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The coniveau filtration and non-divisibility for algebraic cycles

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0 Introduction

(0.1) Let X be a smooth projective algebraic variety of dimension d defined over a number field K. Let $B^p(X_{\bar{K}}) \subset CH^p(X_{\bar{K}})$ denote the subgroup of the Chow group of $X_{\bar{K}}$ consisting of codimension p algebraic cycles homologically equivalent to zero. Here $\bar{K}$ denotes the algebraic closure of K, and a codimension p cycle is said to be homologically equivalent to zero if its image in the étale cohomology group $H^{2p}(X_{\bar{K}}, \mathbf{Z}_\ell(p))$ is zero (or equivalently, choosing an embedding $\bar{K} \subset \mathbf{C}$, if the pullback cycle on $X_{\mathbf{C}}$ is homologically equivalent to 0 in the usual sense). We will give examples of complete intersections X of dimension 3 in $\mathbb{P}^5$ and rational primes ℓ for which

(0.1.1) The group $B^2(X_{\bar{K}})$ is not ℓ-divisible.

(0.1.2) The group $CH^2(X_{\bar{K}})\{\ell\}$ of ℓ-power torsion cycles is vanishing, whereas $H^3(X_{\bar{K}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$ is not. In particular the natural map [B1]

$$CH^2(X_{\bar{K}})\{\ell\} \to H^3(X_{\bar{K}}, \mathbf{Q}_\ell/\mathbf{Z}_\ell(2))$$

is vanishing.

The referee points out that our examples have no torsion in $H^4(X_{\bar{K}}, \mathbf{Z}_\ell(2))$. Thus

(0.1.1.a) The cycle map

$$CH^2(X_{\bar{K}})/\ell CH^2(X_{\bar{K}}) \to H^4(X_{\bar{K}}, \mathbf{Z}/\ell\mathbf{Z}(2))$$

is not injective, the kernel containing $B^2(X_{\bar{K}})/\ell$.

The hypotheses required for our examples (ordinary reduction, irreducible galois action) are "generic" in character, which suggests that (0.1.1) and (0.1.2) represent the usual state of affairs.

Let S be a smooth affine curve over $\mathbf{C}$, $f : \mathcal{W} \to S$ a smooth, proper morphism with fibre dimension 3. Write $V(s) = f^{-1}(s)$ for a fibre. Assume given a cycle Z of codimension 2 on $\mathcal{W}$ which is "primitive" in the sense that $Z \cdot V(s)$ is homologous to 0. The Leray spectral sequence gives a class $[Z] \in H^1(S, R^3 f_*(\mathbf{Z}))$. Assume given an integer $n \geq 2$. Using the fact that the kernel and cokernel of multiplication by n on $R^3 f_*(\mathbf{Z})$ are finite local systems, it is easy to show that there exists a finite surjective map $\pi : T \to S$ such that $\pi^*[Z] = n \cdot x$ in $H^1(T, R^3 f_{T*}(\mathbf{Z}))$. By a specialization argument, our example (0.1.1) yields an example for which

(0.1.3) There does not exist a finite cover $\pi : T \to S$ such that $\pi^* Z = n \cdot Z'$ in $CH^2(\mathcal{W} \times_S T_{\mathbf{C}})$. That is, the cycle Z does not become divisible in the Chow group.

(0.2) In recent years, the (conjectural) theory of mixed motives has served as a heuristic guide in arithmetic algebraic geometry. From this point of view, one expects a map

$$\rho : B^p(X_{\bar{K}}) \to \mathrm{Ext}^1(\mathbf{Z}(0), H^{2p-1}(X_{\bar{K}}, \mathbf{Z}(p))) .$$

Here the Ext is taken in the category of mixed motives. In the case $p = d$ the right hand side can be identified (using Deligne's theory of 1-motives [D2]) with the $\bar{K}$-points of the Albanese variety of X, and it seems plausible to conjecture that ρ is an isomorphism. Note, however, one expects $\mathrm{Ext}^i = 0$ over $\bar{K}$ for $i \geq 2$. In particular, Ext^1 should be divisible, so for $p < d$, (0.1.1) and (0.1.2) above make such a conjecture unattractive.

On the other hand, the Beilinson conjectures require that the map ρ be an isomorphism tensor $\mathbf{Q}$. (More precisely, replacing $\bar{K}$ by a finite extension M of K, the domain and range of ρ should both have rank equal to the order of zero at $s = p$ of the L-function associated to the representation of $Gal(\bar{K}/M)$ on H^{2p-1}.) Lichtenbaum [L1], [L2], has introduced two-term complexes of étale sheaves $\Gamma(2)$ on X. As objects in the derived category, one has a distinguished triangle

$$(0.2.1) \qquad \Gamma(2) \xrightarrow{\ell^n} \Gamma(2) \to \mu_{\ell^n}^{\otimes 2} .$$

In addition, [L2, Th. 2.13],

$$(0.2.2) \qquad CH^2(X_{\bar{K}}) \subset H^4_{\acute{e}t}(X_{\bar{K}}, \Gamma(2)) ,$$

and this inclusion is an isomorphism tensor $\mathbf{Q}$. Define

$$\mathbf{B}^2(X_{\bar{K}}) = \ker(H^4_{\acute{e}t}(X_{\bar{K}}, \Gamma(2)) \to H^4_{\acute{e}t}(X_{\bar{K}}, \hat{\mathbf{Z}}(2)) .$$

Properties (0.2.1) and (0.2.2) above make $\mathbf{B}^2(X_{\bar{K}})$ an excellent candidate for the motivic Ext.

The subgroup $A^p(X_{\bar{K}}) \subset B^p(X_{\bar{K}})$ consisting of cycles algebraically equivalent to 0 is divisible. The quotient

$$\mathrm{Griff}^p(X_{\bar{K}}) = B^p(X_{\bar{K}})/A^p(X_{\bar{K}})$$

is called the Griffiths group. C. Schoen, [Sc], has found interesting examples of varieties X defined over the algebraic closure of a finite field whose Griffiths group has a non-trivial divisible piece. One might still fantasize for X smooth and projective over $\overline{\mathbb{F}}_p$ that $B^r(X) \cong H^{2r-1}(X, \mathbf{Q}/\mathbf{Z}(r))$. For some important ideas in this direction applicable to products of curves, abelian varieties, and related schemes, see [So].

(0.3) Let X be a proper variety defined over an algebraically closed field. Let n be prime to the residue characteristic. The *coniveau* filtration $N^\bullet H^*(X, \mathbf{Z}/n\mathbf{Z})$ is defined by

$$N^r H^*(X, \mathbf{Z}/n\mathbf{Z}) = \left\{ x \in H^*(X, \mathbf{Z}/n\mathbf{Z}) \,\middle|\, \begin{array}{l} \exists\, Y \subset X \text{ closed of codim. } r, \\ x \mapsto 0 \in H^*(X - Y, \mathbf{Z}/n\mathbf{Z}) \end{array} \right\}.$$

Similarly, one can define $N^\bullet H^*(X)$ for any cohomology theory. For example, Deligne's mixed Hodge theory [D1] implies that if $H^0(X, \Omega^r_X) \neq 0$ then $N^1 H^r_{DR}(X) \neq H^r_{DR}(X)$. Katz gave a criterion in [K] for

$$N^1 H^*(X, \mathbf{Q}_\ell) \neq H^*(X, \mathbf{Q}_\ell)$$

for X over $\overline{\mathbb{F}}_p$. Central to our work is the analogous problem with finite coefficients. We show how the p-adic étale vanishing cycles spectral sequence constructed in [BK] implies, under certain conditions involving *ordinary reduction* at a prime dividing ℓ, that for X proper and smooth over $\bar{\mathbf{Q}}$,

$$N^1 H^r(X, \mathbf{Z}/\ell\mathbf{Z}) \neq H^r(X, \mathbf{Z}/\ell\mathbf{Z}). \tag{0.3.1}$$

Note, however, the coniveau filtration does not behave well under projective limits. Schoen's work [Sc] shows one can have a $\mathbf{Z}_\ell$-cohomology class which is a limit of $\mathbf{Z}/\ell^n\mathbf{Z}$-classes in N^1 but which is not, itself, in N^1. In fact, to our knowledge it is still possible that one has an equality in (0.3.1) for any smooth, proper X defined over $\overline{\mathbb{F}}_p$, with $p \neq \ell$.

1 Ordinary reduction

(1.0) Let Y be a smooth, proper variety of dimension d over a perfect field k of characteristic $p > 0$. Write Ω^r_Y for the Kähler r-forms on Y, and let $B^r \subset \Omega^r_Y$ be the exact r-forms. Cohomology groups will be for the étale cohomology unless noted. We have [BK, def. 7.2]

Definition (1.1). *Y is ordinary if $H^q(X, B^r) = (0)$ for all q and r.*

Let K be a complete discrete valuation field with valuation ring A and residue field k. We assume K has characteristic 0 and k is perfect of characteristic $p > 0$. Let V be a smooth, projective variety over K. Fix an integer m, and let $N^\bullet H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z})$ denote the coniveau filtration (0.3).

Theorem (1.2). *With notation as above, assume:*

(i) *V has good, ordinary, reduction in the sense that there exists a cartesian diagram*

$$\begin{array}{ccccc} V & \xrightarrow{j} & X & \xleftarrow{i} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & S = \mathrm{Spec}(\Lambda) & \longleftarrow & s = \mathrm{Spec}(k) \end{array}$$

with X smooth and proper over S and $Y = X(s)$ ordinary.

(ii) *Either the crystalline cohomology of Y has no torsion, or*

$$d < (p-1)/\mathrm{gcd}(e, p-1).$$

Here e denotes the absolute ramification degree of Λ.

(iii) $\Gamma(Y, \Omega_Y^m) \neq (0)$.

Then

$$N^1 H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}) \neq H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}).$$

In other words, writing $\bar{K}(V)$ for the function field, the natural map

$$H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}) \to H^m(\bar{K}(V), \mathbf{Z}/p\mathbf{Z})$$

is non-zero.

Proof. Let $\bar{\Lambda}$ denote the integral closure of Λ in the algebraic closure $\bar{K}$. Write a bar over items in (1.2)(i) to indicate passage to this integral closure. Thus, for example, $\bar{Y} = Y \times_k \bar{k}$. Following [BK, Sect. 8], write

$$\bar{M}_n^q = \bar{i}^* R^q \bar{j}_*(\mathbf{Z}/p^n\mathbf{Z}(q)). \tag{1.2.1}$$

This is an étale sheaf on $\bar{Y}$, and there is a spectral sequence

$$E_2^{s,t} = H^s(\bar{Y}, \bar{M}_n^t(-t)) \Rightarrow H^{s+t}(V_{\bar{K}}, \mathbf{Z}/p^n\mathbf{Z}), \tag{1.2.2}$$

equivariant for the action of $\mathrm{Gal}(\bar{K}/K)$.

Of course, the spectral sequence (1.2.2) exists by general nonsense. Even if we drop the hypothesis that V, Y, and X are proper over their respective base schemes, we still have the spectral sequence, providing we remove the $\bar{i}$ from the term $\bar{M}_n^0$ and calculate the $E^{s,0}$-term via cohomology on X. The observation in [op. cit.] is that one can to some extent calculate the structure of the sheaves $\bar{M}_n^q$ in terms related to the de Rham-Witt sheaves $W_r\Omega_{\bar{Y}}^s$ and their logarithmic or Cartier-fixed subsheaves $W_r\Omega_{\bar{Y},\log}^s$. In particular, when Y is ordinary one gets [BK, 9.2]

$$H^s(\bar{Y}, \bar{M}_n^t(-t)) \cong H^s(\bar{Y}, W_n\Omega_{\bar{Y},\log}^t)(-t), \tag{1.2.3}$$

and

$$H^s(\bar{Y}, \bar{M}_n^t(-t)) \otimes W_n(\bar{k}) \cong H^s(\bar{Y}, W_n\Omega_{\bar{Y}}^t)(-t) \tag{1.2.4}$$

as $\mathrm{Gal}(\bar{K}/K)$-modules. For a survey of these results, the reader is referred to [B5].

We claim that either of the hypotheses (ii) imply that the spectral sequence (1.2.2) degenerates at E_2. The hypothesis on $d = \dim V$ implies degeneration by [op. cit. Cor (9.4)]. Torsion-freeness for crystalline cohomology implies (in this ordinary case) torsion-freeness for the Hodge groups $H^s(X, \Omega^t_{X/S})$ [op. cit. 9.5] and hence [op. cit. 9.3] torsion-freeness for $\varprojlim H^s(\bar{Y}, \bar{M}^t_n(-t))$ and an isomorphism

$$(1.2.5) \qquad H^s(\bar{Y}, \bar{M}^t_n(-t)) \cong (\varprojlim H^s(\bar{Y}, \bar{M}^t_n(-t)))/p^n .$$

Since the groups in (1.2.3) are finite, $\varprojlim$ is exact. Torsion-freeness enables us to get degeneration in the inverse limit by a weight argument. (1.2.5) then implies degeneration for each value of n.

We now have

$$(1.2.6) \qquad H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}) \twoheadrightarrow H^0(\bar{Y}, \bar{M}^m_1(-m)) \hookrightarrow H^0(\bar{Y}, \Omega^m_{\bar{Y}})(-m) .$$

Write α for the composition (1.2.6). By (1.2.4), α is zero if and only if $H^0(\bar{Y}, \Omega^m_{\bar{Y}}) = (0)$. Further, α is constructed locally. In fact, $\bar{M}^m_1(-m)$ is generated by $\bar{M}^1_1(-1) = i^* j_* \mathcal{O}^*_{\bar{V}}/p$ and α is the d log map [op. cit. 1.4]. In other words, writing A for the strict henselization of the local ring on $\bar{X}$ at the generic point of $\bar{Y}$ (so the residue field of A is the separable closure L of $\bar{k}(Y)$), we have a commutative diagram

(1.2.7)

$$\begin{array}{ccccc}
H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}) & = & H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z}) & \xrightarrow{\alpha} & H^0(\bar{Y}, \Omega^m_{\bar{Y}})(-m) \\
\beta \downarrow & & \downarrow & & \gamma \downarrow \\
H^m(\bar{K}(V), \mathbf{Z}/p\mathbf{Z}) & \longrightarrow & H^m(\mathrm{Spec}(A[1/p]), \mathbf{Z}/p\mathbf{Z}) & \longrightarrow & \Omega^m_L(-m)
\end{array}$$

with γ injective. It follows that $\alpha \neq 0 \Rightarrow \beta \neq 0$ as claimed. QED

Corollary (1.3). *Let V be a smooth projective variety defined over a number field F. Let $\mathfrak{p}$ be a prime of F, and write $K = F_{\mathfrak{p}}$. Assume V_K satisfies the hypotheses of* (1.2). *Then $N^1 H^m(V_{\bar{F}}, \mathbf{Z}/p\mathbf{Z}) \neq H^m(V_{\bar{F}}, \mathbf{Z}/p\mathbf{Z})$.*

Proof. The point is that under the identification $H^m(V_{\bar{F}}, \mathbf{Z}/p\mathbf{Z}) \cong H^m(V_{\bar{K}}, \mathbf{Z}/p\mathbf{Z})$ the coniveau filtrations coincide, or in other words that passage from one algebraically closed base field to a larger one preserves the coniveau filtration. This follows from an easy specialization argument. Alternately, the map $H^m(V_{\bar{F}}, \mathbf{Z}/p\mathbf{Z}) \to H^m(\bar{F}(V), \mathbf{Z}/p\mathbf{Z})$ cannot be zero, as composed with

$$H^m(\bar{F}(V), \mathbf{Z}/p\mathbf{Z}) \to H^m(\bar{K}(V), \mathbf{Z}/p\mathbf{Z}) ,$$

it is not. QED

Remark (1.4). In our application, $\mathrm{Gal}(\bar{F}/F)$ acts irreducibly on $H^m(V_{\bar{F}}, \mathbf{Z}/p\mathbf{Z})$, so it follows from (1.3) that $N^1H^m(V_{\bar{F}}, \mathbf{Z}/p\mathbf{Z}) = (0)$. We will want to conclude that

$$N^1H^m(V_{\bar{F}}, \mathbf{Z}/p^r\mathbf{Z}) = (0) \tag{1.4.1}$$

for all r. It would probably be possible to prove this by strengthening the argument in (1.2), replacing $\Omega^m_{\bar{Y}}$ with $W_r\Omega^m_{\bar{Y}}$. However, we will only need the case $m = 3$. As explained in [B3], p. 383 for the Betti cohomology, it follows from [MS] that $H^3(\bar{F}(V), \mathbf{Z}/p^{r-1}\mathbf{Z})$ injects into $H^3(\bar{F}(V), \mathbf{Z}/p^r\mathbf{Z})$, and by induction hypothesis that $H^3(V_{\bar{F}}, \mathbf{Z}/p^{r-1}\mathbf{Z})$ injects into $H^3(\bar{F}(V), \mathbf{Z}/p^{r-1}\mathbf{Z})$. Looking at the exact sequence $0 \to \mathbf{Z}/p \to \mathbf{Z}/p^r \to \mathbf{Z}/p^{r-1} \to 0$, one concludes that $H^3(V_{\bar{F}}, \mathbf{Z}/p^r\mathbf{Z})$ injects into $H^3(\bar{F}(V), \mathbf{Z}/p^r\mathbf{Z})$. This shows (1.4.1).

2 Hilbert irreducibility and cycles

(2.0) We will use the Hilbert Irreducibility Theorem, for which our basic references is [La]. Let k be a field. Let $f \in k(t)[X]$ be an irreducible polynomial which is monic in X. Let

$$U_{f,k} = \{\tau \in k | f(\tau, X) \text{ is defined and irreducible}\} .$$

A *Hilbert subset* $H \subset k$ [La, p. 225] is a set of the form

$$H = U(k) \cap \bigcap_{i=1}^{i=n} U_{f_i,k}$$

where $U \subset \mathbf{A}^1_k$ is non-empty and Zariski open.

Theorem (2.1). *Let H be a Hilbert subset of a number field F. Let $\mathfrak{p}$ be a prime of F. Then H is dense for the $\mathfrak{p}$-adic topology on F.*

Proof. When $F = \mathbf{Q}$, this is [La, Sect. 9, Cor 2.5]. In general, given $\alpha \in F^\times$, we will show $H \cap \mathbf{Q} \cdot \alpha$ contains a set $H_\alpha \cdot \alpha$ for a Hilbert set $H_\alpha \subset \mathbf{Q}$. Since H_α is p-adically dense in $\mathbf{Q}$ where $\mathfrak{p}|p$, the assertion of the theorem will follow. Suppose $H = \bigcap_{i=1}^{i=n} U_{f_i,F} - S$ for some finite set S, where $f_i(t,X) \in F(t)[X]$ is monic in X and irreducible. Let $g_i(t,X) = f_i(\alpha \cdot t, X)$. Let $G_i(t,X) \in \mathbf{Q}(t)[X]$ be a product of distinct conjugates of g_i. For all but finitely many $t_0 \in \mathbf{Q}$ we see that $G_i(t_0, X)$ irreducible implies $f_i(t_0 \cdot \alpha, X)$ irreducible. Let $H_\alpha := \bigcap U_{G_i,\mathbf{Q}} - T$ for a suitable finite set T. We have $H_\alpha \cdot \alpha \subset H$ as claimed. QED

We now recall a theorem of Terasoma [T]. Let F be a number field, and let $U \subset \mathbf{A}^1_F$ be a non-empty Zariski-open subset. Let $\bar{\eta}$ be a geometric generic point of U, and let $\pi_1(U, \bar{\eta})$ be the algebro-geometric fundamental group of U. Let $G \subset GL_n(\mathbf{Q}_\ell)$ be a closed subgroup for the ℓ-adic topology. Assume given a continuous surjective group homomorphism

$$\varphi : \pi_1(U, \bar{\eta}) \twoheadrightarrow G .$$

Given $u \in U(F)$, we choose $\bar{u}$ a geometric point lying over u, and a "path" [D3], p. 220 from $\bar{u}$ to $\bar{\eta}$. The resulting homomorphism

$$a_u : \mathrm{Gal}(\bar{F}/F) = \pi_1(u, \bar{u}) \to \pi_1(U, \bar{\eta})$$

depends upto conjugation only on u. Let

$$J = \{u \in U(F) | \varphi \circ a_u : \mathrm{Gal}(\bar{F}/F) \twoheadrightarrow G\} .$$

Theorem (2.2). *$J \supset H$ for some Hilbert subset H of F.*

Proof. Since Terasoma does not formulate his theorem in terms of Hilbert sets, we recall his argument briefly. He remarks that the fact the G is an ℓ-adic Lie group implies there exists an open subgroup $\mathscr{G} \subset G$ such that a homomorphism $\Gamma \to G$ from a group Γ is surjective if and only if the composed map (of sets) $\Gamma \to G/\mathscr{G}$ is surjective. Let $K = \varphi^{-1}(\mathscr{G}) \subset \pi_1(U, \bar{\eta})$, and let $g : W \to U$ be the corresponding étale cover. It is easy to check now that given $u \in U(F)$, $g^{-1}(u)$ irreducible implies $u \in J$. Shrinking U if necessary, we may arrange for the coordinate ring of W to be defined by the vanishing of $f(t, X)$ monic in X, where t is the coordinate on U. Then J contains the Hilbert set $U(F) \cap U_{f,F}$.
QED

(2.3) As was remarked in [B4], Terasoma's theorem has implications for the Griffiths group. These we now recall. Let W be a smooth, projective variety of dimension $2r$ defined over a number field F. Assume $H^{2r}(W_{\bar{F}}, \mathbf{Q}_\ell(r))$ contains a nontrivial primitive algebraic cycle class $[Z]$ with Z defined over F. (We have in mind the case W a quadric in $\mathbb{P}^5$.) Let $\mathscr{W}$ be W blown up along the base of a Lefschetz pencil defined over F, so we have $h : \mathscr{W} \to \mathbb{P}^1_F$. Let $U \subset \mathbb{P}^1_F$ be a non-empty affine over which h is smooth. Write $M_{\mathbf{Q}_\ell}$ for the ℓ-adic vanishing cycle representation of $\pi_1(U, \bar{\eta})$. Replacing our projective embedding by a multiple if necessary, we may assume $M_{\mathbf{Q}_\ell} \neq 0$. (In our application, $M_{\mathbf{Q}_\ell} = R^{2r-1}h_*(\mathbf{Q}_{\ell,\mathscr{W}})_{\bar{\eta}}(r)$.) Let $\rho : \pi_1(U, \bar{\eta}) \to \mathrm{Aut}(M_{\mathbf{Q}_\ell})$.

The class $[Z]$ carries a group cohomology class $a \in H^1(\pi_1(U, \bar{\eta}), M_{\mathbf{Q}_\ell})$. Analogously to Griffiths' work, note that this class is non-zero. Indeed, for it to be trivial would mean that $[Z]$ was carried, as a homology class, on the union of the bad fibres of the Lefschetz pencil. For a bad fibre with a single isolated ordinary double point to carry an extra homology class of dimension $2r$, it is necessary that the corresponding vanishing cycle δ be trivial in $M_{\mathbf{Q}_\ell}$. But $M_{\mathbf{Q}_\ell}$ is generated by the vanishing cycles, and they are all conjugate, so triviality of δ implies triviality of $M_{\mathbf{Q}_\ell}$.

A 1-cocycle α representing a gives rise to a homomorphism

$$\sigma : \pi_1(U, \bar{\eta}) \to M_{\mathbf{Q}_\ell} \rtimes \rho(\pi_1(U, \bar{\eta})) ,$$

which agrees with ρ on the right hand factor. We apply (2.2) with $G = \mathrm{image}(\sigma)$ to conclude there exists $J \subset U(F)$ containing a Hilbert subset such that for $u \in J$ the map $\sigma \circ a_u : \mathrm{Gal}(\bar{F}/F) \to G$ is onto. Note that for such a u, the map

(2.3.1)
$$\Gamma_u := \ker(\mathrm{Gal}(\bar{F}/F) \to \mathrm{image}(\rho)) \to \mathrm{image}(\sigma) \cap M_{\mathbf{Q}_\ell} = \mathrm{image}(\sigma|\ker(\rho))$$

is also surjective.

From the Hochschild-Serre spectral sequence we get a commutative diagram

$$\begin{array}{ccccccc} 0 \to & H^1(\mathrm{image}(\rho), M_{\mathbf{Q}_\ell}) & \to & H^1(\pi_1(U, \bar{\eta}), M_{\mathbf{Q}_\ell}) & \to & \mathrm{Hom}_{\mathrm{image}(\rho)}(\ker(\rho)^{\mathrm{ab}}, M_{\mathbf{Q}_\ell}) \\ & \| & & \downarrow & & \downarrow \\ 0 \to & H^1(\mathrm{image}(\rho), M_{\mathbf{Q}_\ell}) & \to & H^1(\mathrm{Gal}(\bar{F}/F), M_{\mathbf{Q}_\ell}) & \to & \mathrm{Hom}_{\mathrm{image}(\rho)}(\Gamma_u^{\mathrm{ab}}, M_{\mathbf{Q}_\ell}) \, . \end{array}$$

The image of α in the homomorphism group on the top right is represented by $\sigma|\ker(\rho)$, so the surjectivity of (2.3.1) implies that α pulls back to a non-trivial cohomology class $\beta(u) \in H^1(\mathrm{Gal}(\bar{F}/F), M_{\mathbf{Q}_\ell})$.

Let $V(u)$ be the fibre of h over $u \in U(F)$. The cycle $Z|V(u)$ is cohomologically trivial, and $\beta(u) \in H^1(\mathrm{Gal}(\bar{F}/F), M_{\mathbf{Q}_\ell})$ is the class defined in [B2], (1.2). In the example we consider, $H^{2r-1}(W_{\bar{F}}, \mathbf{Q}_\ell(r)) = (0)$ and $M_{\mathbf{Q}_\ell} \cong H^{2r-1}(V(u)_{\bar{F}}, \mathbf{Q}_\ell(r))$. One has a cycle map

(2.3.2)
$$\gamma : \left\{ \begin{array}{c} \text{cod. } r \text{ cycles on } V(u) \text{ def.} \\ \text{over } F \text{ hom. eq. to 0 over } \bar{F} \end{array} \right\} \to H^1(\mathrm{Gal}(\bar{F}/F), H^{2r-1}(V(u)_{\bar{F}}, \mathbf{Z}_\ell(r)))$$

and

$$\beta(u) = \gamma(Z \cdot V(u)) \otimes \mathbf{Q}_\ell \, .$$

As a consequence of the above discussion we have

Theorem (2.4). *With notation as above, $\{u \in U(F) | \gamma(Z \cdot V(u)) \neq 0\}$ contains a Hilbert subset of F.*

3 Hilbert irreducibility and Galois action

(3.0) Suppose now V is a smooth hyperplane section of a smooth variety W over $\mathbf{C}$ and $\dim(V) = d = 2r - 1$. We choose a Lefschetz pencil through V to get an action of $\pi := \pi_1(U_{\mathbf{C}})$ on $H^d(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$, for some open $U \subset \mathbb{P}^1$.

We have defined the vanishing cycles $\delta \in H^d(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$. Let $\mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$ be the span of these in $\mathbf{Z}/\ell\mathbf{Z}$-cohomology. By [D3 4.3.3]

$$\mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})^{\perp} = \mathrm{image}(H^d(W, \mathbf{Z}/l\mathbf{Z}) \to H^d(V, \mathbf{Z}/l\mathbf{Z})) \, .$$

Lemma (3.1). *Let $x \in Van(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$. Then either x is π-invariant, or x spans $Van(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$ as a π-module.*

Proof. The monodromy group is generated by transvections

$$x \mapsto x + \langle x, \delta\rangle\delta;\ \ \delta \text{ a vanishing cycle}.$$

If x is not π-invariant, there exists a vanishing cycle δ such that

$$\langle x, \delta\rangle \not\equiv 0 \bmod(\ell).$$

It follows that the π-module spanned by x contains δ. Since all the δ are conjugate, the π-module equals $\mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$. QED

Lemma (3.2). *$Van(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$ is irreducible for almost all ℓ.*

Proof. Let $\delta_1, \ldots, \delta_n$ freely generate an abelian subgroup of $\mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z})$ of finite index N. For $(\ell, N) = 1$, an element

$$x = \sum a_i\delta_i \in \mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$$

is π-invariant if and only if

$$\sum a_i\langle\delta_i, \delta_j\rangle \equiv 0 \bmod(\ell), \quad 1 \leqq j \leqq n.$$

Because there are no invariant vanishing cycles tensor $\mathbf{Q}$, $D := \det(\langle\delta_i, \delta_j\rangle) \neq 0$. Thus, for $(\ell, ND) = 1$,

$$\mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})^{\pi} = (0).$$

It follows from (3.1) that for these values of ℓ, $\mathrm{Van}(V_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$ is an irreducible π-module. QED

Proposition (3.3). *Let W be a smooth, projective variety of dimension $d = 2r$ defined over a number field F. Let $\{V(u)\}_{u\in\mathbf{P}^1}$ be a Lefschetz pencil which we assume also to be defined over F. Let η be the generic point of U over F, $\bar{\eta}$ be a $\mathbf{C}$ valued point above η, ℓ be a rational prime such that* $\mathrm{Van}(V(\bar{\eta}_{\mathbf{C}}), \mathbf{Z}/\ell\mathbf{Z}))$ *is an irreducible $\pi_1(U_{\mathbf{C}}, \bar{\eta})$-module. Then there exists a Hilbert set $H \subset U(F)$ such that for $u \in H$, the representation of $Gal(\bar{F}/F)$ on* $\mathrm{Van}(V(u)_{\bar{F}}, \mathbf{Z}/\ell\mathbf{Z})$ *is irreducible.*

Proof. The topological fundamental group π acts on $\mathrm{Van}(V(\bar{\eta}_{\mathbf{C}}), \mathbf{Z}/\ell\mathbf{Z}))$ through its profinite completion, a subgroup of $\pi_1(U, \bar{\eta})$. A "path" [D2], p. 220, from a $\bar{\mathbf{Q}}$ valued point $\bar{u}$ of U to $\bar{\eta}$ defines isomorphisms from $\mathrm{Van}(V(\bar{u})_{\bar{\mathbf{Q}}}, \mathbf{Z}/\ell\mathbf{Z})$ to $\mathrm{Van}(V(\bar{\eta})_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$ and from $\pi_1(U, \bar{\eta})$ to $\pi_1(U, \bar{u})$. The action of $\pi_1(U, \bar{u})$ on $\mathrm{Van}(V(\bar{u})_{\bar{\mathbf{Q}}}, \mathbf{Z}/\ell\mathbf{Z})$ becomes an action of $\pi_1(U, \bar{\eta})$ on $\mathrm{Van}(V(\bar{\eta})_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$, factorizing the action of π. Thus it is irreducible. As in the proof of (2.2), let $\varphi : T \to U$ be the finite étale Galois cover corresponding to the kernel of the action of $\pi_1(U, \bar{\eta})$ on $\mathrm{Van}(V(\bar{\eta})_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$. A point $u \in U(F)$ such that $\varphi^{-1}(u)$ remains irreducible induces an irreducible representation of $\mathrm{Gal}(\bar{\mathbf{Q}}/F) = \pi_1(u, \bar{u})$ on $\mathrm{Van}(V(\bar{\eta})_{\mathbf{C}}, \mathbf{Z}/\ell\mathbf{Z})$, and the set of such points contains a Hilbert set. QED

4 The example

Proposition (4.0). *Let $W \subset \mathbb{P}^5$ be a smooth quadric of dimension 4 defined over $\mathbf{Q}$. Let Z be the primitive codimension 2 algebraic cycle defined by taking the difference of the two "rulings". We assume W is "split" so Z is defined over $\mathbf{Q}$. Let $n \geqq 4$ be an integer. Then there exists a Lefschetz pencil $\{V(u)\}_{u \in \mathbb{P}^1}$ on W defined over $\mathbf{Q}$ in the linear system defined by n times the hyperplane class; a rational prime ℓ; a non-empty open set $U \subset \mathbb{P}^1_{\mathbf{Q}}$; an algebraic number field F; and an infinite set $\mathscr{S} \subset U(F)$ such that for $u \in \mathscr{S}$ we have:*

(i) *$V(u)$ has good, ordinary reduction at ℓ.*
(ii) *The representation of* $\mathrm{Gal}(\bar{\mathbf{Q}}/F)$ *on $H^3(V(\bar{u}), \mathbf{Z}/\ell\mathbf{Z})$ is irreducible.*
(iii) *The cycle class $\gamma(Z \cdot V(u)) \in H^1(\mathrm{Gal}(\bar{\mathbf{Q}}/F), H^3(V(\bar{u}), \mathbf{Q}_\ell(r)))$ is non-zero.*

Proof. Let ℓ be an odd prime. The quadric W has good, ordinary reduction at ℓ, so, by a theorem of Illusie [I], (2.2), (2.3.2), there exists a non-empty open set $\mathscr{U}$ in the projective space over $\mathbb{F}_\ell$ parametrizing intersections of W with degree n hypersurfaces such that for $\mu \in \mathscr{U}$, the corresponding complete intersection $V(\mu)$ of multidegree $(2,n)$ over $\overline{\mathbb{F}}_\ell$ is smooth and ordinary. We may choose our Lefschetz pencil on W to have good reduction mod ℓ and such that the reduced pencil meets $\mathscr{U}$. Let $r \geqq 1$ be such that this intersection contains a closed point μ defined over $\mathbb{F}_{\ell^r}$. Let F be a number field with a non-archimedean place λ with residue field $\mathbb{F}_{\ell^r}$. Let $U \subset \mathbb{P}^1_{\mathbf{Q}}$ be the smooth locus of the pencil. From (2.2) and (2.4) there exists a Hilbert set $H \subset U(F)$ of points u satisfying (ii) and (iii). By (2.1), the set $\mathscr{S}$ of points of H whose reduction mod λ equals μ is infinite. QED

Proposition (4.1). *With notation as above, let $V = V(u) \subset W$ for some $u \in \mathscr{S}$, so V satisfies* (4.0)(i), (ii), *and* (iii)*. Then the ℓ-power torsion subgroup $CH^2(V_{\bar{\mathbf{Q}}})\{\ell\} = 0$. Moreover, $Z \cdot V$ is not infinitely ℓ-divisible in the Chow group $CH^2(V_{\bar{\mathbf{Q}}})$, i.e. there exists an $n \geqq 1$ such that $Z \cdot V \neq \ell^n \cdot A$ for any algebraic cycle.*

Proof. Let $M = H^3(V_{\bar{\mathbf{Q}}}, \mathbf{Z}_\ell(2))$ and let $M_n = M/\ell^n M$. By [MS, § 18], the map $CH^2(V_{\bar{\mathbf{Q}}})\{\ell\} \to \varinjlim M_n$ is injective with image $\varinjlim N^1 M_n$. Our hypothesis that V has multi-degree $(2,n)$ with $n \geqq 4$ implies $H^0(V, \Omega^3) \neq (0)$ so by (1.3), $N^1 M_1 \neq M_1$. Since $N^\bullet M_1$ is G-stable, it follows from (4.0)(ii) that $N^1 M_1 = (0)$. As remarked in (1.4), this implies $N^1 M_r = (0)$ so $CH^2(V_{\bar{\mathbf{Q}}})\{\ell\} = (0)$.

Let $G =$ $\mathrm{Gal}(\bar{\mathbf{Q}}/F)$, where F is any number field, and let $H \subset G$ be a normal subgroup of finite index. Define

$$H^1(H, M) := \varprojlim_n H^1(H, M/\ell^n M) .$$

We claim

$$H^1(G, M) \cong H^1(H, M)^{G/H} . \tag{4.1.1}$$

The Hochschild-Serre spectral sequence yields an exact sequence

$$0 \to H^1(G/H, M_n^H) \to H^1(G, M_n) \xrightarrow{\alpha_n} H^1(H, M_n)^{G/H} \to H^2(G/H, M_n^H).$$

The group on the left is finite, so by Mittag-Leffler we get exactness in the limit as indicated

$$(4.1.2) \qquad 0 \to \varprojlim_n H^1(G/H, M_n^H) \to \varprojlim_n H^1(G, M_n) \to \varprojlim_n \mathrm{image}(\alpha_n) \to 0$$
$$0 \to \varprojlim_n \mathrm{image}(\alpha_n) \to \varprojlim_n H^1(H, M_n)^{G/H} \to \varprojlim_n H^2(G/H, M_n^H).$$

The groups M_n^H are finite so the inverse system $\{M_n^H\}$ satisfies Mittag-Leffler. Also, $\varprojlim M_n^H = (\varprojlim M_n)^H = (0)$ since the weights on M are non-zero and M is torsion-free. By standard results [J 2.1 and 2.2]

$$(4.1.3) \qquad (0) = H^i_{cont}\left(G/H, \varprojlim_n M_n^H\right) = \varprojlim_n H^i(G/H, M_n^H).$$

Combining (4.1.2) and (4.1.3) yields (4.1.1).

As a consequence of (4.1.1), the cycle map γ in (2.3.2) yields a map

$$B^2(V_{\bar{\mathbf{Q}}})^G \xrightarrow{\tau} H^1(G, M).$$

Suppose now that $Z \cdot V$ is divisible in $B^2(V_{\bar{\mathbf{Q}}})$. Fix $r \geqq 1$ and write

$$(4.1.4) \qquad Z \cdot V = \ell^r \cdot A_r$$

in $CH^2(V_{\bar{\mathbf{Q}}})$. Let N be a positive number annihilating the torsion of $H^4(V_{\bar{\mathbf{Q}}}, \mathbf{Z}_\ell(2))$. Then replacing $Z \cdot V$ by $N(Z \cdot V)$ fulfilling iii) as well, and A_r by $N\, A_r$, we may assume that $A_r \in B^2(V_{\bar{\mathbf{Q}}})$. Let $H \subset G = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ be normal of finite index such that A_r is defined over the fixed field of H. For $\sigma \in G/H$, $\sigma(A_r) - A_r \in B^2(V_{\bar{\mathbf{Q}}})\{\ell\} = (0)$. Thus

$$\sigma\gamma(A_r) - \gamma(A_r) = \gamma(\sigma(A_r) - A_r) = 0,$$

so

$$\gamma(A_r) \in H^1(H, M)^{G/H} = H^1(G, M).$$

But now from (4.1.4) it follows that $\gamma(Z \cdot V)$ dies in $H^1(G, M_r)$. Since r was arbitrary and $H^1(G, M) = \varprojlim H^1(G, M_r)$ we get $\gamma(Z \cdot V) = 0$, contradicting (4.0)(iii). QED

Proposition (4.2). *Let $f : \mathscr{W} \to S$ be the morphism obtained by blowing up the base locus of the pencil in* (4.0) *and restricting to the open curve S of smooth fibres. Let $\mathscr{Z} \in CH^2(\mathscr{W}_{\mathbf{C}})$ be the class of the pullback of Z. There exists an $n \geqq 2$ such that for any finite surjective map of smooth curves $\pi : T_{\mathbf{C}} \to S_{\mathbf{C}}$, it is* not *the case that $\pi^* \mathscr{Z} = n \cdot \mathscr{Y}$ for some $\mathscr{Y} \in CH^2(\mathscr{W} \times_S T_{\mathbf{C}})$.*

Proof. Let f be defined over $F \subset \bar{\mathbf{Q}}$, and let $u \in S(F)$ be such that $V(u) := f^{-1}(u)$ satisfies (4.0)(i), (ii), and (iii). By (4.1) there exists an n such that $Z \cdot V(u)$ is not divisible by n in $CH^2(V(u)_{\mathbf{C}})$. Indeed, if $Z \cdot V(u) = n \cdot Y$ in $CH^2(V(u))_{\mathbf{C}}$, then Y is defined over some subfield L of $\mathbf{C}$ of finite type over $\bar{\mathbf{Q}}$. Taking a corresponding $\bar{\mathbf{Q}}$ rational point of the variety defined by L leads to a solution defined over $\bar{\mathbf{Q}}$. If $\mathscr{Y}$ existed as in the statement of (4.2), one could pull back to some $\mathbf{C}$-point of T lying over u and get a contradiction.

QED

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