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Reflexive Modules Over Rational Double Points

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0. Introduction

(0.0) This note is an appendix to the article of the same title by Artin and Verdier [1], whose main content is the following.

Let $(X, 0)$ be a germ of a rational double point over an algebraically closed field k, let $\pi: \tilde{X} \to X$ be the minimal desingularization and let E be the exceptional locus. There is a one-to-one correspondence $[M_i] \leftrightarrow E_i$ between isomorphism classes of non trivial indecomposable reflexive modules M_i on $(X, 0)$ and the irreducible components E_i of E: For such a module M_i let $\mathcal{M}_i$ be the locally free sheaf on $\tilde{X}$ defined by $\mathcal{M}_i := \pi^*(M_i)/\text{torsion}$. If r_i is the rank of M_i one has $\left(\bigwedge^{r_i} \mathcal{M}_i\right) \cdot E_j = \delta_{ij}$ and $r_i = \left(\bigwedge^{r_i} \mathcal{M}_i\right) \cdot Z$, where Z denotes the fundamental cycle of $\tilde{X}$.

(0.1) We assume moreover that $(X, 0)$ is a germ of a *quotient singularity*. This means that there exists a cover $q: (\mathbf{A}_k^2, 0) \to (X, 0)$ with Galois group $G \subset SL(2, k)$ whose order is prime to the characteristic of k. This is always the case in characteristic 0.

(0.2) For each M_i as in (0.0) define N_i to be the reflexive hull of $M_i \otimes \Omega_X^1$, where $\Omega_X^1 = i_* \Omega_{(X-0)}^1$, $i: X - \{0\} \to X$ the inclusion.

(0.3) The aim of this note is to prove the *multiplication formula*

Theorem.

$$\bigwedge^{2r_i} \mathcal{N}_i \cong \left(\bigwedge^{r_i} \mathcal{M}_i\right)^{\otimes 2} \otimes \mathcal{O}_{\tilde{X}}(E_i).$$

(0.4) Originally this multiplication formula was proven case by case by Gonzalez-Springberg and Verdier [2]. Recently they computed examples of rational double points in characteristic 2, 3, 5 where this multiplication formula does not hold.

1. Proof of Theorem (0.3)

(1.1) Let α be the Euler differential on $(\mathbf{A}_k^2, 0)$:

$$\alpha = x \cdot dy - y \cdot dx.$$

As α is invariant under $SL(2,k)$ one can think of α as being a differential one-form on X and on $\tilde{X}$ as well. One has the exact sequence

$$(1.1.1)\qquad 0 \longrightarrow \mathcal{O}_{(\mathbf{A}^2,0)} \xrightarrow{\otimes\alpha} \Omega^1_{(\mathbf{A}^2,0)} \xrightarrow{\wedge\alpha} \mathfrak{m}_{(\mathbf{A}^2,0)} \otimes \omega_{(\mathbf{A}^2,0)} \longrightarrow 0,$$

where $\mathfrak{m}$ denotes the maximal ideal and ω the dualizing module. Applying q_* and taking the G-invariants one obtains the exact sequence

$$(1.1.2)\qquad 0 \longrightarrow \mathcal{O}_{(X,0)} \xrightarrow{\otimes\alpha} \Omega^1_{(X,0)} \xrightarrow{\wedge\alpha} \mathfrak{m}_{(X,0)} \otimes \omega_{(X,0)} \longrightarrow 0.$$

(1.2) Let F_i be the non trivial indecomposable representation of G such that $M_i \cong (\mathcal{O}_{(\mathbf{A}^2,0)} \otimes_k F_i)^G$. Tensorize (1.1.1) by F_i:

$$(1.2.1)\qquad 0 \to (\mathcal{O}_{(\mathbf{A}^2,0)} \otimes_k F_i) \to (\Omega^1_{(\mathbf{A}^2,0)} \otimes_k F_i) \to (\mathfrak{m}_{(\mathbf{A}^2,0)} \otimes \omega_{(\mathbf{A}^2,0)} \otimes F_i) \to 0.$$

Apply q_* and take the G-invariant parts. Since the representation F_i is not trivial there are no invariants of degree 0. Therefore one obtains the exact sequence on $(X,0)$:

$$(1.2.2)\qquad 0 \longrightarrow M_i \xrightarrow{\otimes\alpha} N_i \xrightarrow{\wedge\alpha} M_i \longrightarrow 0$$

where N_i is as in (0.2) and we identify $\omega_{(X,0)}$ with $\mathcal{O}_{(X,0)}$.

(1.3) Pulling back (1.1.2) on $\tilde{X}$ one obtains the complex

$$(1.3.1)\qquad 0 \longrightarrow \mathcal{O}_{\tilde{X}} \xrightarrow{\otimes\alpha} \tilde{\Omega} \xrightarrow{\wedge\alpha} \mathcal{O}_{\tilde{X}}(-Z) \longrightarrow 0$$

where $\tilde{\Omega} := \pi^*\Omega^1_{(X,0)}/\text{torsion}$. This complex is exact away from E, and also left and right exact. Let $\mathcal{K}$ be the kernel of $(\wedge\alpha)$ in (1.3.1).

Claim. (i) *$\mathcal{K}$ is locally free.*
(ii) *$\mathcal{K} \cong \mathcal{O}_{\tilde{X}}(R)$ for an effective divisor R supported on E.*

Proof. Since $\mathcal{O}_{\tilde{X}}(-Z)$ is locally free one has the exact sequence

$$0 \leftarrow \mathcal{K}^\vee \leftarrow \tilde{\Omega}^\vee \leftarrow \mathcal{O}_{\tilde{X}}(Z) \leftarrow 0.$$

By definition $\mathcal{K}^\vee$ is reflexive on $\tilde{X}$, and therefore $\mathcal{K}^\vee$ is locally free. Dualizing once again one obtains the exact sequence

$$0 \to \mathcal{K}^{\vee\vee} \to \tilde{\Omega} \to \mathcal{O}_{\tilde{X}}(-Z) \to 0,$$

and therefore $\mathcal{K} = \mathcal{K}^{\vee\vee}$ is locally free. Since the inclusion $\mathcal{O}_{\tilde{X}} \hookrightarrow \mathcal{K}$ is an isomorphism outside of E one has $\mathcal{K} \cong \mathcal{O}_{\tilde{X}}(R)$ for an effective R supported on E.

(1.4) **Claim.** *The complex* (1.3.1) *is exact.*

Proof. From (1.3) one obtains

$$\bigwedge^2 \tilde{\Omega} \cong \mathcal{O}_{\tilde{X}}(-Z+R).$$

Since $\left(\bigwedge^2 \tilde{\Omega}\right) \cdot E_l$ is non-negative for all l and $(-Z)$ is the largest vertical divisor intersecting each E_l non negatively one has $R=0$, or $R=Z$. Assume $R=Z$. Then by the theorem of Artin-Verdier (0.0), one has $\Omega^1_{\tilde{X}} = \mathcal{O}_X \oplus \mathcal{O}_X$. Therefore $\mathfrak{m}_{(X,0)}$ has two generators by (1.1.2), and $(X,0)$ has to be smooth.

(1.5) We now pull (1.2.2) back to $\tilde{X}$. One obtains the complex

$$0 \longrightarrow \mathcal{M}_i \xrightarrow{\otimes\alpha} \mathcal{N}_i \xrightarrow{\wedge\alpha} \mathcal{M}_i \longrightarrow 0. \tag{1.5.1}$$

This complex is left exact since $\mathcal{M}_i$ is torsion free, and it is right exact as π^* is. In addition the complex is exact away from E. Let $\mathcal{K}_i$ be the kernel of $(\wedge\alpha)$ in (1.5.1).

Claim. *$\mathcal{K}_i$ is locally free.*

The proof is the same as for (1.3).

(1.6) **Claim.** $\left(\bigwedge^{2r_i}\mathcal{N}_i\right) \cong \left(\bigwedge^{r_i}\mathcal{M}_i\right) \otimes \left(\bigwedge^{r_i}\mathcal{M}_i\right) \otimes \mathcal{O}_{\tilde{X}}(R_i)$ *for an effective divisor R_i supported on E.*

Proof. One has

$$\left(\bigwedge^{2r_i}\mathcal{N}_i\right) \cong \left(\bigwedge^{r_i}\mathcal{M}_i\right) \otimes \left(\bigwedge^{r_i}\mathcal{K}_i\right)$$

and the inclusion $\mathcal{M}_i \hookrightarrow \mathcal{K}_i$ is an isomorphism outside of E.

(1.7) **Claim.** *One has $R_i = E_i + R_i'$ for an effective divisor R_i' supported on E.*

Proof. Assume that R_i does not contain E_i. Then $R_i \cdot E_i \geqq 0$. Since $\left(\bigwedge^{r_i}\mathcal{M}_i\right) \cdot E_j = 0$ for $j \neq i$ one has $R_i \cdot E_j = \left(\bigwedge^{2r_i}\mathcal{N}_i\right) \cdot E_j \geqq 0$ for $i \neq j$. Therefore $R_i^2 \geqq 0$. As the intersection matrix of E is negative definite one gets $R_i = 0$. Therefore $\left(\bigwedge^{2r_i}\mathcal{N}_i\right) \cong \left(\bigwedge^{r_i}\mathcal{M}_i\right)^{\otimes 2}$, and by the theorem of Artin-Verdier (0.0) we obtain

$$\mathcal{N}_i \cong \mathcal{M}_i \oplus \mathcal{M}_i.$$

Restrict this isomorphism to $U := X - \{0\} = \tilde{X} - E$ and tensor with $M_i^\vee|_U$. One obtains

$$\operatorname{End}(M_i)|_U \otimes \Omega_U^1 \cong \operatorname{End}(M_i)|_U \otimes (\mathcal{O}_U \otimes \mathcal{O}_U). \tag{1.7.1}$$

The trace map $\operatorname{End}(M_i)|_U \to \mathcal{O}_U$ defines a natural splitting since the characteristic of k is prime to the order of G and hence to the rank r_i of M_i. Taking traces on both sides of (1.7.1) one gets

$$\Omega_U^1 \cong \mathcal{O}_U \oplus \mathcal{O}_U$$

which contradicts (1.4).

(1.8) By the normal basis theorem one has

$$q_* \mathcal{O}_{(\mathbf{A}^2, 0)} \cong \mathcal{O}_{(X,0)} \oplus \bigoplus_i r_i M_i$$

and

$$q_* \Omega^1_{(\mathbf{A}^2, 0)} \cong \Omega^1_{(X,0)} \oplus \bigoplus_i r_i N_i.$$

As $\Omega^1_{(\mathbf{A}^2,0)}$ is isomorphic to $\mathcal{O}_{(\mathbf{A}^2,0)} \oplus \mathcal{O}_{(\mathbf{A}^2,0)}$ as an $\mathcal{O}_{(\mathbf{A}^2,0)}$-module one has

$$\Omega^1_{(X,0)} \oplus \bigoplus_i r_i N_i \cong 2\mathcal{O}_{(X,0)} \oplus 2 \bigoplus_i r_i M_i. \tag{1.8.1}$$

(1.9) Pull (1.8.1) back to $\tilde{X}$ and apply (1.7) and (1.4). This gives

$$\mathcal{O}_{\tilde{X}}(-Z+\sum r_i(E_i+R_i'))\otimes \bigotimes_i \Big(\bigwedge^{r_i} \mathcal{M}_i\Big)^{\otimes 2r_i} \cong \bigotimes_i \Big(\bigwedge^{r_i} \mathcal{M}_i\Big)^{\otimes 2r_i}.$$

Therefore $\mathcal{O}_{\tilde{X}}(\sum r_i R_i')\cong \mathcal{O}_{\tilde{X}}$. As each R_i' is effective we have $R_i'=0$ for all i, and the theorem (0.3) is proven.

References

1. Artin, M., Verdier, J.-L.: Reflexive modules over rational double points. Math. Ann. **270**, 79–82 (1985)
2. Gonzalez-Sprinberg, G., Verdier, J.-L.: Construction géométrique de la correspondance de McKay. Ann. Sci. Ec. Norm. Supér. IV Ser. **16**, 409–449 (1983)

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Note added in proof. M. Auslander and I. Reiten recently showed that the Auslander-Reiten-quiver of any rational double point X over an algebraically closed field is an extended Dynkin quiver. This implies that (1.8.1) holds if one replaces Ω_X^1 by the unique non trivial extension of m_X by ω_X and N_i by the middle term of the almost split exact sequence starting and ending with M_i. Using almost split exact sequences instead of (1.2.2) one can proceed as above to show that the multiplication formula (0.3) holds with these modified definitions, which coincide with the ones given in this note if X is a quotient singularity.