} \quad \text{and} \quad M_0 = M_0(\mu, R) := \mathcal{O}_X^{\oplus rR+1}.$$

We are interested in M_1 and M_0 because of the following lemma.

Lemma 2.1 *Let E be a semistable vector bundle of slope $\mu(E) \in [-g-1,-g)$. Furthermore, we fix an integer $R \geq 1$, and the vector bundles M_1 and M_0 as above. Then the following three conditions are equivalent:*
(i) E is not a base point of $|R\cdot\Theta|$.
(ii) For some morphism $M_1 \xrightarrow{\phi} M_0$ we have $H^0(E\otimes \mathrm{coker}(\phi)) = 0$.
(iii) For some morphism $M_1 \xrightarrow{\phi} M_0$ the resulting morphism $H^1(E\otimes M_1) \to H^1(E\otimes M_0)$ is injective.

Proof: Suppose there exists a vector bundle F such that $H^*(E\otimes F) = 0$ with $\mathrm{rk}(F) = rR$, and $\det(F) \cong L_1^{\otimes rR(g-1)-dR}$. This implies that F itself is a semistable bundle of slope $\mu(F) = (g-1)-\mu$. Hence, F is semistable of slope greater than $2g-1$. Thus, F is globally generated. Indeed, we can generate this vector bundle by $rR+1$ global sections. This way, we obtain a surjection: $\mathcal{O}^{\oplus rR+1} \xrightarrow{\pi} F$. The kernel of π is a line bundle M_1. The determinant of this line bundle is isomorphic to $M_1 \cong \det(M_0)\otimes\det(F)^{-1} \cong \det(F)^{-1}$. Thus we have shown, that (i) $\Longrightarrow$ (ii).
To see the implication (ii) $\Longrightarrow$ (i), we remark that setting $F := \mathrm{coker}(\phi)$ we obtain a sheaf of rank rR and determinant $L_1^{\otimes rR(g-1)-dR}$. From Riemann-Roch we deduce that $\chi(E\otimes F) = 0$. Thus, $H^0(E\otimes F) = 0$ implies $H^*(E\otimes F) = 0$.
The equivalence of (ii) and (iii) follows directly from the exact sequence

$$H^0(E\otimes M_0) \to H^0(E\otimes \mathrm{coker}(\phi)) \to H^1(E\otimes M_1) \to H^1(E\otimes M_0)$$

and the fact that $H^0(E\otimes M_0) = 0$, because the semistable bundle $E\otimes M_0$ has negative slope. □
We consider the vector space $V := \mathrm{Hom}(M_1, M_0)$. Its dimension is $v := \dim(V) = (rR+1)(1-g+rR(g-1-\mu))$. We consider the product space

$$X \xleftarrow{p} X\times\mathbb{P}(V^\vee) \xrightarrow{q} \mathbb{P}(V^\vee)$$

Combining the universal morphism $M_1 \to V^\vee\otimes M_0$ on X and the morphisms $\mathcal{O}_{\mathbb{P}(V^\vee)}\otimes V^\vee \to \mathcal{O}_{\mathbb{P}(V^\vee)}(1)$ on $\mathbb{P}(V^\vee)$ we obtain a morphism

$$p^*M_1 \xrightarrow{\alpha} p^*M_0 \otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(1) \quad \text{on } X\times\mathbb{P}(V^\vee).$$

If we consider $\mathbb{P}(V^\vee)$ as the moduli space of morphisms different from zero from $M_1(\mu, R)$ to $M_0(\mu, R)$ modulo scalars, then this morphism is the universal family over $\mathbb{P}(V^\vee)$. Since

$p^*M_1(\mu,R)$ is a line bundle and α is not trivial, we deduce that α is injective. We denote its cokernel by $G(\mu,R)$. Twisting the the short exact sequence

$$0 \to p^*M_1(\mu,R) \to p^*M_0(\mu,R) \otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(1) \to G(\mu,R) \to 0$$

by $q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(m)$ for any $m \geq 0$, we obtain a short exact sequence of sheaves possessing no higher direct images with respect to p. Thus, we obtain a short exact sequence on X.

$$0 \to \mathrm{Sym}^m(V^\vee) \otimes M_1(\mu,R) \to \mathrm{Sym}^{m+1}(V^\vee) \otimes M_0(\mu,R) \to p_*(G(\mu,R) \otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(m)) \to 0$$

We define the sheaf $S_{\mu,R,m} := p_*(G(\mu,R) \otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(m))$. From the construction of $S_{\mu,R,m}$ we conclude the following the first properties of the sheaf $S_{\mu,R,m}$, namely

Proposition 2.2 *There exists a short exact sequence*

$$0 \to \mathrm{Sym}^m(V^\vee) \otimes M_1(\mu,R) \to \mathrm{Sym}^{m+1}(V^\vee) \otimes M_0(\mu,R) \to S_{\mu,R,m} \to 0.$$

Proposition 2.3 *The numerical invariants of the sheaf $S_{\mu,R,m}$ are given by*

$$\begin{aligned} \mathrm{rk}(S_{\mu,R,m}) &= \binom{m+v-1}{m}\left((rR+1)\frac{v+m}{m+1} - 1\right) \\ \deg(S_{\mu,R,m}) &= \binom{m+v-1}{m} rR(g-1-\mu) \\ \mu(S_{\mu,R,m}) &= \frac{(m+1)rR(g-1-\mu)}{(m+1)rR+(v-1)(rR+1)} = g-1-\mu - \frac{(v-1)(rR+1)(g-1-\mu)}{(m+1)rR+(v-1)(rR+1)} \end{aligned}$$

Remark. Considered as a function depending on $m \in \mathbb{N}$ the slope of $S_{\mu,R,m}$ is of the form $\mu(S_{\mu,R,m}) = g-1-\mu-\frac{a}{m+b}$ for some positive $a,b \in \mathbb{Q}$.

2.2 Properties of $S_{\mu,R,m}$ for $\mu \in [-g-1,-g)$

We keep the notation of 2.1. In particular we still assume that $\mu = \frac{d}{r} \in [-g-1,-g)$, R and the bundles M_1 and M_0 are fixed in §2.2. We need the following result.

Lemma 2.4 *([9, Lemma 13]) Let U and W be k-vector spaces of finite dimension. Suppose that a given morphism $U \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\rho} W \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ is not injective. Then for any integer $m \geq (\dim(U)-1)n$ we have $H^0(\ker(\rho)(m)) \neq 0$.*

Note, that the sheaf E in the following proposition is not necessarily of slope μ. However, semistable vector bundles of negative slope fulfill the premise of the proposition.

Proposition 2.5 *Let E be a sheaf on X with the property that $H^0(X,E)=0$. For any $m \geq (v-1)(h^1(M_1 \otimes E)-1)$, the following two conditions are equivalent:*
(i) There exists a short exact sequence $0 \to M_1 \to M_0 \to F \to 0$ with $H^0(F \otimes E) = 0$.
(ii) $H^0(S_{\mu,R,m} \otimes E) = 0$.

Proof: From $H^0(E) = 0$, we deduce that E is a vector bundle and the equalities $h^0(M_0 \otimes E) = 0 = h^0(M_1 \otimes E)$. Furthermore, the dimension $h^1(M_1 \otimes E)$ can be computed by Riemann-Roch and is positive.
We consider the short exact sequence

$$0 \to p^*(M_1 \otimes E) \to p^*(M_0 \otimes E) \otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(1) \to p^*E \otimes G(\mu,R) \to 0$$

on $X \times \mathbb{P}(V^\vee)$. Since $H^0(M_0 \otimes E) = 0$ we obtain on $\mathbb{P}(V^\vee)$ an exact sequence

$$0 \to q_*(p^*E \otimes G(\mu,R) \to H^1(M_1 \otimes E) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \xrightarrow{\beta} H^1(M_0 \otimes E) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \to$$

By base change, condition (i) is equivalent to the injectivity of the morphism β. This is by lemma 2.4 equivalent to $H^0(q_*(p^*E \otimes G(\mu,R)) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(m)) = 0$. Thus, (i) is equivalent to $H^0(p^*E \otimes G(\mu,R) \otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(m)) = 0$. This implies the result by definition of $S_{\mu,R,m}$, and the projection formula. □

As a corollary of the proof of proposition 2.5 we obtain the

Corollary 2.6 *For any sheaf E on X the assignment $m \mapsto h^0(S_{\mu,R,m} \otimes E)$ is the Hilbert function of the torsion free sheaf $q_*(p^*E \otimes G(\mu,R))$. In particular, $h^0(S_{\mu,R,m_0} \otimes E) \neq 0$ implies $h^0(S_{\mu,R,m} \otimes E) \neq 0$ for all $m \geq m_0$.*

Corollary 2.7 *For any $m \geq 0$ the sheaf $S_{\mu,R,m}$ is a vector bundle on X.*

Proof: Take a stable vector bundle F with $\det(F) \cong L_1^{\otimes rR(g-1-\mu)}$. As seen in Lemma 2.1, there exists a short exact sequence $0 \to M_1 \xrightarrow{\phi} M_0 \xrightarrow{\pi} F \to 0$. We take a (sufficiently negative) line bundle E on X, such that $h^0(E) = 0 = h^0(E \otimes F)$. The line bundle E fulfills the assumption of Proposition 2.5 and condition (i) of Proposition 2.5 is satisfied. Thus, we conclude $H^0(S_{\mu,R,m} \otimes E) = 0$ for $m \gg 0$. By Corollary 2.6 this implies $H^0(S_{\mu,R,m} \otimes E) = 0$ for all $m \geq 0$. Hence, $S_{\mu,R,m}$ is torsion free. □

2.3 Definition and properties of $S^{\mathrm{rk}}_{\mu,R}$ for $\mu \in [-g-1,-g)$

In this part 2.3 we still assume that $\mu = \frac{d}{r} \in [-g-1,-g)$ and d and r are coprime. Thus, for a vector bundle E of slope $\mu(E) = \mu$ we have $\mathrm{rk}(E) := h \cdot r$ for some natural number h. Remember, the number $v = (rR+1)(1-g+rR(g-1-\mu))$. For any number rk which is a multiple of r we define

$$S^{\mathrm{rk}}_{\mu,R} := S_{\mu,R,(v-1)(\mathrm{rk}\cdot(g-1-\mu)(rR+1)-1)}\,.$$

Proposition 2.8 *For a semistable vector bundle E of slope $\mu(E) = \mu \in [-g-1,-g)$ on the curve X we have an equivalence.*

$$E \textit{ is not a base point of } |R \cdot \Theta| \iff H^0(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E) = 0.$$

Proof: First we note that $h^0(E) = 0$ because E is semistable of negative slope. Thus, $h^0(M_1 \otimes E) = 0$ and we can compute $h^1(M_1 \otimes E) = \mathrm{rk}(E) \cdot (g-1-\mu)(rR+1)$ by the Riemann-Roch theorem. Now we deduce the equivalence from Propositions 2.1 and 2.5 because we took the number m in the definition of $S^{\mathrm{rk}}_{\mu,R}$ to be the smallest possible m such that Proposition 2.5 applies to $S_{\mu,R,m}$ and E. □

Lemma 2.9 *If E is a coherent sheaf of slope $\mu(E) = \mu \in [-g-1,-g)$ on X with the property $H^0(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E) = 0$, then E is semistable.*

Proof: First, we note that $H^0(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E) = 0$ implies that E is torsion free. Now suppose that E is not semistable. Let $E' \subset E$ be a destabilizing subbundle. We have $\mu(E') \geq \mu(E) + \frac{1}{\mathrm{rk}(E)(\mathrm{rk}(E)-1)}$. By proposition 2.3 and the choice of m we derive the inequality

$$\mu(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E') = \mu(S^{\mathrm{rk}(E)}_{\mu,R}) + \mu(E') > g-1.$$

This implies $\chi(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E') > 0$. Hence, we deduce $0 \neq H^0(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E') \subset H^0(S^{\mathrm{rk}(E)}_{\mu,R} \otimes E)$, which contradicts our assumption. □

2.4 Definition and properties of S_μ^{rk} for $\mu \in [-g-1,-g)$

We define the vector bundle S_μ^{rk} to be $S_{\mu,\tilde{R}}^{\mathrm{rk}}$ with $\tilde{R} = \lceil \frac{(\mathrm{rk}+1)^2}{4} \rceil$. Still assuming, that $\mu \in [-g-1,-g)$, $\mu = \frac{d}{r}$, with $\mathrm{rk} = rh$ for some integer h we conclude the following result.

Proposition 2.10 *For a coherent sheaf E of slope $\mu \in [-g-1,-g)$ and rank* rk*, we have the equivalence*

$$E \text{ is a semistable vector bundle} \iff H^0(S_\mu^{\mathrm{rk}} \otimes E) = 0.$$

Proof: We have seen in Lemma 2.9, $h^0(E \otimes S_\mu^{\mathrm{rk}}) = 0$ implies that E is a semistable vector bundle. Suppose now that E is a semistable vector bundle. Since the generalized Θ-divisor $|R \cdot \Theta|$ is base point free for $R \geq \frac{(\mathrm{rk}+1)^2}{4}$ (see Theorem 4.1 in Popa's article [14]) we have by Proposition 2.8, that $h^0(S_{\mu,R}^{\mathrm{rk}} \otimes E) = 0$ for all $R \geq \frac{(\mathrm{rk}+1)^2}{4}$. By definition of S_μ^{rk} this proves the claimed statement. □

2.5 The vector bundles S_μ^{rk} and R_μ^{rk}

Let $\mu = \frac{d}{r}$ be a rational number expressed as quotient of two coprime integers with $r \geq 1$. In contrast to parts 2.1–2.4 there exists no restriction on μ. We take an integer rk which is a multiple of r.
We define the vector bundle S_μ^{rk} through

$$S_\mu^{\mathrm{rk}} := L_{-1}^{\otimes(\lfloor\mu\rfloor+1+g)} \otimes S_{\mu-(\lfloor\mu\rfloor+1+g)}^{\mathrm{rk}}.$$

This is well defines, as $\mu - (\lfloor\mu\rfloor + 1 + g) \in [-g-1,-g)$. Now, we have the

Proposition 2.11 *If E is a coherent sheaf of positive rank* rk *and of slope μ, then*

$$E \text{ is a semistable vector bundle} \iff H^0(S_\mu^{\mathrm{rk}} \otimes E) = 0.$$

Proof: We have E is a semistable vector bundle, if and only if $E \otimes L_{-1}^{\otimes(\lfloor\mu\rfloor+1+g)}$ is a semistable vector bundle. Since $\mu(L_{-1}^{\otimes(\lfloor\mu\rfloor+1+g)} \otimes E) = \mu - \lfloor\mu\rfloor - g - 1 \in [-g-1,-g)$, we can apply Proposition 2.10 to obtain that E is a semistable vector bundle, if and only if the cohomology group $H^0(S_{\mu-(\lfloor\mu\rfloor+g+1)}^{\mathrm{rk}} \otimes L_{-1}^{\otimes(\lfloor\mu\rfloor+1+g)} \otimes E)$ is zero. By definition this happens exactly when $H^0(S_\mu^{\mathrm{rk}} \otimes E) = 0$. □
We define the vector bundle R_μ^{rk} to be the dual of S_μ^{rk}. We have the

Theorem 2.12 *Let E be a coherent sheaf on the smooth projective curve X of rank* $\mathrm{rk} > 0$ *and slope $\mu = \frac{d}{r}$. The following conditions are equivalent:*

(i) *E is a semistable vector bundle.*
(ii) *There exists a sheaf F of rank $\lceil \frac{(\mathrm{rk}+1)^2}{4} \rceil r$ such that $H^*(E \otimes F) = 0$.*
(iii) *There exists a sheaf $F \neq 0$ such that $H^*(E \otimes F) = 0$.*
(iv) $H^0(S_\mu^{\mathrm{rk}} \otimes E) = 0$.
(v) $\mathrm{Hom}(R_\mu^{\mathrm{rk}}, E) = 0$.

Proof: The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i), and (iv) $\iff$ (v) are standard. The equivalence of (i) and (ii) is shown in Theorem 4.1 of [14]. The equivalence of (i) and (iv) was shown in Proposition 2.11. □

2.6 Further remarks

Let R_μ^{rk} be a Raynaud bundle constructed above. We remark that for any unstable E of slope μ and rank rk we have $\hom(R_\mu^{\mathrm{rk}}, E') - \mathrm{ext}^1(R_\mu^{\mathrm{rk}}, E') > 0$ for all destabilizing $E' \subset E$ (see Lemma 2.9). Suppose R_μ^{rk} is not stable. Then we have a surjection to a stable bundle $R_\mu^{\mathrm{rk}} \to \overline{R_\mu^{\mathrm{rk}}}$ with $\mu(R_\mu^{\mathrm{rk}}) \geq \mu(\overline{R_\mu^{\mathrm{rk}}})$. The last inequality implies $\hom(\overline{R_\mu^{\mathrm{rk}}}, E') - \mathrm{ext}^1(\overline{R_\mu^{\mathrm{rk}}}, E') > 0$ for all E' as above. Since $\mathrm{Hom}(R_\mu^{\mathrm{rk}}, E) = 0$ for all semistable E, we deduce that $\mathrm{Hom}(\overline{R_\mu^{\mathrm{rk}}}, E) = 0$. As a consequence we note:

Proposition 2.13 *There are stable Raynaud bundles R_μ^{rk}.*

Remark. The semicontinuity Theorem (III.12.8 in [7]) implies that semistability is an open condition. Indeed, take any vector bundle R and define R-semistability of E by the condition $\mathrm{Hom}(R, E) = 0$. From the semicontinuity Theorem we deduce that R-semistability is an open condition in flat families.
Question. What is the smallest possible rank for a Raynaud bundle R_μ^{rk}? As we see in Section 3, there can be constructed Raynaud bundles of smaller rank. However, these bundles still have huge rank as we can see in the small table after Corollary 3.4. It is the author's believe that these ranks are still far from being optimal.

3 Base points of $|R \cdot \Theta|$ on $\mathrm{U}_X(r, r(g-1))$

Throughout this section 3 E is a coherent sheaf of rank r and slope $\mu(E) = g-1$. It turns out that in this case we can construct vector bundles $S_R^r(M_0)$ with the same property as the bundle $S_{(g-1),R}^r$ given in Proposition 2.8, but having a significantly smaller rank.

3.1 A Raynaud bundle for $\mu = g - 1$

Let us fix the notation: We consider a smooth projective curve X of genus $g \geq 2$ over an algebraically closed field k. Furthermore, we fix a natural number $R \geq 2$.

Lemma 3.1 *There exists a short exact sequence of vector bundles on X*

$$0 \to M_1 \xrightarrow{\phi} M_0 \xrightarrow{\psi} F \to 0$$

with the following properties:
(i) F *is stable with* $\mathrm{rk}(F) = R$, *and* $\det(F) \cong \mathcal{O}_X$.
(ii) M_0 *is stable with* $\mathrm{rk}(M_0) = R + 1$, *and* $\deg(M_0) = (R+1)(1-g) - 1$.
(iii) $\mathrm{Ext}^1(M_0, F) = 0$.

Proof: Considering all triples $M_0 \xrightarrow{\psi} F$ we see that there exist surjections $M_0 \xrightarrow{\psi} F$ with the given numerical invariants and F stable (see Proposition 7.3 and Theorem 7.7 in [6]). Take a pair $(\tilde{M}_0, F)$ of stable sheaves with $\det(F) \cong \mathcal{O}_X$. $\mathrm{rk}(F) = R$, $\deg(\tilde{M}_0) = (R+1)(1-g)$, $\mathrm{rk}(\tilde{M}_0) = R + 1$ such that $H^*(F \otimes \tilde{M}_0^\vee) = 0$. The existence of such a pair is well known (cf. Beauville's survey article [3]). The stability of $\tilde{M}_0$, and $\mu(\tilde{M}_0) \in \mathbb{Z}$ imply that for any surjection $\pi : \tilde{M}_0 \to k(P)$ the kernel M_0 is also stable. From the short exact sequence $0 \to M_0 \to \tilde{M}_0 \to k(P) \to 0$, and $H^*(F \otimes \tilde{M}_0^\vee) = 0$ we deduce that M_0 satisfies (ii) and (iii). Since the properties (i)–(iii) are open properties on the irreducible moduli space of triples $M_0 \xrightarrow{\psi} F$ (again Theorem 7.7 in [6]) we deduce the claim. □

Notation. From now on we take fixed vector bundles M_1 and M_0 from a short exact sequence $0\to M_1 \stackrel{\phi_0}{\to} M_0 \stackrel{\psi_0}{\to} F \to 0$ like in Lemma 3.1. Compare the following result with Lemma 2.1.

Lemma 3.2 *Let E be a semistable vector bundle of slope $\mu(E)=g-1$. Furthermore, we fix an integer $R\geq 2$, and the vector bundles M_1 and M_0 as above. Then the following three conditions are equivalent:*

(i) *E is not a base point of $|R\cdot\Theta|$.*

(ii) *For some morphism $M_1 \stackrel{\phi}{\to} M_0$ we have $H^0(E\otimes \mathrm{coker}(\phi))=0$.*

(iii) *For some morphism $M_1 \stackrel{\phi}{\to} M_0$ the resulting morphism $H^1(E\otimes M_1)\to H^1(E\otimes M_0)$ is injective.*

Proof: The implications (iii) $\iff$ (ii) $\implies$ (i) follow like in the proof of 2.1. The problem with (i) $\implies$ (ii) is that not all semistable vector bundles F of rank R and determinant $\mathcal{O}_X$ are quotients of M_0. Applying $\mathrm{Hom}(-,F)$ to the short exact sequence of lemma 3.1 yields the long exact sequence

$$\to \mathrm{Hom}(M_1,F) \stackrel{\alpha}{\to} \mathrm{Ext}^1(F,F)\to \mathrm{Ext}^1(M_0,F)\to \mathrm{Ext}^1(M_1,F)\to 0$$

The consequences of the vanishing of $\mathrm{Ext}^1(M_0,F)$ (see Lemma 3.1 (iii)) we express in terms of the Quot scheme $\mathrm{Quot}=\mathrm{Quot}(M_0)_X^{R,0}$ of rank R, degree zero quotients of M_0. First we conclude, that $\mathrm{Ext}^1(M_1,F)=0$. This is the obstruction space of the Quot scheme at $[\psi]=[\psi : M_0\to F]\in \mathrm{Quot}(k)$. Thus, there exists a smooth open neighborhood U of $[\pi]$ which parameterizes semistable vector bundles.
Secondly we deduce surjectivity of α. This is the tangent map at $[\psi]$ of the morphism $U\to \mathrm{U}_X(R,0)$ from U to the moduli space of rank R bundles of degree zero. Passing to a smaller open subset of U we may assume that $U\to \mathrm{U}_X(R,0)$ is a smooth morphism. The image V of U is open and contains a vector bundle with trivial determinant. We conclude, that a dense open subset $V_{\mathcal{O}_X}$ of the moduli space $\mathrm{SU}_X(R,\mathcal{O}_X)$ of rank R bundles with trivial determinant is parameterized by points of our Quot scheme.
Now assume (i). Thus, there exists a vector bundle F of rank R with trivial determinant, such that $H^*(X,E\otimes F)=0$. Thus, the vector bundles G parameterized by $\mathrm{SU}_X(R,\mathcal{O}_X)$ with $h^1(E\otimes G)\neq 0$ form a divisor which can not contain the open set $V_{\mathcal{O}_X}$. This shows that (ii) holds. $\square$

Now we set $V:=\mathrm{Hom}(M_1,M_0)$. Since the difference of the slopes $\mu(M_0)-\mu(M_1)>2g-2$, we have $\mathrm{Ext}^1(M_1,M_0)=0$ and can compute the dimension v of $\mathrm{Hom}(M_1,M_0)$ by Riemann-Roch to be $v=(R+1)(R-1)(g-1)+R$.
We follow the construction in 2.1: We consider the projections $X \stackrel{p}{\longleftarrow} X\times\mathbb{P}(V^\vee)\stackrel{q}{\longrightarrow}\mathbb{P}(V^\vee)$ and the morphism $\alpha : p^*M_1 \to p^*M_0\otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1)$ to obtain for any $m\geq 0$ the bundle $S_{R,m}(M_0):=p_*(\mathrm{coker}(\alpha)\otimes q^*\mathcal{O}_{\mathbb{P}(V^\vee)}(m))$. We set

$$S^r_R(M_0):=S_{R,w} \quad \text{with } w:=((R-1)(R+1)(g-1)+R-1)(r(R+1)(g-1)+r-1)\,.$$

$$\text{and } S^r(M_0):=S^r_u(M_0) \quad \text{with } u:=\left\lceil\frac{(r+1)^2}{4}\right\rceil, \text{ and } R^r(M_0)=(S^r(M_0))^\vee\,.$$

Theorem 3.3 (Properties of $S_{R,m}(M_0)$, $S^r_R(M_0)$, and $S^r(M_0)$)

(i) *$S_{R,m}(M_0)$, $S^r_R(M_0)$, and $S^r(M_0)$ are vector bundles on X.*

(ii) *The numerical invariants of $S_{R,m}(M_0)$ are given by*

$$\begin{aligned} \deg(S_{R,m}(M_0)) &= ((R+1)(1-g)-1)\tfrac{v-1}{m+1}\tbinom{v+m-1}{m} \\ \mathrm{rk}(S_{R,m}(M_0)) &= \tfrac{Rv+Rm+v-1}{m+1}\tbinom{v+m-1}{m} \\ \mu(S_{R,m}(M_0)) &= \tfrac{((R+1)(1-g)-1)(v-1)}{Rv+Rm+v-1} \end{aligned}$$

where $v := (R+1)(R-1)(g-1)+R$.

(iii) *For $m \geq 0$, and any coherent sheaf E on X we have $H^0(S_{R,m}(M_0) \otimes E) \neq 0$ implies $H^0(S_{R,M}(M_0) \otimes E) \neq 0$ for all $M \geq m$.*

(iv) *For a semistable sheaf E of rank r with $\chi(E) = 0$ we have an equivalence*

$$E \text{ is a base point of } |R \cdot \Theta| \text{ on } \mathrm{U}_X(r, r(g-1)) \iff H^0(S^r_R(M_0) \otimes E) \neq 0.$$

(v) *For a coherent sheaf E of rank r with $\chi(E) = 0$ we have an equivalence*

$$E \text{ is semistable} \iff H^0(S^r(M_0) \otimes E) = 0 \iff \mathrm{Hom}(R^r(M_0), E) = 0.$$

Proof: The results follow straightforward by applying Lemma 3.2 instead of Lemma 2.1. In particular: (i) follows from Corollary 2.7, (ii) from Proposition 2.3, (iii) is Corollary 2.6, Proposition 2.8 gives (iv), and (v) is just Proposition 2.11. □

Corollary 3.4 *The slope $\mu(S_{R,m}(M_0))$ of the vector bundle $S_{R,m}(M_0)$ considered as a function of m is of type $\mu(S_{R,m}(M_0)) = \frac{-a}{Rm+b}$ for positive integers $a, b \in \mathbb{N}$. In particular, we have*

$$\mu(S_{R,m}(M_0)) \geq 1-g \iff m \geq (R-1) + \frac{R-g}{R(g-1)}.$$

We list the rank and the slopes of the Raynaud bundles $R^r(M_0)$ which we obtained for $\mu = g-1$ by the methods of this subsection for $r, g \in \{2, 3, 4\}$.

g	r	$\mathrm{rk}(R^r(M_0))$	$\mu(R^r(M_0))$
2	2	59539855602920	$\frac{50}{313}$
2	3	64175219835983462023160614286	$\frac{54}{659}$
2	4	$5.78978673052 \cdot 10^{106}$	$\frac{486}{13669}$
3	2	48350526022102866304247716226	$\frac{54}{331}$
3	3	$4.88907844550 \cdot 10^{63}$	$\frac{363}{4393}$
3	4	$2.18037666849 \cdot 10^{230}$	$\frac{1734}{48661}$
4	2	1824638833656411997322692606728754378288789766	$\frac{338}{2057}$
4	3	$5.06529456824 \cdot 10^{100}$	$\frac{192}{2317}$
4	4	$1.52141697065 \cdot 10^{364}$	$\frac{3750}{105157}$

These values show that even for small values of g and r the help a computer program (`bc` in my case) is needed to compute the rank and slopes of the Raynaud bundles.

3.2 Base points of $|2 \cdot \Theta|$ on $\mathrm{U}_X(r, r(g-1))$ and of $|\Theta|$ on $\mathrm{SU}_x(r)$

Lemma 3.5 *Let F be a vector bundle of rank r_F and slope $\mu(F) \leq g-1$. If $r_E \geq r_F$ is an integer, then there exists a semistable vector bundle E of rank r_E and slope $\mu(E) = g-1$*

with $\mathrm{Hom}(F,E)\neq 0$. *Moreover, if* $r_E > r_F$ *or* $\mu(F) < g-1$, *then the S-equivalence classes of the bundles* E *with* $\mathrm{Hom}(F,E)\neq 0$ *form a positive dimensional subset in the moduli space* $\mathrm{U}_X(r_E, r_E(g-1))$.

Proof: The proof works by induction on the rank r_F. We take an elementary transformation $0\to F\to\tilde F\to T\to 0$ such that $\tilde F$ is a vector bundle of rank r_F and $\mu(\tilde F)=g-1$. Now we distinguish two cases:
Case 1: $\tilde F$ is stable. In this case we may take $E=\tilde F\oplus E'$ to be a sum of two stable vector bundles of slope $g-1$.
Case 2: If $\tilde F$ is not stable there exists a surjection $\tilde F\to F''$ to a stable bundle F'' with $\mu(F'')\leq g-1$ and $\mathrm{rk}(F'')<\mathrm{rk}(F)$. Thus, by the induction hypothesis we are done.
We remark that for $r_F=1$ we are always in the situation of case 1. The statement about the dimensions is trivial (we may change the determinant of $\tilde F$ by varying the support of T or vary the bundle E'). □

Proposition 3.6 *For any smooth projective curve* X *of genus* $g\geq 2$ *the linear system* $|2\cdot\Theta|$ *on the moduli space* $\mathrm{U}_X(r,r(g-1))$ *has base points for* $r\geq\frac{27g^2-15g+2}{2}$. *Furthermore, the base locus is of positive dimension for* $g>2$ *or* $r>\frac{27g^2-15g+2}{2}$

Proof: The dual vector bundle $(S_2,1(M_0))^\vee$ has slope $\mu((S_2,1(M_0))^\vee)\leq g-1$ by Corollary 3.4 and is of rank $\frac{27g^2-15g+2}{2}$ by Proposition 3.3.(ii). Thus, for all $r\geq\frac{27g^2-15g+2}{2}$ we find semistable vector bundles E with $\mathrm{Hom}((S_2,1(M_0))^\vee,E)\neq 0$. This is equivalent to $H^0(S_2,1(M_0)\otimes E)\neq 0$ and implies by (iii) and (iv) of Proposition 3.3 that E is a base point of $|2\cdot\Theta|$. □

Proposition 3.7 *For any smooth projective curve* X *of genus* $g\geq 2$ *the linear system* $|\Theta|$ *on* $\mathrm{SU}_X(r)$ *has base points for* $r\geq\frac{27g^2-15g+2}{2}$. *The base locus is of positive dimension for* $r>\frac{27g^2-15g+2}{2}$.

Proof: We take a base point $[E]\in\mathrm{U}_X(r,r(g-1))$. There exists a line bundle $M\in\mathrm{Pic}^{1-g}(X)$ such that $\det(E\otimes M)\cong\mathcal{O}_X$. We claim that $E\otimes M$ is a base point of $|\Theta|$ on $\mathrm{SU}_X(r)$. Indeed if it were not a base point, we would have a proper divisor $D\subset\mathrm{Pic}^{g-1}$ such that for all $L\in Pic^{g-1}(X)\setminus D$ we have $H^*(E\otimes M\otimes L)=0$. Take $L\in\mathrm{Pic}^{g-1}(X)$, such that neither L nor $(M^{-2}\otimes L^{-1})$ are in D. Then it follows that $H^*(E\otimes M\otimes(L\oplus(M^{-2}\otimes L^{-1})))=0$. However, $\det(M\otimes(L\oplus(M^{-2}\otimes L^{-1})))\cong\mathcal{O}_X$. Thus it would not be a base point. This proves the claim. □

3.3 Base points of $|R\cdot\Theta|$ on $\mathrm{U}_X(r,r(g-1))$

As in the subsection before we remark that $S_{R,R}(M_0)$ has slope at least $1-g$ (see Corollary 3.4). Thus, by Lemma 3.5 we obtain that for all $r\geq\mathrm{rk}(S_{R,R}(M_0))$ the linear system $|R\cdot\Theta|$ is not base point free on $\mathrm{U}_X(r,r(g-1))$. Moreover, if $g\geq R$, then this holds for all $r\geq\mathrm{rk}(S_{R,R-1}(M_0))$.

In the table we have computed for small R and g the minimal ranks r for which $\mathrm{U}_X(r,r(g-1))$ is known to have base points by our method.

	g=2	g=3	g=4	g=5
R=2	40	100	187	301
R=3	3718	5130	14238	30450
R=4	160930	2443665	1332800	3786640

The big values of r explain why we do not include an explicit formula in the next corollary. However, the interested reader can extract the rank using Theorem 3.3 (ii).

Corollary 3.8 *For any* $R\geq 2$ *there exists a polynomial* p_R *of degree* R *such that for all* $r\geq p_R(g)$ *the linear system* $|R\cdot\Theta|$ *on* $\mathrm{U}_X(r,r(g-1))$ *is not base point free.* □

References

[1] D. Arcara, *A lower bound for the dimension of the base locus of the generalized theta divisor*, Bull. Inst. Math. Acad. Sin. (N.S.) **2** no. 1 (2007) 1–27.

[2] A. Beauville, *Vector bundles on curves and generalized theta functions: recent results and open problems*, in H. Clemens (ed.) et al., *Current topics in complex algebraic geometry*, Cambridge Univ. Press, New York, 1995.

[3] A. Beauville, *Vector bundles on curves and theta functions*, in: Moduli spaces and arithmetic geometry, Adv. Stud. Pure Math. **45** Math. Soc. Japan, Tokyo, 2006, 145–156.

[4] P. Belkale, *The strange duality conjecture for generic curves*, J. Amer. Math. Soc. **21** no. 1 (2008) 235–258.

[5] P. Belkale, *Strange duality and the Hitchin/WZW connection*, `math.AG/0705.0717`.

[6] S. Bradlow, O. Garcia-Prada, and P. Gothen, *Moduli spaces of holomorphic triples over compact Riemann surfaces*, Math. Ann. **328** (2004) 299–351.

[7] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977.

[8] G. Hein and D. Ploog, *Stable bundles and Fourier-Mukai transforms on elliptic curves*, Contributions to Algebra and Geometry **46** no. 2 (2005) 423-434.

[9] G. Hein and D. Ploog, *Postnikov-Stability for Complexes*, `math.AG/0704.2512`.

[10] G. Hein, *Raynaud's vector bundles and base points of the generalized Theta divisor*, Mathematische Zeitschrift **257** (2007) 597–611.

[11] A. Marian, and D. Oprea, *The level-rank duality for non-abelian theta functions*, Invent. Math. **168** no. 2 (2007) 225–247.

[12] C. Pauly, *On the base locus of the linear system of generalized theta functions*, `math.AG/0707.3364`.

[13] M. Popa, *On the base locus of the generalized theta divisor*, Comptes Rendus de l'Academie des Sciences de Paris **329** (1999) 507–512.

[14] M. Popa, *Dimension estimates for Hilbert schemes and effective base point freeness on moduli spaces of vector bundles on curves*, Duke Math J. **107** (2001) 469–495.

[15] M. Popa, *Generalized Theta Linear Series on Moduli Spaces of Vector Bundles on Curves*, `math.AG/0712.3192`.

[16] M. Raynaud, *Section des fibrés vectoriels sur une courbe*, Bull. Soc. math. France **110** (1982) 103–125.

[17] O. Schneider, *Sur la dimension de l'ensemble des points base du fibré déterminant sur* $\mathrm{SU}_C(r)$, Ann. Inst. Fourier (Grenoble) **57** no.2 (2007) 481–490.

Georg Hein, Universität Duisburg-Essen, Fachbereich Mathematik, D-45117 Essen, Germany
email: `georg.hein@uni-due.de`