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Inventiones mathematicae

Varieties over a finite field with trivial Chow group of 0-cycles have a rational point

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1. Introduction

Let X be a smooth projective variety of dimension d over a field k. Let $\overline{k(X)}$ be the algebraic closure of its function field. If the Chow group of 0-cycles $CH_0(X \times_k \overline{k(X)})$ is equal to $\mathbb{Z}$, then S. Bloch shows in [4], Appendix to Lecture 1, that the diagonal $\Delta \in CH^d(X \times_k X) \otimes_{\mathbb{Z}} \mathbb{Q}$ decomposes. This means there are a $N \in \mathbb{N} \setminus \{0\}$, a 0-dimensional subscheme $\xi \subset X$, a divisor $D \subset X$, a dimension d cycle $\Gamma \subset X \times D$ such that

$$N \cdot \Delta \equiv \xi \times X + \Gamma. \tag{1.1}$$

For sake of completeness, we briefly recall his argument. Using the norm, one sees that the kernel of $CH_0(X \times_k k(X)) \to CH_0(X \times_k \overline{k(X)})$ is torsion. Thus up to torsion, the cycle Spec $k(X)$ is equivalent in $CH_0(X \times_k k(X))$ to a $\bar{k}$-rational point of X, thus, up to torsion, to a ξ as above. On the other hand, $CH_0(X \times_k k(X))$ is the inductive limit of $CH_0(X \times_k (X \setminus D))$ as D runs over the divisors of X.

This has various consequences on the shape of de Rham or Hodge cohomologies in characteristic 0. Let $(\Delta) \in H^{2d}(X \times X)$ be the cycle class of Δ. One applies the correspondence $[\Delta]_* = p_{2,*}((\Delta) \cup p_1^*) = [\xi \times X]_* + [\Gamma]_*$ to, for example, $H^i_{DR}(X)$. Then $[\xi \times X]_* H^i_{DR}(X) = 0$ for $i \geq 1$ as the correspondence factors through $H^i_{DR}(\xi) = 0$, while $[\Gamma]_* H^i_{DR}(X) \subset H^i_{DR}(X)$ dies via the restriction map $H^i_{DR}(X) \to H^i_{DR}(X \setminus D)$. Using the surjection $H^i_{DR}(X) \to H^i(X, \mathcal{O}_X)$ to lift classes, and the factorization $H^i_{DR}(X \setminus D) \to H^i(X, \mathcal{O}_X)$ coming from Deligne's Hodge theory ([7]), one concludes that $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. S. Bloch developed this argument, and variants of it for étale cohomology, to kill the algebraic part of H^2 under the representability assumption of the Chow group of 0-cycles over $\overline{k(X)}$ (Mumford's theorem).

The purpose of this note it to observe that P. Berthelot's rigid cohomology [1] has the required properties to make the above argument work in this framework. If k has characteristic $p > 0$, let $W(k)$ be its ring of Witt vectors, K be the quotient field of $W(k)$. Let X be smooth proper over k, $Z \subset X$ be a closed subvariety, and $U = (X \setminus Z)$ be its complement. The rigid cohomology, which coincides with the crystalline cohomology on X, fulfills a localization sequence ([1], 2.3.1)

$$(1.2) \qquad \ldots \to H^i_Z(X/K) \to H^i(X/K) \to H^i(U/K) \to \ldots$$

which is compatible with the Frobenius action ([5], Theorem 2.4). By [3] and [13], the slope $[0 \ \ 1[$ part of $H^i(X/K)$ is $H^i(X, W\mathcal{O}_X) \otimes_{W(k)} K$. One has

Theorem 1.1. *Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$. If the Chow group of 0-cycles $CH_0(X \times_k \overline{k(X)})$ is equal to $\mathbb{Z}$, then the slope $[0 \ \ 1[$ part of $H^i(X/K)$ is vanishing for $i > 0$.*

On the other hand, if one now assumes that $k = \mathbb{F}_q$ is a finite field, with $q = p^n$, the Lefschetz trace formula for crystalline cohomology (see e.g. [11], Sect. II. 1)

$$(1.3) \qquad |X(k)| = \sum_i (-1)^i \mathrm{Trace}(\mathrm{Frob}^n | H^i(X/K))$$

implies in particular that if all the slopes of Frob are ≥ 1 and X is geometrically connected, then

$$(1.4) \qquad |X(k)| \equiv 1 \text{ modulo } q.$$

Thus one has

Corollary 1.2. *Let X be a smooth, projective, geometrically connected variety over a finite field k. If the Chow group of 0-cycles $CH_0(X \times_k \overline{k(X)})$ is equal to $\mathbb{Z}$, then X has a rational point over k.*

An example of application is provided by Fano varieties. A variety X is said to be Fano if it is smooth, projective, geometrically connected and the dual of the dualizing sheaf ω_X is ample. By [15], Theorem V. 2.13, Fano varieties on any algebraically closed field are chain rationally connected ([15], Definition IV. 3.2) in the sense that any two rational points can be joined by a chain of rational curves. This implies in particular that $CH_0(X \times_k \overline{k(X)}) = \mathbb{Z}$. Thus one concludes

Corollary 1.3. *Let X be a Fano variety over a finite field k, or more generally, let X be a smooth, projective, geometrically connected variety over a finite field k, which is chain rationally connected over $\overline{k(X)}$. Then X has a rational point.*

This corollary answers positively a conjecture by S. Lang [16] and Yu. Manin [17]. This is the reason why we write down the argument for Theorem 1.1, while it is a direct adaption of S. Bloch's argument [4] to crystalline and rigid cohomologies.

That a study of crystalline cohomology should yield via the congruence (1.3) the existence of a rational point on Fano varieties over finite fields is entirely due to M. Kim. His idea was to kill the whole cohomology $H^i(X, W\omega_X)$ for $i < \dim(X)$, using solely the structure of crystalline cohomology on X, together with its Verschiebung and Frobenius operators, and using the ampleness of ω_X^{-1}. The point of Corollary 1.2 is that Kollár-Miyaoka-Mori's and Campana's theorem ([15], loc. cit.), which is anchored in geometry, together with Bloch's type Chow group argument, force (weaker) cohomological consequences. In a way, the difficult theorem is the geometric one.

Acknowledgements. This note relies on P. Berthelot's work on rigid cohomology, with which the author is not familiar. It is a pleasure to thank P. Berthelot, S. Bloch and O. Gabber for their substantial help and for their encouragement. I thank the IHES for support during the preparation of this work.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Thus X is a smooth projective variety over k. For any codimension d cycle $Z \subset X \times X$, the correspondence

$$[Z]_* = p_{2,*}((Z) \cup p_1^*) \tag{2.1}$$

is well defined on $H^i(X/K)$. One needs for this the existence of the cycle class

$$(Z) \in H^{2d}((X \times X)/K) \tag{2.2}$$

which is provided by [1], Corollaire 5.7, (ii), and by [18], 6.2, for the factorization through the Chow group, the contravariance for p_1^* and the covariance for $p_{2,*}$, applied to crystalline cohomology of smooth proper varieties. In particular, this correspondence factors through $H^i(\xi/K)$ if $Z = (\xi \times X)$, which shows via formula (1.1) that

$$N[\Delta]_* = [\Gamma]_* \tag{2.3}$$

on $H^i(X/K)$ for $i > 0$. On the other hand, since $\Gamma \subset X \times D$, $[\Gamma]_*$, seen as a correspondence of X to $(X \setminus D)$, is trivial. One has

$$\begin{aligned}[\Gamma]_*(H^i(X/K)) &\subset \mathrm{Ker}(H^i(X/K) \to H^i((X \setminus D)/K) \\ &= \big(\text{by } (1.2)\big)\ \mathrm{Im}(H^i_D(X/K) \subset H^i(X/K)).\end{aligned} \tag{2.4}$$

Thus, using [5], Theorem 2.4, Theorem 1.1 is a consequence of the following

Lemma 2.1. *(P. Berthelot) Let X be a smooth, geometrically connected, quasi-compact and separated scheme over a perfect field k of characteristic $p > 0$ and let $Z \subset X$ be a non-empty subvariety of codimension ≥ 1. Then the slopes of $H^i_Z(X/K)$ are ≥ 1.*

Proof. Let $\ldots \subset Z_i \subset Z_{i-1} \subset \ldots \subset Z_0 = Z$ be a finite stratification by closed subsets such that $Z_{i-1} \setminus Z_i$ is smooth. The localization [1], 2.5.1

$$\ldots \to H^i_{Z_i}(X/K) \to H^i_{Z_{i-1}}(X/K) \to H^i_{(Z_{i-1}\setminus Z_i)}((X \setminus Z_i)/K) \to \ldots \tag{2.5}$$

allows to reduce to the case where Z is smooth. If X is affine, then the Gysin isomorphism (purity) $H^{i-2\cdot\mathrm{codim}(Z)}(Z) \xrightarrow{\cong} H^i_Z(X)$ commutes to $p^{\mathrm{codim}(Z)}\cdot$ Frob on $H^{i-2\cdot\mathrm{codim}(Z)}(Z/K)$ and Frob on $H^i_Z(X/K)$ ([5], Theorem 2.4). Since the slopes on $H^i(Z/K)$ are all ≥ 0, we conclude that the slopes of the cohomology with support are ≥ 1. In general, one considers a finite affine covering $X = \cup_{i=0}^N U_i$. The spectral sequence

$$E_2^{ab} = H^a(\ldots \to H^b_Z(U^{a-1}/K) \to H^b_Z(U^a/K) \to H^b_Z(U^{a+1}/K) \to \ldots) \tag{2.6}$$

converges to $H^{a+b}(X/K)$. Here the open sets U^a are the $(a+1)$ by $(a+1)$ intersections of the U_i and the maps are the restriction maps. If $H^b_Z(U^a/K) \neq 0$, then Z meets U^a and its slopes are ≥ 1 by the previous case. Thus the spectral sequence has only contributions with ≥ 1 slopes. This finishes the proof. □

3. Comments

Corollary 1.3 can be compared to the main theorems of [12], resp. [6], where the finite field is replaced by $k = F(\text{curve})$ for $F = \mathbb{C}$, resp. an algebraically closed field in characteristic $p > 0$. There the authors show the existence of a rational point on a smooth Fano variety defined over k. In the latter case, one has to add "separably" to "chain rationally connected".

On the other hand, if X is a hypersurface $\subset \mathbb{P}^n$ of degree $\leq n$ over a field k of characteristic 0, then $CH_0(X \times_k \overline{k(X)}) = \mathbb{Z}$ by Roitman's theorem ([19]), whether X is smooth or not. It suggests that perhaps there is a stronger version of Theorem 1.1, requiring X projective but not smooth. This would be compatible with the congruence results of Ax and Katz [14], and their Hodge theoritic counter-part ([8], [9], [10]). In this case it says $H^i(X', \mathcal{O}_{X'}) = 0$, where $X' = \pi^{-1}(X)_{\mathrm{red}}$ and $\pi : \mathbb{P} \to \mathbb{P}^n$ is a birational map with $\mathbb{P}$ smooth, such that X' is a normal crossings divisor. The method used here does not apply to the singular situation.

After receiving this note, G. Faltings and O. Gabber explained to me that one could use a similar argument in étale cohomology in order to obtain the same conclusion, and M. Kim replaced the use of rigid cohomology in the localisation argument presented here by the one of de Rham-Witt cohomology with logarithmic poles. Since it does not yield a stronger result, we do not develop their arguments here.

Finally let us observe that, replacing D in the argument by a subvariety of codimension $\kappa \geq 1$, replaces the congruence (1 mod q) by (1 mod q^κ). But it is hard to understand what are the geometric conditions which guarantee higher codimension. Again, hypersurfaces of very low degree fulfill the correct congruence ([14]), but I do not know whether it is reflected by this strong Chow group property. Without this, the method presented here gives a different proof of the main result of [14] in the smooth case when $\kappa = 1$.

References

1. Berthelot, P.: Finitude et pureté en cohomologie rigide. Invent. Math. **128**, 329–377 (1997)
2. Berthelot, P.: Dualité de Poincaré et formule de Künneth en cohomologie rigide. C. R. Acad. Sci., Paris, Sér. I, Math. **325**, 493–498 (1997)
3. Bloch, S.: Algebraic K-theory and crystalline cohomology. Publ. Math., Inst. Hautes Étud. Sci. **47**, 187–268 (1977)
4. Bloch, S.: Lecture on Algebraic cycles. Duke University Mathematics Series, IV (1980)
5. Chiarellotto, B.: Weights in rigid cohomology and applications to unipotent F-isocrystals. Ann. Sci. Éc. Norm. Supér., IV. Ser. **31**, 683–715 (1998)
6. de Jong, J., Starr, J.: Every rationally connected variety over the function field of a curve has a rational point. Preprint 2002
7. Deligne, P.: Théorie de Hodge II. Publ. Math., Inst. Hautes Étud. Sci. **40**, 5–57 (1972)
8. Deligne, P.: Dimca, A.: Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières. Ann. Sci. Éc. Norm. Supér., IV. Ser. **23**, 645–656 (1990)
9. Esnault, H.: Hodge type of subvarieties of $\mathbb{P}^n$ of small degrees. Math. Ann. **288**, 549–551 (1990)
10. Esnault, H., Nori, M., Srinivas, V.: Hodge type of projective varieties of low degree. Math. Ann. **293**, 1–6 (1992)
11. Etesse, J.-Y.: Rationalité et valeurs de fonctions L en cohomologie cristalline. Ann. Sci. Inst. Fourier **38**, 33–92 (1988)
12. Graber, T., Harris, J., Starr, J.: Families of rationally connected varieties. Preprint 2002
13. Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline. Ann. Sci. Éc. Norm. Supér., IV. Ser. **12**, 501–661 (1979)
14. Katz, N.: On a theorem of Ax. Am. J. Math. **93**, 485–499 (1971)
15. Kollár, J.: Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge **32** (1996). Springer-Verlag, Berlin 1996
16. Lang, S.: Cyclotomic points, very anti-canonical varieties, and quasi-algebraic closure. Preprint 2000
17. Manin, Yu.: Notes on the arithmetic of Fano threefolds. Compos. Math. **85**, 37–55 (1993)
18. Petrequin, D.: Classes de Chern et classes de cycles en cohomologie rigide. Preprint 2001, 69 pages. To appear in Bull. Soc. Math. de France
19. Roitman, A.: Rational equivalence of zero-dimensional cycles. Mat. Zametki **28**, 85–90 (1980)