SOME EXAMPLES OF COMPUTATION OF A REGULATOR MAP ON SINGULAR VARIETIES

by

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Let $X$ be a complex algebraic variety. In [E2] we have defined a regulator map $c_{nn} : K^M_n \to \mathcal{K}^n(n)$ from the Zariski sheaf of Milnor K-theory to some sheaf $\mathcal{K}^n(n)$, which coincides with Bloch-Beilinson's regulator if $X$ is smooth. In this little note, we compute examples for which $c_{nn}$ helps to detect elements in the kernel of $K^M_n \to K^M_n(\mathcal{C}(X))$, where $\mathcal{C}(X)$ is the function field of $X$, as well as in the cokernel of $K^M_{nX} \to \pi_* K^M_{nY}$, where $\pi : Y \to X$ is a desingularization of $X$. It turns out that in the two cases, those elements are generalized (or "Loday") symbols as defined in [E]. In [E2] we have computed explicitly the image of generalized symbols in $H^n_{\mathcal{D}}(Y,E; \mathcal{Z}(n))$, the Deligne-Beilinson cohomology, relative to some subvariety $E$. As we may relate $\mathcal{K}^n(n)$ on $X$ and $H^n_{\mathcal{D}}(Y,E; \mathcal{Z}(n))$ on $Y$ for some $E$, we basically make the computation in the later group.

Except for (2.2) 1, where we slightly improve the sheaf $\mathcal{K}^n(n)$, the main facts used in this note are proved in [E1] and [E2]: we emphasize how to use the methods developed there to compute examples.

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1.1 Let $Y$ be an algebraic variety over $\mathbb{C}$, the field of complex numbers. We denote by $H^q_\mathcal{O}(p)$ the Deligne-Beilinson cohomology groups [6], [E.V]. Let $a_1, ..., a_n$ be regular functions on $Y$, $f$ be an invertible regular function on $Y$, whose value is 1 along the subvariety $T$ defined by the reduced ideal associated to $a_1 ... a_n$. Define $S$ to be the subvariety of $Y$ such that $S + T$ is the subvariety of $Y$ associated to $(f-1)$. One has $f \in H^1_\mathcal{O}(Y, S+T; \mathbb{Z}(1))$, $a_i \in H^0(Y, \mathcal{O}_Y)$. In [E1], we give explicit formulae for the generalized symbol $\{f, a_1, ..., a_n\}_S \in H^{n+1}_\mathcal{O}(Y, S; \mathbb{Z}(n+1))$ mapping to the cup product $(f \cup a_1 \cup ... \cup a_n)_S$ in $H^{n+1}_\mathcal{O}(Y - T, S \cap Y - T; \mathbb{Z}(n+1))$. (To be precise, we define an element $\{f, a_1, ..., a_n\}_{S+T} \in H^{n+1}_\mathcal{O}(Y, S+T; \mathbb{Z}(n+1))$ whose image in $H^{n+1}_\mathcal{O}(Y, S; \mathbb{Z}(n+1))$ is the generalized symbol as defined by A. Beilinson in [B].)

1.2 If $a_i \in H^1_\mathcal{O}(Y, \mathbb{Z}(1))$, that is if $a_i$ is invertible, then

$$\{f, a_1, ..., a_n\}_S = (f \cup a_1 \cup ... \cup a_n)_S \in H^{n+1}_\mathcal{O}(Y, S; \mathbb{Z}(n+1))$$

maps to the cup product

$$(f \cup a_1 \cup ... \cup a_n) \in H^{n+1}_\mathcal{O}(Y, \mathbb{Z}(n+1))$$

So whenever the map

$$H^{n+1}_\mathcal{O}(Y, S; \mathbb{Z}(n+1)) \to H^{n+1}_\mathcal{O}(Y, \mathbb{Z}(n+1))$$

is not injective, $\{f, a_1, ..., a_n\}_S$ will contain a priori more information than $(f \cup a_1 \cup ... \cup a_n)$.

1.3 Recall briefly how to define $\{f, a_1, ..., a_n\}_{S+T}$. 
We choose an analytic open cover $Y_i$ of $Y$ such that $\log_f$ is single valued on $Y_i$, vanishes along $S+T$ (which implies that $z_i^{n-1} := (\delta \log f)^{i_0}_{i_1} := \log_{i_1} f - \log_{i_0} f$ is identically zero on $Y_{i_0 i_1}^{n-1}$ whenever $Y_{i_0 i_1}$ meets $S+T$), and $\log_{i_0 \cdots i_k} a_k$ is single valued on $Y_{i_0 \cdots i_k}$ whenever $Y_{i_0 \cdots i_k}$ does not meet $S+T$ ([E1], (1.4)). Then we define a "product" ([E1], (1.5)), show that its restriction to $Y-T$ is homotop to the Deligne-Beilinson product ([E1], §2), and that the element so defined in the cohomology $H^{n+1}_\mathcal{O}(Y, S + T; \mathbb{Z}(n + 1))$ does not depend on the choices made above ([E1], (3.8)).

Then $\{f, a_1, \ldots, a_n\}_{S+T}$ is represented by a Čech cocycle

$$(-1)^n \sum_{z_{i_0 \cdots i_k}} \frac{da_{k+1}}{a_{k+1}} \wedge \cdots \wedge \frac{da_n}{a_n} \in H^0(Y_{i_0 \cdots i_k}, \Omega^{n-k}_{Y, S+T})$$

where $\Omega^k_{Y, S+T}$ is the sheaf of Kähler $k$-forms vanishing along $S+T$, $z^{n-k}$ is defined inductively by $z^{n-k} = \delta(z_{i_0 \cdots i_k}^{n-k-1})$, and $z^{n-k}$ is identically zero if $Y_{i_0 \cdots i_k}$ meets $S + T$ and lies in $\mathbb{Z}(k)$ otherwise.

1.4 To be honest, we were considering in [E1] only smooth varieties $Y$. The formulae in (1.3) define a class in $H^{n+1}_\mathcal{O}(Y, j_! \mathbb{Z}(n + 1) \to \Omega^0_{Y, S+T} \to \cdots \to \Omega^n_{Y, S+T})$, where $j$ is the open embedding $Y-S-T \to Y$. If $Y$ is smooth, then this group is $H^{n+1}_\mathcal{O}(Y, S + T; \mathbb{Z}(n + 1))$, which contains $H^{n+1}_\mathcal{O}(Y, S + T; \mathbb{Z}(n + 1))$ as the subgroup of classes $\chi$ whose curvature $dx$ has logarithmic growth at infinity. Recall that if $Y$ is smooth, then $H^n_\mathcal{O}(Y, j_! \mathbb{C}/\mathbb{Z}(n + 1))$ is the subgroup of $H^{n+1}_\mathcal{O}(Y, S + T; \mathbb{Z}(n + 1))$ of curvature zero, and that

$$d\{f, a_1, \ldots, a_n\}_{S+T} = \frac{df}{f} \wedge \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n}$$

([E1], (1.2) (1.3)).
1.5 Consider $b_1 \in H^0(Y, \mathcal{O}_Y)$, and assume moreover that $f = 1$ on $T_{b_1}$ defined by the reduced ideal associated to $b_1 = 0$. As $(f, a_1, \ldots, a_n)_{S+T}$ does not depend on the cover chosen in (1.3) with the properties explained there, one obtains

**Proposition**

$$(f, a_1, \ldots, a_{i-1}, a_i b_i, a_{i+1}, \ldots, a_n)_{S+T+b_1} = (f, a_1, \ldots, a_n)_{S+T+b_1} + (f, a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)_{S+T+b_1}$$

in

$$H^{n+1}_\mathcal{O}(Y, S + T + b_1; \mathbb{Z}(n + 1))$$

1.6 Similarly, let $g \in H^1_\mathcal{O}(Y, S + T; \mathbb{Z}(1))$. Then one has

**Proposition**

$$(fg, a_1, \ldots, a_n)_{S+T} = (f, a_1, \ldots, a_n)_{S+T} + (g, a_1, \ldots, a_n)_{S+T}$$

in

$$H^{n+1}_\mathcal{O}(Y, S + T; \mathbb{Z}(n + 1))$$

1.7 One has also obviously

$$(f^{-1}, a_1, \ldots, a_n)_{S+T} = (f, a_1, \ldots, a_n)_{S+T}$$

$$(f, a_1, \ldots, a_{i-1}, a_i^{-1}, a_{i+1}, \ldots, a_n)_{S+T} = (f, a_1, \ldots, a_n)_{S+T}$$

if $a_i$ is invertible.

1.8 Let us compute a very simple example.

Set $Y = \mathbb{C} - \{0\} = \text{Spec } \mathbb{C}[t, \frac{1}{t}]$

$S = \{1, -1\}$

$f = t^2$, $a_1 = \epsilon t$ with $\epsilon = +1$ or $-1$

$n = 2$.
COMPUTATION OF A REGULATOR MAP

One has a commutative diagram

\[
\begin{array}{ccc}
K_2(Y, S) & \rightarrow & K_2(Y) \\
\downarrow c_{22} & & \downarrow c_{22} \\
H^1(Y, j_! \mathbb{C}/\mathbb{Z}(2)) = H^2_{\mathcal{D}}(Y, S; \mathbb{Z}(2)) & \rightarrow & H^1_{\mathcal{D}}(Y, \mathbb{C}/\mathbb{Z}(2)) = H^1(Y; \mathbb{C}/\mathbb{Z}(2))
\end{array}
\]

We denote by \( <, >_S \) the generalized symbols in \( K_2(Y, S) \) and by \( \{, \} \) the Steinberg symbols in \( K_2(Y) \).

We consider \( <t^2, \epsilon t>_S \) in \( K_2(Y, S) \).

Its image \( \{t^2, \epsilon t\} = 2(-\epsilon t, \epsilon t) \) in \( K_2(Y) \) vanishes. Therefore \( c_{22} <t^2, \epsilon t>_S = \{t^2, \epsilon t\}_S \) lies in

\[
K := \ker \left( H^1(Y, j_! \mathbb{C}/\mathbb{Z}(2)) \rightarrow H^1(Y, \mathbb{C}/\mathbb{Z}(2)) \right) = \mathbb{C}/\mathbb{Z}(2)
\]

Let \([\gamma] \in H_1(Y, S; \mathbb{Z})\) be the homology cycle such that \( <[\gamma], K> \) generates \( \mathbb{C}/\mathbb{Z}(2) \). We may take a representative \( \gamma \) of the following shape:

\[
\begin{align*}
\gamma & \colon [0, \pi] \rightarrow Y \\
\theta & \rightarrow e^{i\theta}
\end{align*}
\]

We want to compute \( x := <[\gamma], \{t^2, \epsilon t\}_S> \) in \( \mathbb{C}/\mathbb{Z}(2) \).

Cover a tubular neighbourhood \( U \) of \( \gamma \) by two open sets \( U_{-1}, U_1 \), with

\[
\begin{align*}
\{1\} & \in U_1 \cdot U_{-1}, \{-1\} \in U_{-1} \cdot U_1, \\
\gamma \cap U_1 & = \{\theta \in [0, \frac{3\pi}{4}]\} \\
\gamma \cap U_{-1} & = \{\theta \in [\frac{\pi}{4}, \pi]\} ;
\end{align*}
\]

Choose \( \log_t t^2 \) with

\[
\log_t t^2 = \log_t t^2 + 2i\pi \quad \text{on} \quad U_{-1}
\]

and
\[
\log_{11} et \text{ on } U_{-11}.
\]

Then \( \{t^2, et\}_S \) is given as a Čech cocycle by

\[
(0, - (\delta \log t^2)_{-11} \log_{11} et, \log_i t^2 \frac{det}{et})
\]

in

\[
\mathfrak{g}^2(u, j! \mathbb{C}/\mathbb{Z}(2)) \times \mathfrak{g}^1(u, \Omega^0_{Y,S}) \times \mathfrak{g}^0(u, \Omega^1_{Y,S})
\]

One has

\[
\log_i t^2 \frac{det}{et} = \frac{1}{4} d((\log_i t^2)^2).
\]

Therefore \( \{t^2, et\}_S \) is given by the Čech cocycle

\[
(0, \bar{x} = - (\delta \log t^2)_{-11} \log_{11} et + \frac{1}{4} \delta ((\log_i t^2)^2), 0),
\]

and one has \( x = \bar{x} \) modulo \( \mathbb{Z}(2) \).

One has

\[
\bar{x} = (\delta \log t^2)_{-11} (- \delta \log_{-11} et + \frac{1}{4} \log_i t^2 + \frac{1}{4} \log_i t^2)
\]

\[
= (2i\pi) (- \delta \log_{-11} et + \frac{1}{2} \log_i t^2 + \frac{i\pi}{2}).
\]

Therefore

\[
0 \neq x = (2i\pi) \frac{i\pi}{2} \in \mathbb{C}/\mathbb{Z}(2) \text{ for } \varepsilon = 1
\]

\[
= - (2i\pi) \frac{i\pi}{2} \in \mathbb{C}/\mathbb{Z}(2) \text{ for } \varepsilon = -1
\]

1.9 Remark.

Let \( V \) be any Zariski open set in \( Y \) containing \( S \). Then the restriction map \( K \rightarrow K_V \) where
$K_V := \text{Ker} \left( H^1(V, j! \mathbb{C}/\mathbb{Z}(2)) \to H^1(V, \mathbb{C}/\mathbb{Z}(2)) \right)$

is obviously an isomorphism.

Therefore the restriction of $\{t^2, et\}_S$ to $V$ does not die in $H^1(V, j! \mathbb{C}/\mathbb{Z}(2))$. We will use this remark in (2.3) in order to construct an element in

$$\text{Ker} \left( (K_2(R) \to K_2(Q(R))) \right),$$

where $R$ is a local domain and $Q(R)$ is its field of fractions.
2.1 Let $X$ be a reduced algebraic variety over $\mathbb{C}$, whose singular locus $\Sigma$ is of dimension $d$. Fix an integer $n$ with $n \geq d + 1$ and $n \geq 2$. In [E2], we construct a Zariski sheaf $\mathcal{H}^n(n)$ on $X$, together with a regulator map $c_{nn} : \mathcal{K}_n^M \to \mathcal{H}^n(n)$, which is functorial and coincides with Bloch-Beilinson's regulator map when $X$ is smooth. (Here $\mathcal{K}_n^M$ is the Zariski sheaf of Milnor $K$-theory).

Roughly, the construction goes as follows.

Let $\pi : Y \to X$ be a desingularization such that $E : = (\pi^{-1} \Sigma)_{\text{red}}$ is a divisor with normal crossings, and such that $\mathcal{F} = \pi^* \Omega^n_X/\text{torsion}$ is a locally free sheaf, where $\Omega^n_X$ are the Kähler differentials. Define $j : Y - E \to Y$ and $i : X - \Sigma \to X$.

One observes that $\mathcal{F}$ embeds into $\Omega^n_X(\log E)(- E)$, and therefore that $\mathcal{F}^{\geq n}$ maps to $j_! \mathcal{G}/\mathbb{Z}(n)$, where

$$(\mathcal{F}^{\geq n})^n = \mathcal{F}, \quad (\mathcal{F}^{\geq n})^\mathcal{F} = \Omega^\mathcal{F}_Y(\log E)(- E), \quad \text{for } \mathcal{F} > n
$$

$$= 0 \quad \text{for } \mathcal{F} < n.$$ 

This gives a map

$$\varphi_i : R\pi_* (\mathcal{F}^{\geq n}) \to i_! \mathcal{G}/\mathbb{Z}(n)$$

and one defines $\mathcal{H}^n(n)_{\text{an},i}$ to be the Zariski sheaf in $X$ associated to $\mathbb{H}^n(\text{cone } \varphi_i [-1])$. It does not depend on the desingularization $\pi$ choosen. Then one defines $\mathcal{H}^n(n)_i$ by taking in $\mathcal{F}^{\geq n}$ those sections which have logarithmic growth at infinity (see (2.2), 1)). Finally, there is a subvariety $\Sigma' \subset \Sigma$ of the shape Sing (Sing $\ldots$ (Sing $\ldots$)), in such a way that if $\mathcal{H}^n(n)$ is the sheaf (with logarithmic growth condition at infinity) associated to $\mathbb{H}^n(\text{cone } \varphi_i [-1])$, where $i' : X - \Sigma' \to X$ and $\varphi_i : R\pi_* (\mathcal{F}^{\geq n}) \to i'_! \mathcal{G}/\mathbb{Z}(n)$, the natural cup product of elements of $\mathcal{K}_1$ lands in ([E2], (1.4)). This defines at the same time $c_{nn}$ ([E2], (2.2)).
2.2 Remarks

1. Let us be more precise on the logarithmic growth at infinity. Let $U$ be an open set in $X$. Take a good compactification of $V = \pi^{-1}(U)$:

$$
\begin{align*}
V & \rightarrow \tilde{V} \\
\pi \downarrow & \quad \downarrow \tilde{\pi} \\
U & \rightarrow \tilde{X}
\end{align*}
$$

such that $\tilde{X}$ is any compactification of $X$, $\tilde{V}$ is smooth and $(\tilde{V} - V)$ is a normal crossing divisor. The one defines

$$
\mathcal{G}^k := \mathcal{E}_* \mathcal{G}^k \cap \Omega^k_V (\log (\tilde{V} - V)),
$$

and $\mathcal{H}^n_i(n)$ is the sheaf associated to

$$
H^n(\tilde{X}, \text{cone } (R \tilde{\pi}_* \mathcal{G}^\infty \rightarrow R\pi_* i_! \mathcal{C}/\mathcal{Z}(n)) [- 1]).
$$

Once again, it does not depend on the choices of $\tilde{X}, V, \tilde{V}$.

One defines similarly $\mathcal{H}^n(n)$ by replacing $i$ by $i'$.

One has for degree reasons

$$
H^n(\tilde{X}, \text{cone } (R \tilde{\pi}_* \mathcal{G}^\infty \rightarrow R\pi_* i_! \mathcal{C}/\mathcal{Z}(n)) [- 1])
$$

$$
= H^n(\tilde{X}, \text{cone } (R \tilde{\pi}_* \mathcal{G}^\infty \rightarrow R\pi_* i_! \mathcal{C}/\mathcal{Z}(n)) [- 1]).
$$

One has maps of sheaves
Define $\Omega_{U,X}^n$ to be the fiber product. As the vertical arrow is injective, $\Omega_{U,X}^n$ is a subsheaf of $k_* \Omega_{U}^n$.

As $H^0(\tilde{X}, R^n \pi_* \mathcal{G}^{2n})$ and $H^0(U, R^n \pi_* \mathcal{G}^{2n})$ do not depend on $\tilde{X}, V, \tilde{V}$, $H^0(\tilde{X}, \Omega_{U,X}^n)$ does not depend on $\tilde{X}, V, \tilde{V}$ either. Define $\mathcal{G}^n_i$ to be the Zariski sheaf associated to

$$H^n(\tilde{X}, \text{cone}(\Omega_{U,X}^n \to Rk_* \text{ i}); \mathbb{C}/\mathbb{Z}(n))[-1]),$$

and similarly for $\mathcal{G}^n_i$ by replacing i by $i'$. One obtains natural maps

$$\mathcal{G}^n(n) \to \mathcal{G}^n(n),$$
$$\mathcal{G}^n_i(n) \to \mathcal{G}^n_i(n).$$

The point is doing that is that one does not lose the torsion in the Kähler differentials.

One can prove along the same line as in [E2] that this definition is functorial and leads to a regulator

$$\tilde{c}_{n,n} = \mathcal{K}_n^M \to \mathcal{G}^n(n)$$

lifting $c_{n,n}$.

We will not use this in the rest of this article.

2. M. Levine [L] defines another Zariski sheaf on $X$. Roughly speaking, he takes the sheaf associated to
\[ H^n(\bar{U}, \Omega^n_U(\log(\bar{U} - U))) \to Rk_{\ast} \text{ cone}(\mathcal{Z}(n) \to \Omega^1) \]

where \( k : U \to \bar{U} \) is a compactification such that \((\bar{U} - U)\) is supported by a Cartier divisor and \( \Omega^n_U(\log(\bar{U} - U)) \) consists of those \( \text{Kähler} \) forms which have logarithmic growth along the normal crossing divisor \((\bar{V} - V)\) where

\[
\begin{align*}
\mathcal{V} & \to \bar{V} \\
\downarrow & \downarrow \\
U & \to \bar{U}
\end{align*}
\]

is a diagram of desingularization. Of course \( \mathcal{K}^n(n) \) maps to M. Levine’s sheaf, whereas \( \mathcal{K}^n(n) \) does not: “my” Betti part lifts ”his”, but I lose the torsion in the forms.

2.3. We will now compute a simple example of \( c_{22} : X \) will be a rational curve with a double point.

Set \( R = \mathbb{C}[1 - t^2, t(1 - t^2), \frac{1}{t}] \) \( \to A = \mathbb{C}[1, 1] \)

\[ = \mathbb{C}[x, y, \frac{1}{1 - x}]/(x^2 - y^2 - x^3) \]

Define \( Y = \text{Spec } A \) \( X = \text{Spec } R \)

\[
\begin{align*}
Y & \xrightarrow{j} X \\
S & \cong X - 0
\end{align*}
\]

where \( 0 = (x = 0, y = 0), S = \{ t = -1, t = 1 \} \).

We consider the commutative diagram

\[
\begin{array}{ccc}
K_2(X, \{0\}) & \xrightarrow{i^\ast} & K_2(X) \\
\downarrow \pi^\ast & & \downarrow \pi^\ast \\
K_2(Y, S) & \xrightarrow{j^\ast} & K_2(Y) \\
c_{22} \downarrow & & \downarrow c_{22} \\
H^1(Y, j_!(\mathcal{O}/\mathcal{Z}(2))) & \to & H^1(Y, \mathcal{O}/\mathcal{Z}(2)).
\end{array}
\]
In $K_2(X, \{0\})$ one has the generalized symbol

$$z := \langle i^2, \varepsilon(1 - i^2) \rangle_{\{0\}} = i^* \langle i^2, \varepsilon(1 - i^2) \rangle_S$$

where $\langle i^2, \varepsilon(1 - i^2) \rangle_S$ is the generalized symbol in $K_2(Y, S)$.

By (1.5), one has

$$j^* \langle i^2, \varepsilon(1 - i^2) \rangle_S = j^* \langle i^2, \varepsilon \rangle_S + j^* \langle i^2, (1 - i^2) \rangle_S \text{ in } K_2(Y).$$

One has $j^* \langle i^2, \varepsilon \rangle_S = \{i^2, \varepsilon\} = 0$ in $K_2(Y)$.

Let $\sigma: Y \to \mathbb{C}^*$

$$t \to i^2 =: \tau$$

Let $j: \mathbb{C}^* \setminus \{1\} \to \mathbb{C}^*$.

Then

$$\langle i^2, (1 - i^2) \rangle_S = \sigma^* \langle \tau, 1 - \tau \rangle_{(1)}$$

where

$$\langle \tau, 1 - \tau \rangle_{(1)} \in K_2(\mathbb{C}^*, \{1\}).$$

By functoriality, one has

$$c_{22} \langle i^2, (1 - i^2) \rangle_S = c_{22} \sigma^* \langle \tau, 1 - \tau \rangle_{(1)}$$

$$= \{i^2, (1 - i^2) \} \sigma^* \langle \tau, 1 - \tau \rangle_{(1)}$$

But one has injections:

$$H^1(\mathbb{C}^*, j(\mathbb{C}/\mathbb{Z}(2))) \to H^1(\mathbb{C}^*, \mathbb{C}/\mathbb{Z}(2))$$

$$\downarrow$$

$$H^1(\mathbb{C}^* \setminus \{1\}, \mathbb{C}/\mathbb{Z}(2))$$
Therefore $\langle t, 1 - t \rangle_{(1)} = 0$ as its image in $H^1(\mathbb{C}^* - \{1\}, \mathbb{C}/\mathbb{Z}(2))$ vanishes (by Bloch's construction of the regulator $!$).

One obtains:

$$c_{22} \left( \langle t^2, et(1 - t^2) \rangle_S \right) = \langle t^2, et \rangle_S$$

By (1.6), it does not die in

$$K = \text{Ker} \left( H^1(Y, j_1 \mathbb{C}/\mathbb{Z}(2)) \to H^1(Y, \mathbb{C}/\mathbb{Z}(2)) \right)$$

Finally, $\pi^* z = \langle t^2, et \rangle_S + \langle t^2, 1 - t^2 \rangle = 0$ in $K_2(\mathbb{C}(t))$.

So we have constructed an element $z \in K_2(\mathbb{C}(X))$, whose image in $K_2(\mathbb{C}(X)) = K_2(\mathbb{C}(t))$ vanishes, and which is non zero. Let $\mathfrak{m}$ be the maximum ideal of $0$ in $R$, and $R_{\mathfrak{m}}$ be the localization of $R$ in $\mathfrak{m}$. It remains to show that the image $\bar{z}$ of $z$ in $K_2(R_{\mathfrak{m}})$ does not vanish.

Apply $c_{22}$; one has

$$c_{22}(\bar{z}) \in \mathcal{H}^2(2)_0 = \mathcal{H}^1(i_1 \mathbb{C}/\mathbb{Z}(2)) \text{ (IE2) (1.4))},$$

where

$$\mathcal{H}^1(i_1 \mathbb{C}/\mathbb{Z}(2)) = \lim_{0 \in U \text{ Zariski}} H^1(U, i_1 \mathbb{C}/\mathbb{Z}(2))$$

$$= \lim_{0 \in U \text{ Zariski}} H^1(\pi^{-1} U, j_1 \mathbb{C}/\mathbb{Z}(2))$$

By (1.9) $c_{22}(\bar{z}) \neq 0$.

**Conclusion.** We have used the regulator $c_{22}$ to detect an explicit element $\bar{z}$ in $K_2(R_{\mathfrak{m}})$, whose image in $K_2(\mathbb{C}(t))$ vanishes.

In [G], the case of a semi-normal curve singularity is treated in general, without use of a regulator.
2.4 Let us now take M. Levine's definition of $c_{22}$ in the example (2.3). One has maps

$$H^1(i_!\mathcal{C}/\mathbb{Z}(2)) \to H^1(\mathcal{C}/\mathbb{Z}(2)) \to H^2(\mathcal{C}/\mathbb{Z}(2)) \to \mathcal{O}_X \to \Omega^1_X).$$

where the first map in an isomorphism and the second one is injective. Therefore one can also see that $z \neq 0$.

2.5 Remark.
Let $X$, $\Sigma$, $\pi$, $Y$, $i$, $i'$ etc... be as in (2.1).
Consider $n = 2$.
The map $\mathfrak{G}^2(2) \to \pi_* \mathfrak{G}^2(2)$ [E2], (1.7), has more precisely the following shape at the presheaf level [E2], (1.4), proof of 1).

There is a commutative diagram of exact sequences:

$$0 \to H^1(U, i'_!\mathcal{C}/\mathbb{Z}(2)) \to H^2(U, 2) \to \text{Ker}(H^0(V, \mathfrak{G})_c) \to H^2(i_!\mathcal{C}/\mathbb{Z}(2)) \to 0$$

(*)

$$0 \to H^1(V, \mathcal{C}/\mathbb{Z}(2)) \to H^2(V, 2) \to \text{Ker}(H^0(V, \Omega^2(\log(V-V))) \to H^2(V, \mathcal{C}/\mathbb{Z}(2))) \to 0$$

As $H^1(U, i'_!\mathcal{C}/\mathbb{Z}(2)) = H^1(U, j'_!\mathcal{C}/\mathbb{Z}(2))$ with $j' : Y \to E'$ where $E' := \pi^{-1} \Sigma'$, one sees that the map $H^2(U, 2) \to H^2(V, 2)$ is injective if and only if $E'$ is connected.

As $H^2(V, 2) = H^0(V, \mathfrak{G}^2(V, 2))$, one obtains that the map $\mathfrak{G}^2(2) \to \pi_* \mathfrak{G}^2(2)$ is injective if and only if $E'$ is connected.

In particular, if $\Sigma = \Sigma'$ and $X$ is normal (e.g. a normal surface singularity), the regulator $c_{22}$ will never detect elements in Ker $(\mathcal{K}_{2X} \to K_2(\mathcal{C}(X)))$.

Consider now $\mathfrak{G}^2(2)$ as defined in (2.2.1). Then in the diagram (*) one has to replace $\mathfrak{G}$ by $\Omega^2_{U,X}$, and one sees that Ker $(\mathfrak{G}^2(2) \to \pi_* \mathfrak{G}^2(2))$ is contained in the torsion of $\Omega^2_X$.

Then $\tilde{c}_{22}$ will detect elements in Ker $(\mathcal{K}_{2X} \to K_2(\mathcal{C}(X)))$ if one can find $x \in \mathcal{K}_{2X}$ such
that \( d\log x \) is torsion, where \( d\log : \mathcal{H}_2^X \to \Omega^2_X \) is the map \( d\log(f, g) = \frac{df}{f} \wedge \frac{dg}{g} \). Of course we knew that already without complicated regulator!

3.1 Keeping the notations of (2.1), we will now be interested in

\[
\mathcal{Q}_l := \pi_* \mathcal{H}^M_{nY}/\mathcal{H}^M_{nX}.
\]

There is a map

\[
\mathcal{Q}_l \to \pi_* \mathcal{R}^n \alpha_* (\Omega^{2n}_Y / \mathcal{G}^{2n})
\]

where \( \alpha : X_{an} \to X_{zar} \) is the continuous map from the classical to the Zariski topology ([E2], (2.2)), simply defined by

\[
d\log : \mathcal{H}^M_{nY} \to \Omega^{2n}_Y [n]
\]

\[
\{f_1, \ldots, f_n\} \to \frac{df_1}{f_1} \wedge \ldots \wedge \frac{df_n}{f_n}.
\]

3.2 We compute a singularity of type \( A_1 \). Set \( X := \text{Spec } \mathbb{C}[x, y, t, \frac{1}{t^2 - xy}] / ((t^2 - xy)) ; \pi : Y \to X \) is the blow up of \( \{0\} = (x = 0, y = 0, t = 0) \), with exceptional line \( E \).

A) Cover \( Y \) by three Zariski open sets \( Y_0, Y_1, Y_2 \) of coordinates and equations

\[
Y_0 : (a, b, t), x = at, y = bt ; 1 - ab
\]

\[
Y_1 : (x, b', T), y = b'x, t = Tx ; T^2 - b'
\]

\[
Y_2 : (a', y, T'), x = a'y, t = T'y ; T^2 - a'
\]

We consider in \( K_2(Y_0) \) the generalized symbol

\[
\alpha_0 := <1 - t, at> \_E.
\]

One has
\[ a_0 |_{Y \cap Y_1} = (1 - Tx, x)_E = a_1 |_{Y \cap Y_1} \text{ with } a_1 := (1 - Tx, x)_E \in K_2 (Y_1) \]
\[ a_0 |_{Y \cap Y_2} = (1 - Ty, T^2y)_E \]
\[ = (1 - Ty, T)y_E + (1 - Ty, Ty)_E \quad (1.5) \]

Consider \[ \sigma : Y \cap Y_2 \rightarrow \mathbb{C}^* \]
\[ (a, y, T) \rightarrow \tau : = 1 - Ty. \]

One has \[ (1 - Ty, Ty)_E = \sigma^* \langle \tau, 1 - \tau \rangle \]

As \[ \langle \tau, 1 - \tau \rangle \in K_2 (\mathbb{C}^*, \{1\}) \] is uniquely determined by its restriction to \[ K_2 (\mathbb{C}^* - \{1\}) \], it is zero.

Therefore \[ a_0 |_{Y \cap Y_2} = a_2 |_{Y_1 \cap Y_2} \text{ with } a_2 := (1 - Ty, T) \in K_2(Y_2). \]

Similarly, one has \[ a_1 |_{Y_1 \cap Y_2} = a_2 |_{Y_1 \cap Y_2} \in K_2 (Y_1 \cap Y_2). \]

Define \( \alpha \in H^0(Y, K_2) \) to be \( \alpha_1 \) on \( Y_1 \).

B) Now one easily computes that \[ \mathcal{F} := \pi^* \Omega^2_X / \text{torsion} = \Omega^2_Y (-E). \]

As \( \mathcal{F} \) is generated by global sections and \( (X, 0) \) is rational singularity, one has \[ \pi_* \Omega^2_Y / \mathcal{F} = \mathbb{C}. \] It is generated by the image in \( \pi_* \Omega^2_Y / \mathcal{F} \) of \[ \text{dlog } \alpha = \frac{dt \wedge da}{1 - t} = \frac{dT \wedge dx}{1 - T} = \frac{dy \wedge dT'}{1 - T'y} \]

3.3 We compute a singularity of type \( A_2 \) ([E2], 2.12), 2))

Set \( X = \text{Spec } \mathbb{C}[x, y, t, \frac{1}{1 - t^2}]/(t^3 - xy) ; \pi : Y \rightarrow X \) is the blow up of \( \{0\} = (x = 0, y = 0, t = 0) \), with exceptional line \( E \). One has \( E = E_1 + E_2, E_1^2 = -2, E_1 \cap E_2 = : p. \)

A) Cover \( Y \) be three Zariski open sets \( Y_0, Y_1, Y_2 \) of coordinates as in (3.2), and equations:

\( Y_0: t - ab, E_1 : <a = 0>, E_2 : <b = 0> \)

\( Y_1: T^3x - b' \)

\( Y_2: T^3y - a' \).
We consider in $K_2(Y_0)$ the two generalized symbols

$$\alpha_0 := <1 - ab, b>_{E_2}, \beta_0 := <1 - (ab)^2, b^2>_{E_2}.$$ 

One has

$$\alpha_0|_{Y_0 \cap Y_1} = <1 - Tx, T^2x>$$
$$= <1 - Tx, T>_{E_2} + <1 - Tx, Tx>_{E_2}.$$ 

As in 3.2, one has $<1 - Tx, Tx>_{E_2} = 0$, and $\alpha_0|_{Y_0 \cap Y_1} = \alpha_1|_{Y_0 \cap Y_1}$ where $\alpha_1 = <1 - Tx, T>_{E_2}$. Similarly, one has $\alpha_0|_{Y_0 \cap Y_2} = \alpha_2|_{Y_0 \cap Y_2}$ where $\alpha_2 = -<1 - Ty, T>_{E_2}$ $\in K_2(Y_2)$. One computes in the same way that $\alpha_1|_{Y_1 \cap Y_2} = \alpha_2|_{Y_1 \cap Y_2}$ in $K_2(Y_1 \cap Y_2)$.

Define $\alpha \in H^0(Y, \mathcal{K}_2)$ to be $\alpha_i$ on $Y_i$.

Similarly, $\beta_0 \in K_2(Y_0)$,

$$\beta_1 := <1 - (Tx)^2, T>_{E_2} \in K_2(Y_1)$$
$$\beta_2 := <1 - (Ty)^2, T^2>_{E_2} \in K_2(Y_2)$$

define a global section in $H^0(Y, \mathcal{K}_2)$.

B) One has $\pi_* \Omega^2_Y/torsion = \mathfrak{p}_* \Omega^2_X(-E)$ where $\mathfrak{p}$ the maximal ideal of $p$. As $\pi_* \Omega^2_Y/torsion$ is generated by global sections and $(X, 0)$ is a rational singularity, one has

$$R^1 \pi_* (\pi^* \Omega^2_X/torsion) = 0.$$ 

Let $\sigma: Z \to Y$ be the blow up of $p$ with exceptional line $F$. Then one has

$$\mathcal{F} = \sigma^* \pi^* \Omega^2_Y/torsion = \sigma^* \Omega^2_Y(-E) \otimes \mathcal{O}_Z(-F).$$
As $R^1 \sigma_* \mathcal{O}_Z (-F) = 0$, one has

$$\pi_* \sigma_* (\Omega^2_Z/G^0) = \pi_* (\Omega^2_Y/T, \Omega^2_Y (-E))$$

$$= \mathcal{C}_p \otimes \mathcal{C}$$

where $\mathcal{C}_p$ is $\Omega^2_Y (-E)/\mathcal{M}, \Omega^2_Y (-E))$

and $\mathcal{C}$ maps isomorphically to $H^0(\omega_E (-E))$. It is obviously generated by the image of

$$\frac{d \log \alpha}{1 - ab} = \frac{dx \wedge dT}{1 - x^2} = \frac{dy \wedge dT'}{1 - y^2}$$

$$\frac{1}{4} \frac{d \log \beta}{1 - (ab)^2} = \frac{xT \frac{dx \wedge dT}{1 - (xT)^2}}{yT \frac{dy \wedge dT'}{1 - (yT')^2}$$
References


