

NOTES ON DELIGNE'S LETTER TO DRINFELD  
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ABSTRACT. We give a detailed account of Deligne's letter to Drinfeld dated March 5, 2007 in which he shows that irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -sheaves with finite determinant on a normal scheme of finite type over  $\mathbb{F}_p$  have local characteristic polynomials in  $E[t]$ , where  $E$  is a number field, answering thereby his own conjecture [7, Conj. 1.2.10 (ii)]. The proof relies on Lafforgue's results for curves. We also explain the motivic background of Deligne's conjectures.

1. DELIGNE'S THEOREM

**1.1. Statement of Deligne's theorem.** Let  $X$  be a normal scheme, separated and of finite type over  $\mathbb{F}_q$ . For a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  we define a function  $f_V$  on the set of closed points  $|X|$  by

$$f_V : |X| \rightarrow \bar{\mathbb{Q}}_\ell[t], \quad f_V(x) = \det(1 - t F_x, V_{\bar{x}}).$$

Let  $E(V)$  be the subfield of  $\bar{\mathbb{Q}}_\ell$  generated by the coefficients of the polynomials  $f_V(x)$  for  $x \in |X|$ . By [5, Prop. IV.6.4.3] this is also the field generated by the traces

$$t_V^n(x) = \mathrm{Tr}(F_x, V_{\bar{x}})$$

for  $x \in X(\mathbb{F}_{q^n})$ ,  $n \geq 1$ . Here  $F_x = F_{x_0}^{n/d}$  if the image  $x_0$  of  $x : \mathrm{Spec} \mathbb{F}_{q^n} \rightarrow X$  has degree  $d$  dividing  $n$ .

We use the terminology of *lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaves* which correspond to continuous  $\bar{\mathbb{Q}}_\ell$ -representations of the Weil group [7, Déf. 1.1.10], and of *lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaves* which correspond to continuous representations of the fundamental group.

In [7, Conjecture 1.2.10] Deligne conjectured the following.

**Theorem 1.1.** *For an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaf  $V$  on  $X$  with determinant of finite order,  $E(V)$  is a number field.*

**Remark 1.2.** An irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  with determinant of finite order is étale, see [7, Prop. 1.3.14].

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We will see that this theorem is equivalent to the following theorem, the formulation of which is better suited for pullback arguments.

**Theorem 1.3.** *For a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  which is pure of weight 0,  $E(V)$  is a number field.*

## NOTATION

Let  $\mathbb{F}_q$  be the finite field with  $q$  elements,  $q$  a power of the prime  $p$ . We fix an algebraic closure  $\mathbb{F}$  of  $\mathbb{F}_q$ . By  $\ell$  we denote a prime different from  $p$ . All schemes are separated and of finite type over some specified field. For a scheme  $X$  over a field we denote by  $|X|$  for the set of closed points of  $X$ .

By a curve we mean a smooth quasi-projective connected scheme of dimension 1 over a field. Let  $x \in X(\mathbb{F})$ . Let  $\pi_1(X, x)$  be Grothendieck's fundamental group based at  $x$ . It maps to  $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$ . Let  $W(X, x)$  be the inverse image in  $\pi_1(X, x)$  of  $\mathbb{Z} \cdot F \subset \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ , where  $F$  is the Frobenius of  $\mathbb{F} \supset \mathbb{F}_q$ . Then  $W(X, x) = \pi_1(X, x) \times_{\text{Gal}(\mathbb{F}/\mathbb{F}_q)} \mathbb{Z} \cdot F$ . The Weil group based at  $x$  is the group  $W(X, x)$  endowed with the product topology [7, 1.1.7]. We denote by  $\pi_1$  the functor from the category of connected schemes to the category of pro-finite groups modulo inner automorphisms and by  $W$  the functor  $W(-) = \pi_1(-) \times_{\text{Gal}(\mathbb{F}/\mathbb{F}_q)} \mathbb{Z} \cdot F$ . It is a functor from the category of connected schemes to the category of topological groups.

For a connected scheme  $X$  we identify the set of isomorphism classes of lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaves with the set of isomorphism classes of continuous representations of  $W(X)$  to finite dimensional  $\bar{\mathbb{Q}}_\ell$  vector spaces, and the set of isomorphism classes of lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaves with the set of isomorphism classes of continuous representations of  $\pi_1(X)$  to finite dimensional  $\bar{\mathbb{Q}}_\ell$  vector spaces.

Let  $X$  be a scheme over  $\mathbb{F}_q$  and  $V$  a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf on  $X$ . We use the notations  $f_V, t_V^n$  introduced in Section 1.1.

Let  $E(V)$  be the subfield of  $\bar{\mathbb{Q}}_\ell$  generated by the coefficients of the polynomials  $f_V(x)$  for all  $x \in |X|$ . Note that by [5, Prop. IV.6.4.3]  $E(V)$  is also the subfield of  $\bar{\mathbb{Q}}_\ell$  generated by the  $t_V^n(x)$  for  $n \geq 1$  and  $x \in X(\mathbb{F}_{q^n})$ .

## 2. MOTIVIC DREAM

We try to explain as well as we can Grothendieck's programmatic dream about the existence of pure isomotives over general bases as sketched in his letter of to Illusie dated May 3, 1973, published in [14, Appendix]. We formulate some precise expectations. Nothing here is

logically needed for later sections and any inaccuracies are entirely due to the authors.

Let  $k$  be a perfect base field. We assume Grothendieck's standard conjectures over all extension fields  $K \supset k$ , see [2, Sec. 5].

Let  $\text{Sch}/k$  be the category of normal schemes separated and of finite type over  $k$ . We expect that for any  $S \in \text{Sch}/k$  there is a graded semi-simple  $\mathbb{Q}$ -linear rigid abelian  $\otimes$ -category  $M(S, \mathbb{Q})$ , with  $\text{End}(\mathbf{1}) = \mathbb{Q}$  if  $S$  is connected.

The following properties should be satisfied.

- (1) The categories  $M(S, \mathbb{Q})$  form an étale stack over  $\text{Sch}/k$ , in particular there are  $\otimes$ -functors

$$f^* : M(S, \mathbb{Q}) \rightarrow M(S', \mathbb{Q})$$

for a morphism  $f : S' \rightarrow S$ .

- (2) For any prime number  $\ell$  different from  $\text{char}(k)$ , there is a faithful  $\mathbb{Q}_\ell$ -linear  $\otimes$ -functor

$$R_\ell : M(-, \mathbb{Q}) \otimes \mathbb{Q}_\ell \rightarrow \text{Sh}(-, \mathbb{Q}_\ell),$$

where  $S \mapsto \text{Sh}(S, \mathbb{Q}_\ell)$  is the étale stack of lisse  $\mathbb{Q}_\ell$ -étale sheaves over  $\text{Sch}/k$ .

- (3) There is a contravariant functor from the category of smooth projective schemes  $f : X \rightarrow S$  to motives  $\mathfrak{h}(X) \in M(S, \mathbb{Q})$  such that

$$R_\ell \circ \mathfrak{h}(X) \cong \bigoplus_n R^n f_* \mathbb{Q}_\ell.$$

For a field  $K \supset k$  we define

$$M(\text{Spec } K, \mathbb{Q}) = 2\text{-}\varinjlim_U M(U, \mathbb{Q})$$

where  $U$  runs through all normal connected affine schemes of finite type over  $k$  with  $k(U) \subset K$ .

- (4) For a perfect field  $K \supset k$  the category  $M(\text{Spec } K, \mathbb{Q})$  is the same as Grothendieck's category, see [2, Sec. 6.1].
- (5) For any connected  $S \in \text{Sch}/k$  there is a 2-Cartesian square

$$\begin{array}{ccc} M(S, \mathbb{Q}) & \longrightarrow & M(\eta, \mathbb{Q}) \\ R_\ell \downarrow & & \downarrow R_\ell \\ \text{Sh}(S, \mathbb{Q}_\ell) & \longrightarrow & \text{Sh}(\eta, \mathbb{Q}_\ell) \end{array}$$

where  $\eta$  is the generic point of  $S$ .

(6) The pullback along the relative Frobenius

$$F_{S/k}^* : M(S^{(p)}, \mathbb{Q}) \rightarrow M(S, \mathbb{Q})$$

is an equivalence of categories.

For the rest of this section we assume the existence of  $M(S, \mathbb{Q})$  with the above properties.

For any field  $F \supset \mathbb{Q}$  define  $M(S, F)$  to be the pseudo-abelian envelope of  $M(S, \mathbb{Q}) \otimes F$ . By [15, Lem. 2]  $M(S, F)$  is a semi-simple abelian category.

For connected  $S$  the category  $M(S, F)$  is tannakian. Using the tannakian formalism and a theorem of Deligne [10, Cor. 6.20] one deduces

**Lemma 2.1.**

- (i) For any  $F \supset \mathbb{Q}$  the categories  $M(S, F)$  form an étale stack over  $\text{Sch}/k$ .
- (ii) For connected  $S$  with generic point  $\eta \in S$  and for fields  $\mathbb{Q} \subset F \subset F'$  the square

$$\begin{array}{ccc} M(S, F) & \longrightarrow & M(\eta, F) \\ \downarrow & & \downarrow \\ M(S, F') & \longrightarrow & M(\eta, F') \end{array}$$

is 2-Cartesian.

- (iii) For fields  $F \subset F'$  of characteristic 0 the functor

$$M(S, F) \rightarrow M(S, F')$$

induces an injection of isomorphism classes of objects. For algebraically closed  $F$  it is bijective on isomorphism classes.

Let  $k = \mathbb{F}_q$  be a finite field, then the Tate conjecture can be formulated as follows.

**Conjecture 2.2.** The functors

$$R_\ell : M(S, \mathbb{Q}_\ell) \rightarrow \text{Sh}(S, \mathbb{Q}_\ell)$$

are fully faithful.

Concerning the image of  $R_\ell$ , Deligne suggests:

**Conjecture 2.3.** The essential image of

$$R_\ell : M(S, \bar{\mathbb{Q}}_\ell) \rightarrow \text{Sh}(S, \bar{\mathbb{Q}}_\ell)$$

consists of direct sums of irreducible sheaves  $V$  which are pure of integral weight, and such that the eigenvalues of  $F_x$  for all closed points  $x \in |X|$  are  $\ell'$ -adic units for all prime numbers  $\ell' \neq p$ .

Conjecture 2.3 and Lemma 2.1(iii) motivate Deligne's Theorem 1.1 as well as the other parts of Deligne's conjecture [7, Conj. 1.2.10] (except the  $p$ -adic part (vi)).

One can also give a motivic interpretation of the field  $E(V)$  for  $V$  in the essential image of  $R_\ell$ .

**Definition 2.4.** For  $P \in M(S, \bar{\mathbb{Q}})$  let  $H \subset \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  consist of those  $h \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  with  $h(P) \cong P$ . Define the number field  $E(P)$  to be  $\bar{\mathbb{Q}}^H$ .

**Lemma 2.5.** *Conjecture 2.2 implies that for  $P \in M(S, \bar{\mathbb{Q}})$  one has  $E(P) = E(R_\ell(P))$ .*

### 3. $\ell$ -ADIC SHEAVES

**3.1. Implications of Langlands.** Lafforgue deduced Deligne's conjecture for curves from the Langlands correspondence for  $\text{GL}_r$  of function fields [18, Chap. VII]. Let  $X/\mathbb{F}_q$  be a smooth quasi-projective connected scheme of dimension 1 (we say curves).

For the reader's convenience we recall Lafforgue's results here. Let  $\ell \neq p = \text{char}(\mathbb{F}_q)$  be a prime number.

**Theorem 3.1.** *For an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  with determinant of finite order the following holds:*

- (i) *The field  $E(V)$  is a finite extension of  $\mathbb{Q}$ .*
- (ii) *For a dense open subscheme  $X' \subset X$  we have  $E(V|_{X'}) = E(V)$ .*
- (iii) *For an arbitrary, not necessarily continuous, automorphism  $\sigma \in \text{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ , there is an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V_\sigma$  on  $X$ , called  $\sigma$ -companion, with determinant of finite order such that*

$$f_{V_\sigma} = \sigma(f_V),$$

*where  $\sigma$  acts on the polynomial ring  $\bar{\mathbb{Q}}_\ell[t]$  by  $\sigma$  on  $\bar{\mathbb{Q}}_\ell$  and by  $\sigma(t) = t$ .*

- (iv)  *$V$  is pure of weight 0.*

*Proof.* Except for (ii) the theorem is contained in [18, Théorème VII.6]. We deduce (ii) from (i) and (iii). Clearly  $E(V|_{X'}) \subset E(V)$ . For a  $\sigma \in \text{Aut}(\bar{\mathbb{Q}}_\ell/E(V|_{X'}))$  we consider  $V_\sigma$  as in (iii). It is sufficient to show that  $V_\sigma \cong V$ . As the Weil group of  $X'$  surjects onto the one of  $X$ , this is equivalent to  $V|_{X'} = V_\sigma|_{X'}$ , which follows from

$$f_V|_{X'} = \sigma f_V|_{X'} = f_{V_\sigma}|_{X'}$$

and Chebotarev density theorem. □

From Lafforgue's theorem one can deduce certain results on higher dimensional schemes. Let  $X$  be a normal scheme geometrically connected, separated and of finite type over  $k$ .

**Corollary 3.2.** *(of Theorem 3.1 (iv)). For an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  the following are equivalent:*

- (i)  $V$  is pure of weight 0,
- (ii) there is a closed point  $x \in X$  such that  $V_{\bar{x}}$  is pure of weight 0,
- (iii) there is a one-dimensional lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $W$  on  $\text{Spec}(\mathbb{F}_q)$  which is pure of weight 0 such that the determinant  $\det(V \otimes W)$  is of finite order.

*Proof.* (iii)  $\Rightarrow$  (i):

Without loss of generality we can assume that  $\det(V)$  is of finite order. Then by Remark 1.2  $V$  is étale. For a closed point  $x \in X$  choose a curve  $C/k$  and a morphism  $\psi : C \rightarrow X$  such that  $x$  is in the set theoretic image of  $\psi$  and such that  $\psi^*V$  is irreducible. A proof of the existence of such a curve is given in the appendix, Proposition 8.1. Then by Theorem 3.1(iv)  $\psi^*V$  is pure of weight 0 on  $C$ , so  $V_{\bar{x}}$  is also pure of weight 0.

(i)  $\Rightarrow$  (ii): Trivially.

(ii)  $\Rightarrow$  (iii):

Set  $r = \text{rank}(V)$ . Choose a one-dimensional lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $W$  on  $\text{Spec}(\mathbb{F}_q)$  such that  $(W|_{k(x)})^{\otimes r} \cong \det(V_{\bar{x}})^\vee$ . By the Katz-Lang finiteness theorem, see for example [7, Prop. 1.3.4], it follows that the determinant  $\det(V \otimes W)$  has finite order.

□

**Corollary 3.3.** *(of Theorem 3.1 (ii)) Assume  $X/\mathbb{F}_q$  is smooth. For an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  and for a dense open subscheme  $X' \subset X$  we have  $E(V|_{X'}) = E(V)$ .*

*Proof.* We have to show that  $f_V(x) \in E(V|_{X'})[t]$  for any  $x \in |X|$ . By [16] we can find a smooth curve  $C/\mathbb{F}_q$ , a morphism  $C \rightarrow X$  and a scheme theoretic splitting  $x \rightarrow C$ . Furthermore we can assume that  $C \times_X X' \neq \emptyset$ . From Theorem 3.1(ii) we get the equality in

$$f_V(x) \in E(V|_C)[t] = E(V|_{C \times_X X'})[t] \subset E(V|_{X'})[t].$$

□

**Corollary 3.4.** (of Theorem 3.1 (iii)) Assume  $\dim(X) = 1$ . For a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$  and an automorphism  $\sigma \in \text{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$ , there is a  $\sigma$ -companion to  $V$ , i.e. a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V_\sigma$  on  $X$  such that

$$f_{V_\sigma} = \sigma(f_V).$$

*Proof.* Without loss of generality we may assume that  $V$  is irreducible. In the same way as in the proof of Corollary 3.2 we find a one-dimensional lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $W$  on  $\text{Spec}(\mathbb{F}_q)$  such that  $V \otimes_{\bar{\mathbb{Q}}_\ell} W$  has determinant of finite order. As the lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $W$  of rank 1 over  $\text{Spec} \mathbb{F}_q$  is just given by the image  $\lambda$  say of the Frobenius  $F$  of the Galois group of  $\mathbb{F}_q$  in  $\bar{\mathbb{Q}}_\ell^\times$ , we trivially construct a  $\sigma$ -companion  $W_\sigma$  to it by sending  $F$  to  $\sigma(\lambda)$ . By Theorem 3.1 (iii), we get a  $\sigma$ -companion  $(V \otimes W)_\sigma$  to  $V \otimes W$ . Then  $(V \otimes W)_\sigma \otimes W_\sigma^\vee$  is a  $\sigma$ -companion to  $V$ .  $\square$

**Remark 3.5.** Drinfeld has shown [11] that Corollary 3.4 stays true for higher dimensional  $X$  if  $X$  is assumed to be smooth. His argument relies on Deligne's Theorem 1.3.

*Proof that Theorem 1.1  $\Leftrightarrow$  Theorem 1.3:*

' $\Rightarrow$ ':

Let  $V$  be as in Theorem 1.3. We may replace  $V$  by its semi-simplification and without loss of generality, we may assume that  $V$  is irreducible. By Corollary 3.2 there is a  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $W$  of rank 1 and of weight 0 on  $\text{Spec} \mathbb{F}_q$  such that  $\det(V \otimes W)$  is of finite order. Since  $W$  has weight 0,  $E(W)$  is a number field. By Theorem 1.1 also  $E(V \otimes W)$  is a number field. It follows that

$$E(V) \subset E(V \otimes W) \cdot E(W)$$

is a number field.

' $\Leftarrow$ ':

Let  $V$  be as in Theorem 1.1. By Corollary 3.2  $V$  is of weight 0. So Theorem 1.3 tells us that  $E(V)$  is a number field.  $\square$

**3.2. Structure of a lisse  $\bar{\mathbb{Q}}_\ell$ -sheaf over a scheme over a finite field.** Let  $X$  be a geometrically connected normal scheme separated and of finite type over  $\mathbb{F}_q$  and let  $V$  be a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf of rank  $r$  on  $X$ .

In case of lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaves the following proposition is shown in [4, Prop. 5.3.9]. The case of Weil sheaves works similarly.

**Proposition 3.6.** Let  $V$  be an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf on  $X$ .

- (i) Let  $m$  be the number of irreducible constituents of  $V_{\mathbb{F}}$ . There is an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V^b$  on  $X_{\mathbb{F}_{q^m}}$  such that

- the pullback of  $V^b$  to  $X_{\mathbb{F}}$  is irreducible,
  - $V \cong b_{m,*}V^b$ , where  $b_m$  is the natural map  $X \otimes \mathbb{F}_{q^m} \rightarrow X$ .
- (ii)  $V$  is pure of weight 0 if and only if  $V^b$  is pure of weight 0.
- (iii) If  $V'$  is another irreducible  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf on  $X$  with  $V'_{\mathbb{F}} \cong V_{\mathbb{F}}$ , then there is a rank 1  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $W$  on  $\text{Spec}(\mathbb{F}_{q^m})$  with
- $$V' \cong b_{m,*}(V^b \otimes W).$$

A special case of the Grothendieck trace formula [19, (1.1.1.3)] says:

**Proposition 3.7.** *Let  $V$  and  $m$  be as in Proposition 3.6. For  $n \geq 1$  and  $x \in X(\mathbb{F}_{q^n})$*

$$t_V^n(x) = \sum_{\substack{y \in X_{\mathbb{F}_{q^m}}(\mathbb{F}_{q^n}) \\ y \mapsto x}} t_{V^b}^n(y).$$

In concrete terms here,  $t_V^n(x) = 0$  if  $m$  does not divide  $n$  and if  $m$  divides  $n$ , then

$$t_V^n(x) = \sum_{\substack{y \in X_{\mathbb{F}_{q^m}}(\mathbb{F}_{q^n}) \\ y \mapsto x}} t_{V^b}^n(y).$$

#### 4. RAMIFICATION THEORY

Because of a lack of a proper reference we review some well known facts from ramification theory, in particular the relation between different and discriminant on normal schemes and a semi-continuity for pullback to curves. The only things from this section that are needed later on are some definitions and Proposition 4.11, a dimension bound for cohomology with compact support on curves.

**4.1. Different and discriminant.** Let  $X$  be a normal noetherian integral scheme. Let  $X' \rightarrow X$  be a finite dominant morphism with  $X'$  integral. We denote by  $K \subset K'$  the corresponding extension of the fields of rational functions. Consider the diagonal morphism  $\phi : X' \rightarrow X' \times_X X'$ . Let  $\mathcal{I} \subset \mathcal{O}_{X' \times_X X'}$  be the coherent ideal sheaf of the diagonal.

The following version of the different was introduced in [3].

**Definition 4.1** (Different). The *homological different* of  $X'$  over  $X$  is defined as the coherent ideal sheaf

$$\text{Diff}_{X'/X} = \phi^{\natural}(\text{Ann}_{\mathcal{O}_{X' \times_X X'}}(\mathcal{I})) \subset \mathcal{O}_{X'}.$$

Here  $\phi^{\natural}$  is the usual pullback of ideal sheaves. Taking the norm we get the coherent ideal sheaf

$$D_{X'/X} = \mathcal{O}_X \text{Nm}_{K'/K}(\text{Diff}_{X'/X}) \subset \mathcal{O}_X.$$



**Definition 4.2** (Discriminant). If  $X' \rightarrow X$  is flat and generically étale we define the *discriminant*  $\Delta_{X'/X}$  to be the invertible ideal sheaf in  $\mathcal{O}_X$  generated locally by

$$\det(\mathrm{Tr}_{K'/K}(x_i x_j))_{i,j}$$

where  $x_1, \dots, x_n$  form a local basis of  $\mathcal{O}_{X'}$  over  $\mathcal{O}_X$ .

**Proposition 4.3.** *If  $X' \rightarrow X$  is flat and generically étale, then:*

- 1) *In codimension 1, one has an inclusion of the ideal sheaves  $\mathcal{D}_{X'/X} \subset \Delta_{X'/X}$ .*
- 2) *If in addition,  $X'$  is normal, then in codimension 1 one has  $\mathcal{D}_{X'/X} = \Delta_{X'/X}$ .*

*Proof.* Without loss of generality, we may assume that  $X$ , and thus  $X'$ , have dimension 1.

Auslander-Buchsbaum ([3, Prop. 3.3]) show that for  $X' \rightarrow X$  flat  $\mathrm{Diff}_{X'/X}$  coincides with the ordinary different  $\mathcal{D}_{X'/X} \subset \mathcal{O}_{X'}$ , which is defined as follows. One defines

$$\mathcal{C}_{X'/X} = \{\beta \in K', \mathrm{Tr}_{K'/K}(\beta \cdot \mathcal{O}_{X'}) \subset \mathcal{O}_X\}.$$

Then

$$\mathcal{D}_{X'/X} = \{\alpha \in K', \alpha \cdot \mathcal{C}_{X'/X} \subset \mathcal{O}_{X'}\}.$$

We prove 1). Let  $\alpha \in \mathcal{D}_{X'/X}$ , thus  $\mathcal{C}_{X'/X} \subset \alpha^{-1} \cdot \mathcal{O}_{X'}$ . By [21, Lemma I.5.3], one has

$$\mathrm{Nm}_{K'/K}(\alpha) \cdot \mathcal{O}_X = \chi_{\mathcal{O}_X}(\mathcal{O}_{X'}/\alpha \cdot \mathcal{O}_{X'}) = \chi_{\mathcal{O}_X}(\alpha^{-1} \cdot \mathcal{O}_{X'}/\mathcal{O}_{X'}).$$

But one has

$$\mathcal{C}_{X'/X}/\mathcal{O}_{X'} \subset \alpha^{-1} \cdot \mathcal{O}_{X'}/\mathcal{O}_{X'},$$

which implies  $\chi_{\mathcal{O}_X}(\alpha^{-1} \cdot \mathcal{O}_{X'}/\mathcal{O}_{X'}) \subset \chi_{\mathcal{O}_X}(\mathcal{C}_{X'/X}/\mathcal{O}_{X'})$ . This finishes the proof of 1).

As for 2), we apply [21, Prop. III.3.6].

□

**4.2. Tameness.** From now on,  $X$  is a normal integral scheme, separated and of finite type over a perfect field  $k$ .

Consider a lisse  $\mathbb{Q}_\ell$ -Weil sheaf  $V$  on  $X$ .

**Definition 4.4.** We say that  $V$  is *tame* if its pullback along any curve  $C \rightarrow X$  is tame in the usual sense.

**Remark 4.5.** One can show, see [17], that for regular  $X$  tameness of  $V$  is a birational invariant and coincides with the Grothendieck-Murre definition of tameness [13, Def. 2.2.2] for a regular compactification with simple normal crossings divisor at infinity.

Let  $\bar{X} \supset X$  be an open immersion with  $\bar{X}$  integral, normal, proper over  $\mathbb{F}_q$ . Let  $D \in \text{Div}^+(\bar{X})$  be an effective Cartier divisor on  $\bar{X}$  supported in  $\bar{X} \setminus X$ .

**Definition 4.6.** We say that *the (wild) ramification of  $V$  is bounded by  $D$*  if there is a connected étale covering  $\phi : X' \rightarrow X$  such that  $\phi^*(V)$  is tame and such that  $\mathcal{O}_{\bar{X}}(-D) \subset D_{\bar{X}'/\bar{X}}$ , where  $\bar{X}'$  is the normalization of  $\bar{X}$  in  $k(X')$ .

**Remark 4.7.** Recall that for any lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaf  $V$ , there is a Galois covering  $\phi : X' \rightarrow X$  such that  $\phi^*(V)$  is tame. Indeed, choose a finite normal extension  $R$  of  $\mathbb{Z}_\ell$  such  $V$  descends to a representation on  $R^{\oplus r}$ . Define  $\phi$  to be the Galois covering which trivializes the quotient representation on  $(R/\mathfrak{m}_R)^{\oplus r}$ , where  $\mathfrak{m}_R \subset R$  is the maximal ideal.

**4.3. Semi-continuity under pullback.** Let  $Y$  be a smooth curve,  $Y \subset \bar{Y}$  be the smooth completion and  $\bar{f} : \bar{Y} \rightarrow \bar{X}$  be a morphism with  $\bar{f}^{-1}(X) = Y$ . Write  $f : Y \rightarrow X$  for the restriction of  $\bar{f}$ .

**Proposition 4.8.** *If the ramification of the  $\bar{\mathbb{Q}}_\ell$ -sheaf  $V$  on  $X$  is bounded by  $D \in \text{Div}^+(\bar{X})$  then the ramification of  $f^*(V)$  is bounded by  $\bar{f}^*(D)$ .*

*Proof.* Let  $\hat{Y}$  be an irreducible component of  $\bar{X}' \times_{\bar{X}} \bar{Y}$  which is dominant over  $\bar{Y}$  with its reduced subscheme structure and let  $\bar{Y}'$  be the normalization of  $\hat{Y}$ . One can easily show that

$$\bar{f}^{\natural}(D_{\bar{X}'/\bar{X}}) \subset D_{\hat{Y}/\bar{Y}}.$$

As both morphisms  $\bar{Y}' \rightarrow \bar{Y}$  and  $\hat{Y} \rightarrow \bar{Y}$  are flat, Proposition 4.3 implies

$$D_{\hat{Y}/\bar{Y}} \subset \Delta_{\hat{Y}/\bar{Y}} \quad \text{and} \quad D_{\bar{Y}'/\bar{Y}} = \Delta_{\bar{Y}'/\bar{Y}}.$$

For the discriminant a short calculation gives

$$\Delta_{\hat{Y}/\bar{Y}} \subset \Delta_{\bar{Y}'/\bar{Y}}.$$

So

$$\mathcal{O}_{\bar{Y}}(-\bar{f}^*D) = \bar{f}^{\natural}(\mathcal{O}_{\bar{X}}(-D)) \subset \bar{f}^{\natural}(D_{\bar{X}'/\bar{X}}) \subset D_{\bar{Y}'/\bar{Y}}.$$

Finally one observes that the pullback of  $V$  to  $Y' = \bar{Y}' \times_{\bar{Y}} Y$  is tame.  $\square$

**4.4. A conductor discriminant formula.** Let now  $X/k$  be a smooth curve over the perfect base field  $k$ . Let  $X \subset \bar{X}$  be a smooth compactification. Let  $V$  be a lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaf on  $X$  and let  $\phi : X' \rightarrow X$  be a connected étale covering. Denote by  $\bar{X}'$  the normalization of  $\bar{X}$  in  $k(X')$ . Let  $D \in \text{Div}^+(\bar{X})$  be the effective divisor associated to the ideal sheaf  $D_{\bar{X}'/\bar{X}}$ . For  $x \in |\bar{X}|$  let  $s_x(V)$  be the Swan conductor of  $V$  at  $x$ , see [19, (2.2.1)].

**Lemma 4.9.** *If  $\phi^*(V)$  is tame the inequality of divisors*

$$\sum_{x \in |\bar{X}|} s_x(V)[x] \leq \text{rank}(V) D$$

*holds on  $\bar{X}$ .*

*Proof.* There is an injective map of sheaves on  $X$

$$V \rightarrow \phi_* \circ \phi^*(V)$$

For any  $x \in |X|$

$$s_x(V) \leq s_x(\phi_* \circ \phi^*(V)) \leq \text{rank}(V) \text{mult}_x(D).$$

The second inequality follows from [20, Prop. 1(c)]. □

**4.5. Bounding dimensions of cohomology groups.** Let  $D \in \text{Div}^+(\bar{X})$  have support in  $\bar{X} \setminus X$ .

**Definition 4.10.** Define the *complexity* of  $D$  to be

$$\mathcal{C}_D = 2g(\bar{X}) + 2 \deg(D) + 1$$

Let  $k$  be an algebraically closed field.

**Proposition 4.11.** *For any lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaf  $V$  on a connected curve  $X/k$  with ramification bounded by  $D \in \text{Div}^+(\bar{X})$ , such that  $\text{supp}(D) = \bar{X} \setminus X$ , the inequality*

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^0(X, V) + \dim_{\bar{\mathbb{Q}}_\ell} H_c^1(X, V) \leq \text{rank}(V) \mathcal{C}_D$$

*holds.*

*Proof.* Grothendieck-Ogg-Shafarevich theorem says that

$$\chi_c(X, V) = (2 - 2g(\bar{X})) \text{rank}(V) - \sum_{x \in \bar{X} \setminus X} \deg(x) (\text{rank}(V) + s_x(V)),$$

see [19, Théorème 2.2.1.2]. Combining this with Lemma 4.9 gives the proposition. □

## 5. TRACE OF FROBENIUS ON CURVES

**5.1. Faithfulness of trace.** Let  $X$  be a separated *normal* scheme of finite type over  $\mathbb{F}_q$ . It is well known that the trace map

$$\{ \text{semi-simple } \bar{\mathbb{Q}}_\ell\text{-sheaves on } X \} / \text{iso} \xrightarrow{\prod_n t^n} \prod_{n>0} \bar{\mathbb{Q}}_\ell^{X(\mathbb{F}_{q^n})}$$

is injective, see [19, Théorème 1.1.2].

One can ask whether under some restrictions on the sheaves a finite number of the trace functions  $t^n$  is sufficient to guarantee injectivity. Deligne's conjectures deal with lisse  $\mathbb{Q}_\ell$ -Weil sheaves of weight 0.

For the analogue question over number fields, with  $\mathbb{Q}_\ell$ -coefficients and no weight condition, an important result was obtained by Faltings in his proof of the Mordell conjecture [9, Theorem 3.1].

In the proof of his theorem, explained below, Deligne relies on weight arguments from Weil II. Let  $X/\mathbb{F}_q$  be a smooth quasi-projective connected curve with smooth completion  $\bar{X}/\mathbb{F}_q$ . Let  $D \in \text{Div}^+(\bar{X}_{\mathbb{F}})$  be an effective divisor with support equal to  $\bar{X}_{\mathbb{F}} \setminus X_{\mathbb{F}}$ .

**Theorem 5.1.** (*Deligne*) *If two semi-simple lisse  $\mathbb{Q}_\ell$ -Weil sheaves  $V$  and  $V'$  of rank  $r$ , pure of weight 0 on a curve  $X$ , such that the ramification of  $(V \oplus V')_{\mathbb{F}}$  is bounded by  $D$ , satisfy  $t_V^n = t_{V'}^n$ , for all*

$$n \leq 4r^2 \lceil \log_q(4r^2 \mathcal{C}_D) \rceil,$$

then

$$V \cong V'.$$

Here for a real number  $w$  we let  $\lceil w \rceil$  be the smallest integer larger or equal to  $w$ .

*Proof.* Let  $J$  be the set of isomorphism classes of lisse irreducible  $\mathbb{Q}_\ell$ -Weil sheaves on  $X$  which are isomorphic to constituents of  $V \oplus V'$ . Consider the set of equivalence classes  $I = J/\sim$ , where for  $j_1, j_2 \in J$ ,  $j_1 \sim j_2$  if and only if the sheaves associated to  $j_1$  and  $j_2$  become isomorphic on  $X_{\mathbb{F}}$ . Choose representative sheaves  $S_i$  on  $X$  for  $i \in I$ . By Proposition 3.6 for each  $i \in I$  we have

$$S_i = b_{m_i, *} S_i^{\flat}$$

for positive integers  $m_i$  and sheaves  $S_i^{\flat}$  on  $X_{\mathbb{F}_{q^{m_i}}}$ .

It follows from Proposition 3.6 that there are semi-simple  $\mathbb{Q}_\ell$ -Weil representations  $W_i$  and  $W'_i$  pure of weight 0 over  $\text{Spec } \mathbb{F}_{q^{m_i}}$  such that

$$V = \bigoplus_{i \in I} b_{m_i, *} (S_i^{\flat} \otimes_{\mathbb{Q}_\ell} W_i)$$

and

$$V' = \bigoplus_{i \in I} b_{m_i, *} (S_i^{\flat} \otimes_{\mathbb{Q}_\ell} W'_i).$$

For  $n > 0$  set

$$I_n = \{i \in I, m_i | n\}.$$

**Lemma 5.2.** *The functions*

$$t_{S_i}^n : X(\mathbb{F}_{q^n}) \rightarrow \bar{\mathbb{Q}}_\ell \quad i \in I_n$$

are linearly independent for  $n \geq 2 \log_q(4r^2 \mathcal{C}_D)$ .

*Proof.* Fix an isomorphism  $\iota : \bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ . Assume we have a linear relation

$$(5.1) \quad \sum_{i \in I_n} \lambda_i t_{S_i}^n = 0, \quad \lambda_i \in \bar{\mathbb{Q}}_\ell,$$

such that not all  $\lambda_i$  are 0. Multiplying by a constant in  $\bar{\mathbb{Q}}_\ell^\times$ , we may assume that  $|\iota(\lambda_{i_o})| = 1$  for one  $i_o \in I_n$  and  $|\iota(\lambda_i)| \leq 1$  for all  $i \in I_n$ . Set

$$\langle S_{i_1}, S_{i_2} \rangle_n = \sum_{x \in X(\mathbb{F}_{q^n})} t_{\text{Hom}(S_{i_1}, S_{i_2})}^n(x)$$

for  $i_1, i_2 \in I_n$ . Observe that

$$t_{\text{Hom}(S_{i_1}, S_{i_2})}^n = t_{S_{i_1}}^n \cdot t_{S_{i_2}}^n.$$

Multiplying (5.1) by  $t_{S_{i_o}}^n$  and summing over all  $x \in X(\mathbb{F}_{q^n})$  one obtains

$$(5.2) \quad \sum_{i \in I_n} \lambda_i \langle S_{i_o}, S_i \rangle_n = 0.$$

**Claim 5.3.** One has

(i)

$$|\iota \langle S_{i_o}, S_i \rangle_n| \leq \text{rank}(S_{i_o}) \text{rank}(S_i) \mathcal{C}_D q^{n/2}$$

for  $i \neq i_o$ ,

(ii)

$$|m_{i_o} q^n - \iota \langle S_{i_o}, S_{i_o} \rangle_n| \leq \text{rank}(S_{i_o})^2 \mathcal{C}_D q^{n/2}.$$

*Proof of (i):*

By [7, Théorème 3.3.1] the eigenvalues  $\alpha$  of  $F^n$  on  $H_c^k(X_{\mathbb{F}}, \text{Hom}(S_{i_o}, S_i))$  for  $k \leq 1$  fulfill

$$|\iota \alpha| \leq q^{n/2}.$$

On the other hand

$$\dim_{\bar{\mathbb{Q}}_\ell}(H_c^0(X_{\mathbb{F}}, \text{Hom}(S_{i_o}, S_i))) + \dim_{\bar{\mathbb{Q}}_\ell}(H_c^1(X_{\mathbb{F}}, \text{Hom}(S_{i_o}, S_i))) \leq \text{rank}(S_{i_o}) \text{rank}(S_i) \mathcal{C}_D$$

by Proposition 4.11. Under the assumption  $i \neq i_o$  one has

$$H_c^2(X_{\mathbb{F}}, \text{Hom}(S_{i_o}, S_i)) = \text{Hom}_{X_{\mathbb{F}}}(S_i, S_{i_o}) \otimes \bar{\mathbb{Q}}_\ell(-1) = 0$$

by Poincaré duality. Putting this together and using Grothendieck's trace formula [19, 1.1.1.3] one obtains (i).

*Proof of (ii):*

It is similar to (i) but this time we have

$$\dim_{\bar{\mathbb{Q}}_\ell} H_c^2(X_{\mathbb{F}}, \text{Hom}(S_{i_\circ}, S_i)) = m_{i_\circ}$$

and for an eigenvalue  $\alpha$  of  $F^n$  on

$$H_c^2(X_{\mathbb{F}}, \text{Hom}(S_{i_\circ}, S_i)) = \text{Hom}_{X_{\mathbb{F}}}(S_i, S_{i_\circ}) \otimes \bar{\mathbb{Q}}_\ell(-1)$$

we have  $\alpha = q^n$ . This finishes the proof of the claim.

Since under the assumption on  $n$  from Lemma 5.2

$$\mathcal{C}_D \text{rank}(S_{i_\circ}) \sum_{i \in I_n} \text{rank}(S_i) < q^{n/2},$$

we get a contradiction to the linear dependence (5.1). □

**Remark 5.4.** In fact, the proof of Lemma 5.2 shows that we can take  $n > \log_q(2r^2 \mathcal{C}_D)$ , but we will not use this slight sharpening.

By Proposition 3.7 for any  $n \geq 0$  we have

$$t_V^n = \sum_{i \in I_n} t_{W_i}^n t_{S_i}^n$$

and

$$t_{V'}^n = \sum_{i \in I_n} t_{W'_i}^n t_{S_i}^n.$$

Under the assumption of equality of traces from Theorem 5.1 and using Lemma 5.2 we get

$$(5.3) \quad \text{Tr}(F^n, W_i) = \text{Tr}(F^n, W'_i) \quad i \in I_n$$

for

$$2 \log_q(4r^2 \mathcal{C}_D) \leq n \leq 4r^2 \lceil \log_q(4r^2 \mathcal{C}_D) \rceil.$$

In particular this means that equality (5.3) holds for

$$n \in \{m_i A, m_i(A+1), \dots, m_i(A+2r-1)\},$$

where  $A = \lceil 2 \log_q(4r^2 \mathcal{C}_D) \rceil$ . So Lemma 5.5 applied to the set  $\{b_1, \dots, b_w\}$  of eigenvalues of  $F^{m_i}$  of  $W_i$  and  $W'_i$  (so  $w \leq 2r$ ) shows that  $W_i = W'_i$  for all  $i \in I$ . □

**Lemma 5.5.** *Let  $k$  be a field and consider elements  $a_1, \dots, a_w \in k$ ,  $b_1, \dots, b_w \in k^\times$  such that*

$$F(n) := \sum_{1 \leq j \leq w} a_j b_j^n = 0$$

for  $1 \leq n \leq w$ . Then  $F(n) = 0$  for all  $n \in \mathbb{Z}$ .

*Proof.* Without loss of generality we can assume that the  $b_j$  are pairwise different for  $1 \leq j \leq w$ . Then the Vandermonde matrix

$$(b_j^n)_{1 \leq j, n \leq w}$$

has non-vanishing determinant, which implies that  $a_j = 0$  for all  $j$ .  $\square$

**5.2. Effective determination of  $E(V)$  on curves.** The following theorem is an effective version of Deligne's conjecture for curves. The non-effective version was shown by Lafforgue, see Theorem 3.1. Deligne needs this effective version in order to extend to higher dimension the relevant part of Lafforgue's theorem on  $\bar{\mathbb{Q}}_\ell$ -sheaves on curves.

Let  $X, \bar{X}$  and  $D$  be as in Theorem 5.1. In particular the support of  $D$  is equal to  $(\bar{X} \setminus X)_\mathbb{F}$ .

**Theorem 5.6.** *Let  $V$  be a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf on  $X$  which is pure of weight 0 and of rank  $r$ , such that the ramification of  $V_\mathbb{F}$  is bounded by  $D$ . Let  $E$  be the number field generated by the coefficients of the polynomials  $f_V(x)$  for  $x \in |X|$  with*

$$\deg(x) \leq 4r^2 \lceil \log_q(4r^2 \mathcal{C}_D) \rceil.$$

*Then  $E = E(V)$ .*

*Proof.* For a not necessarily continuous automorphism  $\sigma \in \text{Aut}(\bar{\mathbb{Q}}_\ell/E)$  we can use Corollary 3.4 to find a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V_\sigma$  on  $X$  such that  $\sigma f_V = f_{V_\sigma}$ . By [8, Théorème 9.8] the ramification of  $(V \oplus V_\sigma)_\mathbb{F}$  is bounded by  $D$ . The definition of  $E$  implies that for

$$n \leq 4r^2 \lceil \log_q(4r^2 \mathcal{C}_D) \rceil$$

we have  $t_V^n = \sigma t_V^n = t_{V_\sigma}^n$ . So by Theorem 5.1 we have  $t_V^n = t_{V_\sigma}^n$  for all  $n \geq 1$  and therefore  $t_V^n = \sigma t_V^n$ . This implies that  $\sigma$  acts trivially on  $E(V)$ , so  $E = E(V)$ .  $\square$

**Remark 5.7.** In fact we will apply Theorem 5.6 only in the special case  $\bar{X} = \mathbb{P}_{\mathbb{F}_q}^1$ .

## 6. PROOF OF MAIN THEOREMS

In this section we prove Theorem 1.3. The idea is to find for each point  $x \in |X|$  of degree  $n$  over  $\mathbb{F}_q$  a curve  $C \rightarrow X$  with a splitting  $x \rightarrow C$  such that the complexity of  $C$  grows linearly in  $n$ . As the degree of the points in Theorem 5.6 necessary to generate  $E$  grows logarithmically in the complexity, this implies that for  $n = \deg(x)$  large the coefficients of  $f_V(x)$  are contained in the number field generated by the coefficients of  $f_V(y)$  for  $y \in C$  with  $\deg(y) < n$ . By a successive argument on  $n$  we see that we can find a number field which contains all coefficients of the  $f_V$ .

**6.1. Projective space.** Consider an open subscheme  $X \subset \mathbb{P}_{\mathbb{F}_q}^d$  and an effective Cartier divisor  $D_{\mathbb{F}} \in \text{Div}^+(\mathbb{P}_{\mathbb{F}}^d)$  with support equal to  $(\mathbb{P}_{\mathbb{F}_q}^d \setminus X)_{\mathbb{F}}$ .

**Proposition 6.1.** *Let  $V$  be a lisse  $\bar{\mathbb{Q}}_{\ell}$ -Weil sheaf on  $X$  of rank  $r$  which is pure of weight 0 and with ramification of  $V_{\mathbb{F}}$  bounded by  $D$ . Let  $E$  be the number field generated by the coefficients of  $f_V(x)$  for  $x \in |X|$  with*

$$\deg(x) \leq 4r^2 \lceil \log_q(8r^2 \deg(x) \deg(D) + 4r^2) \rceil.$$

*Then  $E = E(V)$ . In particular  $E(V)$  is a number field.*

*Proof.* We prove by induction on  $n$  that for  $x \in |X|$  with  $\deg(x) \leq n$  we have  $f_V(x) \in E[t]$ . Consider a point  $x$  with  $\deg(x) = n$ . If

$$n \leq 4r^2 \lceil \log_q(8r^2 n \deg(D) + 4r^2) \rceil$$

there is nothing to show. So assume the contrary.

Let  $x \in \mathbb{A}_{\mathbb{F}_q}^d \subset \mathbb{P}_{\mathbb{F}_q}^d$  be an open subscheme with

$$\mathbb{A}_{\mathbb{F}_q}^d = \text{Spec}(\mathbb{F}_q[T_1, \dots, T_d]).$$

The point  $x$  gives rise to a homomorphism  $\mathbb{F}_q[T_1, \dots, T_d] \rightarrow \mathbb{F}_q^n$ . We choose an embedding  $x \hookrightarrow \mathbb{A}_{\mathbb{F}_q}^1 = \text{Spec}(\mathbb{F}_q[T])$  and a lifting

$$\phi : \mathbb{F}_q[T_1, \dots, T_d] \rightarrow \mathbb{F}_q[T]$$

with  $\deg(\phi(T_i)) < n$  ( $1 \leq i \leq d$ ). By projective completion we get a morphism  $\psi : \mathbb{P}_{\mathbb{F}_q}^1 \rightarrow \mathbb{P}_{\mathbb{F}_q}^d$  of degree less than  $n$  extending the map  $x \rightarrow X$ .

Consider the curve  $C = \psi^{-1}(X)$  and the divisor  $D_C = \psi^*(D)$  on  $\mathbb{P}_{\mathbb{F}}^1$ . By Proposition 4.8 the ramification of the sheaf  $\psi^*(V_{\mathbb{F}})$  is bounded by  $D_C$ . Clearly  $\mathcal{C}_{D_C} \leq 2n \deg(D) + 1$ . By Theorem 5.6 the coefficients of  $f_{\psi^*V}(x)$  are contained in the field generated by the coefficients of the  $f_{\psi^*V}(z)$  with  $z \in C$  and

$$\deg(z) \leq 4r^2 \lceil \log_q(4r^2 \mathcal{C}_{D_C}) \rceil.$$

The latter coefficients are contained in  $E$  by induction, using that

$$4r^2 \lceil \log_q(4r^2 \mathcal{C}_{D_C}) \rceil < n$$

by our assumption on  $x$ .

□



## 6.2. General schemes.

*Proof of Theorem 1.3.* In case  $X$  is an open subscheme of  $\mathbb{P}_{\mathbb{F}_q}^d$  the theorem is shown in Proposition 6.1. We will reduce to this case by induction on  $\dim(X)$ . We can assume  $X$  is integral. By noetherian induction, we may replace  $X$  by a dense open subscheme. So we can assume that there is a closed immersion  $X \hookrightarrow \mathbb{A}^1 \times_{\mathbb{F}_q} Y$  with  $Y$  an open subscheme of  $\mathbb{A}_{\mathbb{F}_q}^d$  such that  $X \rightarrow Y$  is finite étale.

We can think of  $V$  as an object of  $D_c^b(\mathbb{A}^1 \times_{\mathbb{F}_q} Y)$  concentrated in degree 0. Fix a nontrivial character  $\psi : \mathbb{F}_p \rightarrow \bar{\mathbb{Q}}_\ell^\times$  and let  $\mathcal{F}(V) \in D_c^b(\mathbb{A}^1 \times_{\mathbb{F}_q} Y)$  be the corresponding Fourier-Deligne transform of  $V$  over the base  $Y$ , see [19]. Clearly  $\mathcal{F}(V)$  lives in degree  $-1$  and is a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf of weight 0 on  $\mathbb{A}^1 \times_{\mathbb{F}_q} Y$ . By Proposition 6.1 the field  $E(\mathcal{F}(V))$  is a number field. By the trace formula [19, Théorème 1.2.1.2] and the Fourier inversion formula [19, Théorème 1.2.2.1] it follows that

$$E(\mathcal{F}(V))(\psi(1)) = E(V)(\psi(1)),$$

so that  $E(V)$  is also a number field.  $\square$

## 7. A LEFSCHETZ TYPE RESULT FOR $E(V)$

The Tate conjecture, Conjecture 2.2, motivates to write down the following simple consequence of a theorem of Drinfeld [11], which itself relies on Deligne's Theorem 1.3.

**Proposition 7.1.** *For  $X/\mathbb{F}_q$  a smooth projective geometrically connected scheme and  $H \hookrightarrow X$  a smooth hypersurface section with  $\dim(H) > 0$  consider a lisse  $\bar{\mathbb{Q}}_\ell$ -Weil sheaf  $V$  on  $X$ . Then  $E(V) = E(V|_H)$ .*

*Proof.* Observe that the Weil group of  $H$  surjects onto the Weil group of  $X$ . By [11] Corollary 3.4 stays true for higher dimensional smooth schemes  $X/\mathbb{F}_q$ , i.e. for any  $\sigma \in \text{Aut}(\bar{\mathbb{Q}}_\ell/\mathbb{Q})$  there exists a  $\sigma$ -companion  $V_\sigma$  to  $V$ . So one can argue as in the proof of Theorem 3.1(ii).  $\square$

## 8. APPENDIX

In the proof of Corollary 3.2 we claim the existence of a curve with certain properties. The Bertini argument given in [18, p. 201] for the construction of such a curve is, as such, not correct. We give a complete proof here relying on Hilbert irreducibility instead of Bertini.

Let  $X$  be a normal connected scheme of finite type over  $\mathbb{F}_q$ .

**Proposition 8.1.** *For an irreducible lisse  $\bar{\mathbb{Q}}_\ell$ -étale sheaf  $V$  on  $X$  and a closed point  $x \in X$ , there is an irreducible smooth curve  $C/\mathbb{F}_q$  and a morphism  $\psi : C \rightarrow X$  such that*

- $\psi^*(V)$  is irreducible,
- $x$  is in the image of  $\psi$ .

**Lemma 8.2.** *For an irreducible  $\bar{\mathbb{Q}}_\ell$ -étale sheaf  $V$  on  $X$  there is a connected étale covering  $X' \rightarrow X$  with the following property:  
For a smooth irreducible curve  $C/\mathbb{F}_q$  and a morphism  $\psi : C \rightarrow X$  the implication*

$$C \times_X X' \text{ irreducible} \implies \psi^*(V) \text{ irreducible}$$

holds.

*Proof.* Choose a finite normal extension  $R$  of  $\mathbb{Z}_\ell$  with maximal ideal  $\mathfrak{m} \subset R$  such that  $V$  is induced by a continuous representation

$$\rho : \pi_1(X) \rightarrow GL(R, r).$$

Let  $H_1$  be the kernel of  $\pi_1(X) \rightarrow GL(R/\mathfrak{m}, r)$  and let  $G$  be the image of  $\rho$ . The subgroup

$$H_2 = \bigcap_{\nu \in \text{Hom}(H_1, \mathbb{Z}/\ell)} \ker(\nu)$$

is open normal in  $\pi_1(X)$  according to [1, Th. Finitude]. Indeed observe that  $H_1/H_2 = H_1^{\text{ab}}/\ell$  is Pontryagin dual to  $H_{\text{ét}}^1(X_{H_1}, \mathbb{Z}/\ell)$ , where  $X_{H_1}$  is the étale covering of  $X$  associated to  $H_1$ . Since the image of  $H_1$  in  $G$  is pro- $\ell$ , and therefore pro-nilpotent, any morphism of pro-finite groups  $K \rightarrow \pi_1(X)$  satisfies:

$$(K \rightarrow \pi_1(X)/H_2 \text{ surjective}) \implies (K \rightarrow G \text{ surjective}).$$

(Use [5, Cor. I.6.3.4].)

Finally, let  $X' \rightarrow X$  be the Galois covering corresponding to  $H_2$ .  $\square$

*Proof of Proposition 8.1.* We can assume that  $X$  is affine. Let  $X'$  be as in the lemma. By Noether normalization, e.g. [12, Corollary 16.18], there is a finite generically étale morphism

$$f : X \rightarrow \mathbb{A}^d.$$

Let  $U \subset \mathbb{A}^d$  be an open dense subscheme such that  $f^{-1}(U) \rightarrow U$  is finite étale. Let  $y \in \mathbb{A}^d$  be the image of  $x$ . Choose a linear projection  $\pi : \mathbb{A}^d \rightarrow \mathbb{A}^1$  and set  $z = \pi(y)$  and consider the map  $h : U \rightarrow \mathbb{A}^1$ . By definition,  $U_{k(\mathbb{A}^1)} \subset \mathbb{A}_{k(\mathbb{A}^1)}^{d-1}$ .

Let  $F = k(\Gamma) \supset k(\mathbb{A}^1)$  be a finite extension such that  $X' \otimes_{k(\mathbb{A}^1)} F$  is irreducible and the smooth curve  $\Gamma \rightarrow \mathbb{A}^1$  contains a closed point  $z'$  with  $k(z') = k(y)$ .

It is easy to see that there is an  $\hat{F}$ -point in  $U_{k(\mathbb{A}^1)}$  which specializes to  $y$ . By Hilbert irreducibility, see [11, Cor. A.2], we find an  $F$ -point

$u \in U_{k(\mathbb{A}^1)}$  which specializes to  $y$  and such that  $u$  does not split in  $X' \times_{\mathbb{A}^1} \Gamma$ .

Let  $v \in X$  be the unique point over  $u$ . By the going-down theorem [6, Thm. V.2.4.3] the closure  $\overline{\{v\}}$  contains  $x$ . Finally, we let  $C$  be the normalization of  $\overline{\{v\}}$ .  $\square$

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