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DECOMPOSITION OF THE DIAGONAL AND EIGENVALUES OF FROBENIUS FOR FANO HYPERSURFACES

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Abstract. Let $X \subset \mathbb{P}^n$ be a possibly singular hypersurface of degree $d \leq n$, defined over a finite field $\mathbb{F}_q$. We show that the diagonal, suitably interpreted, is decomposable. This gives a proof that the eigenvalues of the Frobenius action on its ℓ-adic cohomology $H^i(\bar{X}, \mathbb{Q}_\ell)$, for $\ell \neq \mathrm{char}(\mathbb{F}_q)$, are divisible by q, without using the result on the existence of rational points by Ax and Katz.

1. Introduction. If X is a variety defined over a finite field $k = \mathbb{F}_q$, one encodes the number of its rational points over all finite extensions $\mathbb{F}_{q^s} \supset \mathbb{F}_q$ in the zeta function, defined by its logarithmic derivative

$$\frac{\zeta'(X,t)}{\zeta(X,t)} = \sum_{s \geq 1} |X(\mathbb{F}_{q^s})| t^{s-1}. \tag{1.1}$$

By the theorem of Dwork [11], we know that $\zeta(t)$ is a rational function

$$\zeta(X,t) \in \mathbb{Q}(t). \tag{1.2}$$

We assume that X is projective and we denote by $U = \mathbb{P}^n \setminus X$ the complement of a projective embedding. The Grothendieck-Lefschetz trace formula [17] gives a cohomological formula for the numerator and the denominator of the rational function

$$\zeta(U,t) = \prod_{i=0}^{2\ \dim(\mathrm{U})} \det(1 - F_i t)^{(-1)^{i+1}}, \tag{1.3}$$

where F_i is the geometric Frobenius acting on the compactly supported ℓ-adic cohomology $H^i_c(\overline{U}, \mathbb{Q}_\ell)$. Letting $H^i_{\mathrm{prim}}(\overline{X}, \mathbb{Q}_\ell)$ denote the primitive cohomology $H^i(\overline{X}, \mathbb{Q}_\ell)/H^i(\overline{\mathbb{P}^n}, \mathbb{Q}_\ell)$ of X, we have

$$H^i_c(\overline{U}, \mathbb{Q}_\ell) \cong \begin{cases} H^{i-1}_{\mathrm{prim}}(\overline{X}, \mathbb{Q}_\ell) & \text{for } (i-1) \leq 2\ \dim(X), \\ H^i(\overline{\mathbb{P}^n}, \mathbb{Q}_\ell) & \text{for } i \geq 2\ \dim(X) + 2. \end{cases} \tag{1.4}$$

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For X smooth and complete, the Weil conjectures [9] assert that the the eigenvalues of F_i in any complex embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ have absolute values $q^{\frac{(i-1)}{2}}$ if $i \leq 2\dim(X) + 1$ and $q^{\frac{i}{2}}$ if $i \geq 2\dim(X) + 2$. In particular, there is no possible cancellation of eigenvalues between the numerator and the denominator of the zeta function. Consequently, the property

$$|X(\mathbb{F}_{q^s})| \equiv |\mathbb{P}^n(\mathbb{F}_{q^s})| \bmod q^{\kappa s} \tag{1.5}$$

for all $s \geq 1$, and some $\kappa \in \mathbb{N} \setminus \{0\}$, is equivalent to the property that

(1.6) the eigenvalues of F_i are divisible by q^κ, as algebraic integers.

However, if X is singular, one does not have in general the purity of weights of Frobenius on H^i_c. Thus, a cancellation between the numerator of the zeta function and its denominator is at least in principle possible, and the property (1.5) is no longer *a priori* equivalent to the property (1.6). The purpose of this article is to study the relation between (1.5) and (1.6) in the case of hypersurfaces of degree $d \leq n$.

Let X be a complete intersection in $\mathbb{P}^n$ defined by r equations of degrees $d_1 \geq d_2 \geq \cdots \geq d_r$ with the property

$$1 \leq \kappa = [\frac{n - d_2 - \cdots - d_r}{d_1}]. \tag{1.7}$$

The theorem of Ax and Katz says precisely that (1.5) holds true. On the other hand, we also know by [7], [10], [12], that if the finite field is replaced by a field of characteristic 0, the Hodge type of X is κ for all cohomology groups of X. (See [4] for a more precise discussion of those theorems). This gives a strong indication that (1.6) should be true as well. Indeed, as explained to us by Daqing Wan, (1.7) is true for $\kappa = 1$. One knows by [8], Theorem 5.5.3, that q divides the eigenvalues of Frobenius acting on $H^a(\overline{X}, \mathbb{Q}_\ell)$ for $a > \dim(X)$, and since this cohomology vanishes for complete intersections for $a < \dim(X)$, the theorem of Ax and Katz implies divisibility by q for $a = \dim(X)$ as well. Similarly, using vanishing and Ax-Katz's result, and replacing [8], Theorem 5.5.3 by the corresponding statement for the slopes of the Frobenius action on rigid cohomology ([22], p. 820), one obtains that the slopes of the Frobenius action on rigid cohomology are ≥ 1.

The purpose of this note is to give a motivic interpretation for Fano hypersurfaces and $\kappa = 1$ of the divisibility result, which does not use the theorem by Ax and Katz.

We now describe our method. Let us first assume that X is smooth. By Roitman's theorem [21], we know that $CH_0(X \times_k K) = \mathbb{Z}$ for any field extension $K \supset k$ which is algebraically closed. By [2, Appendix to lecture 1], this implies that the class of the diagonal in $CH_{n-1}(X\times X)$ goes to zero in $CH_{n-1}(X\times X\setminus(\xi\times X\cup X\times A))_{\mathbb{Q}}$ for some divisor A on X and some 0-cycle ξ. Letting this class act as a correspondence, it follows that the restriction map $H^i(\overline{X},\mathbb{Q}_\ell)\to H^i(\overline{X\setminus A},\mathbb{Q}_\ell)$ is zero for $i\geq 1$. This shows divisibility, as in [13], Lemma 2.1.

For singular varieties, the proof of the Hodge-type statement in the complete intersection singular case ([12]) shows that the cohomology with compact support $H^i_c(U) =: H^i(\mathbb{P}^n, X)$ carries the necessary information, and is easier to deal with than its dual $H^j(U)$. To carry out the argument used in the smooth case, one needs a version of the Chow groups which is related to compactly supported cohomology. If X is a strict normal crossing divisor, one can use the *relative motivic cohomology* $H^{2n}_M(\mathbb{P}^n\times U, X\times U,\mathbb{Z}(n))$, as defined in [20], chapter 4, 2.2 and p. 209; this relative motivic cohomology acts as correspondences on $H^*_c(\overline{U},\mathbb{Q}_\ell)$. Due to the lack of resolution of singularities in positive characteristic, we will in general need an *alteration* π: $(\mathbb{P}, Y)\to(\mathbb{P}^n, X)$ of $(\mathbb{P}^n, X)$, that is, a projective, generically finite morphism π: $\mathbb{P}\to\mathbb{P}^n$, with $\mathbb{P}$ smooth, such that $Y := \pi^{-1}(X)$ is a strict normal crossing divisor. We then use the relative motivic cohomology $H^{2n}_M(\mathbb{P}\times U, Y\times U,\mathbb{Z}(n))$.

Recall that $H^m_M(\mathbb{P}\times U, Y\times U,\mathbb{Z}(n))$ is the homology $H_{2n-m}(\mathcal{Z}^n(\mathbb{P}\times U, Y\times U, *))$, where $\mathcal{Z}^n(\mathbb{P}\times U, Y\times U, *)$ is the single complex associated to the double higher Chow cycle complex

(1.8)
$$\begin{tikzcd}
\cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "\partial"'] & \cdots \arrow[d, "\partial"'] \\
\mathcal{Z}^n(\mathbb{P}\times U,1) \arrow[r, "\mathrm{rest}"] \arrow[d, "\partial"'] & \mathcal{Z}^n(Y^{(1)}\times U,1) \arrow[r, "\mathrm{rest}"] \arrow[d, "\partial"'] & \mathcal{Z}^n(Y^{(2)}\times U,1) \arrow[d, "\partial"'] \\
\mathcal{Z}^n(\mathbb{P}\times U,0) \arrow[r, "\mathrm{rest}"] & \mathcal{Z}^n(Y^{(1)}\times U,0) \arrow[r, "\mathrm{rest}"] & \mathcal{Z}^n(Y^{(2)}\times U,0).
\end{tikzcd}$$

Here $Y^{(a)}$ is the normalization of all the strata of codimension a, $\mathcal{Z}^n(Y^{(a)}\times U, b)$ is a group of cycles on $Y^{(a)}\times U\times S^b$ where $S^\bullet$ is the cosimplicial scheme $S^n = \mathrm{Spec}\,(k[t_0,\ldots,t_n]/(\sum t_i - 1))$ with face maps $S^n\hookrightarrow S^{n+1}$ defined by $t_i = 0$. More precisely, $\mathcal{Z}^n(Y^{(a)}\times U, b)$ is generated by the codimension n subvarieties $Z\subset Y^{(a)}\times U\times S^b$ such that, for each face F of S^b, and each irreducible component $F'\subset Y^{(a)}$ of the strata of Y we have $\mathrm{codim}_{F'\times U\times F}(Z\cap(F'\times U\times F))\geq n$. The horizontal restriction maps are the intersection with the smaller strata, the vertical ∂'s are the boundary maps.

For technical reasons, we find it convenient to use a subcomplex $\mathcal{Z}^n(\mathbb{P}\times U,\mathcal{I}(Y\times U), *)$ of $\mathcal{Z}^n(\mathbb{P}\times U, Y\times U, *)$. For T a smooth k-scheme of finite type, and A a closed subset, let $\mathcal{Z}^n(T,\mathcal{I}(A), m)$ be the subgroup of $\mathcal{Z}^n(T, m)$ consisting of the cycles $W\in\mathcal{Z}^n(T, m)$ with $\mathrm{Supp}(W)\cap(A\times S^m) = \emptyset$. The $\mathcal{Z}^n(T,\mathcal{I}(A), m)$ evidently form a subcomplex $\mathcal{Z}^n(T,\mathcal{I}(A), *)$ of $\mathcal{Z}^n(T, *)$, functorial for flat pullback and proper push-forward. Set

$$H^m_M(T, \mathcal{I}(A), \mathbb{Z}(n)) := H_{2n-m}(\mathcal{Z}^n(T, \mathcal{I}(A), *)). \tag{1.9}$$

The inclusion $\mathcal{Z}^n(\mathbb{P} \times U, \mathcal{I}(Y \times U), *) \to \mathcal{Z}^n(\mathbb{P} \times U, *)$ extends to a map of complexes $\mathcal{Z}^n(\mathbb{P} \times U, \mathcal{I}(Y \times U), *) \to \mathcal{Z}^n(\mathbb{P} \times U, Y \times U, *)$. We call $H^m_M(T, \mathcal{I}(A), \mathbb{Z}(n))$ the *motivic cohomology with modulus*.

Let $\Delta \subset \mathbb{P} \times U$ be the inverse image of the diagonal $\subset \mathbb{P}^n \times U$, i.e., Δ is the graph of the alteration π restricted to $\mathbb{P} \times U$. Since $\mathrm{rest}(\Delta) = 0$, Δ yields a class

$$[\Delta] \in H^{2n}_M(\mathbb{P} \times U, \mathcal{I}(Y \times U), \mathbb{Z}(n)). \tag{1.10}$$

We show:

THEOREM 1.1. *Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d \leq n$ over a field k. Then there is an alteration π: $(\mathbb{P}, Y) \to (\mathbb{P}^n, X)$ and a divisor $A \subset \mathbb{P}$ which cuts all the strata of Y in codimension ≥ 1 and such that the image of* $[\Delta]$ *in* $H^{2n}_M((\mathbb{P} \setminus A) \times U, \mathcal{I}((Y \setminus A) \times U), \mathbb{Q}(n))$ *is zero.*

The main idea behind this geometric statement relies on the following. By a counting argument, Roitman [21] shows that, for a hypersurface $X \subset \mathbb{P}^n$ of degree $d \leq n$, the correspondence

$$\{(x, \ell) \in X \times \mathrm{Grass}(1, n) \mid \ell \subset X \text{ or } \ell \cap X = \{x\} \text{ for some } x \in X\} \tag{1.11}$$

dominates X. It follows that the map $\mathbb{Z} \cong CH_1(\mathbb{P}^n) \to CH_0(X)$ has cokernel killed by multiplication by $d = \deg X$, where $CH_0(X)$ is Fulton's homological Chow group ([16]). This implies $CH_0(X) \otimes \mathbb{Q} = \mathbb{Q}$.

We replace Roitman's correspondence by

$$\begin{aligned} P := \{&(y, \ell) \in \mathbb{P} \times \mathrm{Grass}(1, n) \mid \pi(y) \in \ell, \\ &\text{and either } \ell \subset X \text{ or } \ell \cap X = \{x\} \text{ for some } x \in X\}. \end{aligned} \tag{1.12}$$

We show that P dominates $\mathbb{P}$, and then use the technique of blowing up strata of Y introduced in [3] to find the rational equivalence relation which holds on the complement of some good divisor A. Finally, we show:

THEOREM 1.2. *Let $X \subset \mathbb{P}^n$ be a projective variety over a field k, and let $U = \mathbb{P}^n \setminus X$. Suppose there is an alteration π: $(\mathbb{P}, Y) \to (\mathbb{P}^n, X)$ and a divisor $A \subset \mathbb{P}$ which cuts all strata of Y in codimension ≥ 1, such that the image of* $[\Delta]$ *in* $H^{2n}_M((\mathbb{P} \setminus A) \times U, \mathcal{I}((Y \setminus A) \times U), \mathbb{Q}(n))$ *is zero.*

(1) *If the characteristic of the ground field k is 0, then* $\mathrm{gr}^F_0 H^i(X) = 0$ *for all* $i \geq 1$.

(2) *If $k = \mathbb{F}_q$ is a finite field, then the eigenvalues of the geometric Frobenius F_i acting on the compactly supported ℓ-adic cohomology $H^i_c(\overline{U}, \mathbb{Q}_\ell)$ are all divisible by q as algebraic integers.*

(3) *If k is a perfect field of characteristic p, then the slopes of the Frobenius operator acting on the rigid cohomology $H^i_c(U/K)$ are all ≥ 1.*

To conclude, we remark that this article solves the natural question posed in the introduction of [15], but only in the case $\kappa = 1$. Thinking of the discussion developed in [4], (5.2) for $\kappa \geq 2$ in the smooth case, it is not entirely clear what the substitute would be for 1.1. One may also try to generalize these results for $\kappa = 1$, replacing the hypersurface X with a more general singular Fano variety. A singular Fano variety X over a field is a gometrically connected, projective, Cohen-Macaulay variety such that the reflexive hull of $\omega^N_{X_{\mathrm{reg}}}$ is invertible and ample for some $N \in \mathbb{Z} \setminus \mathbb{N}$. Examples are hypersurfaces of degree $d \leq n$. The question is then whether a Fano variety fulfills Theorem 1.1. If yes, as in [13], this would show that a singular Fano variety over a finite field has a rational point.

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2. The proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. We fix a base-field k and write $\mathbb{P}^n$ for $\mathbb{P}^n_k$. We want to show that a certain class $[\Delta]$ in motivic cohomology with modulus is trivial. Suppose for a moment we know this vanishing for k an infinite field. If k is a finite field, there exist Galois extensions k_ℓ/k with Galois group $\mathbb{Z}_\ell$ for any prime ℓ. In particular, k_ℓ is infinite, so $[\Delta_{k_\ell}] = 0$ by hypothesis. Since the motivic cohomology with modulus over k_ℓ is a direct limit over motivic cohomology with modulus over finite subfields $k' \subset k_\ell$, and since motivic cohomology with modulus admits a norm, we conclude $[\Delta]$ is killed by some power of ℓ. Since this is true for two different ℓ, and the union of Galois translates of A in good position with respect to Y is still in good position, the theorem follows. Thus, we may assume k is infinite. In particular, we will use without comment various general position arguments.

Fix $X \subset \mathbb{P}^n$ a hypersurface of degree $d \leq n$. We want to define a closed subvariety Z inside the Grassmann of lines $\mathrm{Grass}(1, n)$ consisting of lines "maximally tangent" to X. We have the incidence correspondence $\mathcal{U} := \{(z, \ell) \mid z \in \ell \in \mathrm{Grass}(1, n)\}$. Define

$$\mathcal{V} := \mathcal{U} \times_{\mathrm{Grass}(1,n)} \mathcal{U} = \{(y, \ell, z) \mid y, z \in \ell\}. \tag{2.1}$$

The $\mathbb{P}^1$-bundle $\mathrm{pr}_2\colon\ \mathcal{V} \to \mathcal{U}, (y, \ell, z) \mapsto (z, \ell)$ has a section $(x, \ell) \mapsto (x, \ell, x)$. Locally on $\mathcal{U}$ we may identify $\mathcal{V} \cong \mathbb{P}^1 \times \mathcal{U}$ with homogeneous coordinates s, t in such a way that the section is given by $t = 0$. The section $\mathcal{O}_{\mathbb{P}^n} \xrightarrow{X} \mathcal{O}_{\mathbb{P}^n}(d)$ pulls back to a section of $p^*\mathcal{O}(d)_{\mathbb{P}^n}$ under the projection $p\colon \mathcal{V} \to \mathbb{P}^n, (y, \ell, z) \mapsto y$, and the section X restricts to an equation $F(s,t) = F_0 s^d + F_1 s^{d-t} t + \cdots + F_d t^d$, where the F_i are (local) functions on $\mathcal{U}$. Note F_0 is a local defining equation of $\mathcal{U} \times_{\mathbb{P}^n} X \subset \mathcal{U}$. We are interested in the closed sets defined locally by $F_0 = \cdots = F_{d-1} = 0$ (resp. $F_0 = \cdots = F_d = 0$). Denote by

$$Z'_X \subset Z' \subset \mathrm{Grass}(1,n) \tag{2.2}$$

the projection on the Grassmannian of these sets. Intuitively, the open set $Z^0 := Z' \setminus Z'_X$ consists of lines d-fold tangent to X at a point. Define $Z \subset Z'$ to be the closure of Z^0, and let $Z_X := Z'_X \cap Z$.

Proposition 2.1. *The projection*

$$p_2\colon Z \times_{\mathrm{Grass}(1,n)} \mathcal{U} \to \mathbb{P}^n$$

is surjective. Intuitively, a general point on $\mathbb{P}^n$ has a line through it maximally tangent to X.

Proof. It suffices to consider geometric points. Let $y \in \mathbb{P}^n \setminus X$ be a geometric point. There is a linear transformation of $\mathbb{P}^n$ such that the equation of X is $x_0^d + x_0^{d-1} f_1(x_1, \ldots, x_n) + \cdots + f_d(x_1, \ldots, x_n)$, with $f_i \in k[x_1, \ldots, x_n]$ homogeneous of degree i, and such that y has homogeneous coordinates $(1\colon 0\colon \cdots\colon 0)$. Thus a line passing through y has parametrization $(s\colon tu_1\colon \cdots\colon tu_n) \in \mathbb{P}^n$, for $(s\colon t) \in \mathbb{P}^1$ and $(u_1\colon \cdots\colon u_n) \in \mathbb{P}^{n-1}$. The intersection of this line with X has equation $s^d + s^{d-1} t f_1(u_1, \ldots, u_n) + \cdots + t^d f_d(u_1, \ldots, u_n)$ and its intersection with X will be d-tangent if and only if this equation has the shape $(s+tu)^d$ with $(u\colon u_1\colon \cdots\colon u_n) \in \mathbb{P}^n$. This is equivalent to the d homogeneous equations $f_i = \binom{d}{i} u^i, i = 1, \ldots, d$ in $(u\colon u_1\colon \cdots u_n) \in \mathbb{P}^n$. Since $d \leq n$ there exists a homogeneous solution. □

Example 2.2. Let $(T_0\colon T_1\colon T_2)$ be homogeneous coordinates on $\mathbb{P}^2$, and let $X\colon T_0 T_1 = 0$, so $d = n = 2$. Clearly in this case Z is simply the variety of lines through $(0\colon 0\colon 1)$, and $Z_X \subset Z$ is the two points corresponding to the components of X.

It follows from Proposition 2.1 that $\dim Z \geq n - 1$. Let $\mathcal{Z} \subset Z$ be a general linear section of dimension $n - 1$, and write $\mathcal{Q} := \mathcal{Z} \times_{\mathrm{Grass}(1,n)} \mathcal{U}$. Recall that an *alteration* $\pi\colon (\mathbb{P}, Y) \to (\mathbb{P}^n, X)$ of $(\mathbb{P}^n, X)$ is a projective, generically finite morphism $\pi\colon \mathbb{P} \to \mathbb{P}^n$, with $\mathbb{P}$ smooth, such that $Y := \pi^{-1}(X)$ is a strict normal crossing divisor.

LEMMA 2.3. *There exists a commutative diagram of schemes*

$$\begin{tikzcd} \mathbb{P} & \mathcal{P} \arrow[l, "f"'] \arrow[d, "g"] \arrow[dr, "q"] & \\ \mathbb{P}^n \arrow[from=u, "\pi"'] & \mathcal{Q} \arrow[l, "\bar{r}"] \arrow[r, "\bar{p}"'] & \mathcal{Z} \end{tikzcd}$$

satisfying the following conditions:

(1) *$\mathcal{Q}$ and $\mathcal{Z}$ are as above, and $\bar{p}$ and $\bar{r}$ are the natural maps. In particular, $\bar{p}$ is a $\mathbb{P}^1$-bundle and $\bar{r}$ is surjective.*

(2) *$\mathcal{P}$ is irreducible and normal.*

(3) *$f\colon \mathcal{P} \to \mathbb{P}$ is projective, generically finite and surjective.*

(4) *$q\colon \mathcal{P} \to \mathcal{Z}$ is surjective.*

(5) *There exists a normal crossings divisor $Y \subset \mathbb{P}$ such that $\pi\colon (\mathbb{P}, Y) \to (\mathbb{P}^n, X)$ is an alteration.*

(6) *There exists a divisor $A \subset \mathbb{P}$ such that A meets all the strata of Y properly, and such that $\mathcal{P} \setminus f^{-1}(A) \to \mathbb{P} \setminus A$ is finite.*

Given a surjection of projective k-schemes $\mathcal{Q}' \twoheadrightarrow \mathcal{Q}$, the map g can be taken to factor $\mathcal{P} \to \mathcal{Q}' \to \mathcal{Q}$.

Proof. Since k is infinite and $\mathcal{Z} \subset Z$ is a general linear space section, the map $\bar{r}$ in 1) is surjective. Let π be an alteration. Then there will be an irreducible component of $\mathbb{P} \times_{\mathbb{P}^n} \mathcal{Q}$ dominating both $\mathcal{Q}$ and $\mathbb{P}$. Taking $\mathcal{P}$ to be the normalization of this component gives (2), (3), and (4). (To see the final assertion, one can replace $\mathcal{Q}'$ by a plane section and assume $\mathcal{Q}' \to \mathcal{Q}$ has finite degree. Then substitute $\mathbb{P} \times_{\mathbb{P}^n} \mathcal{Q}'$ in the above.) Condition (5) comes from the work of de Jong [5].

To prove (6), we use the following result ([3], Theorem 2.1.2):

THEOREM 2.4. *Let $Y \subset \mathbb{P}$ be a normal crossings divisor in a smooth variety. Let $f\colon W \to \mathbb{P}$ be a finite type morphism, and assume that $W \setminus f^{-1}(Y) \subset W$ is dense. Given $p\colon \mathbb{P}' \to \mathbb{P}$ the blowup of a face (stratum) of Y, let $Y' = p^*(Y)_{\mathrm{red}}$ be the reduced pullback, and let $f'\colon W' \to \mathbb{P}'$ be the strict transform of f. Then $(\mathbb{P}', Y', f')$ satisfy the same hypotheses as $(\mathbb{P}, Y, f)$. There exists a composition of such blowups, $(\mathbb{P}_N, Y_N) \to \cdots \to (\mathbb{P}, Y)$ such that the strict transform morphism $f_N\colon W_N \to \mathbb{P}_N$ meets the faces of Y_N properly, i.e. for $Z \subset \mathbb{P}_N$ a face of codimension r, $f_N^{-1}(Z) \subset W_N$ has codimension $\geq r$.*

Replacing the alteration $\mathbb{P} \to \mathbb{P}^n$ with a composition $\mathbb{P}_N \to \cdots \to \mathbb{P} \to \mathbb{P}^n$ and changing notation, we may assume $f\colon \mathcal{P} \to \mathbb{P}$ meets faces properly. Since f has finite degree, this amounts to saying that the fibre of f over the generic point of any face is finite. The existence of a divisor A as in (6) is now clear. $\square$

LEMMA 2.5. *Let $p\colon Q \to B$ be a smooth projective morphism of k-schemes, with geometrically connected fibers of dimension one. Let $s_0, s_\infty\colon B \to Q$ be sections, take $\tilde{t} \in H^0(Q, \mathcal{O}_Q(s_\infty(B) - s_0(B)))$ and suppose that the rational function t on Q determined by $\tilde{t}$ satisfies*

$$\mathrm{div}(t) = s_0(B) - s_\infty(B).$$

Let $\bar{B} \subset B$ be the closed subscheme of B defined by the equation $s_0 = s_\infty$. Then the restriction $\bar{t}$ of t to $p^{-1}(\bar{B})$ is a unit, and there is a unit $\bar{u}$ on $\bar{B}$ with $\bar{t} = p^*(\bar{u})$.

Proof. The hypotheses imply $p_*\mathcal{O}_Q^\times = \mathcal{O}_B^\times$ and this continues to hold after pullback. Smoothness of p implies that $s_i(B) \subset Q$ are Cartier divisors. We view $\tilde{t}$ as an isomorphism $\tilde{t}\colon\ \mathcal{O}(s_\infty(B)) \cong \mathcal{O}(s_0(B))$. By definition, the Cartier divisors agree over $\bar{B}$, so there is a tautological identification $\tau\colon\ \mathcal{O}(s_0(B))_{p^{-1}(\bar{B})} \cong \mathcal{O}(s_\infty(B))_{p^{-1}(\bar{B})}$. The composition

$$\tilde{t} \circ \tau \in \Gamma\Big(p^{-1}(\bar{B}), \mathrm{Aut}\big(\mathcal{O}(s_0(B))\big)\Big) = \Gamma\Big(p^{-1}(\bar{B}), \mathcal{O}^\times_{p^{-1}(\bar{B})}\Big) \cong \Gamma(\bar{B}, \mathcal{O}^\times_{\bar{B}})$$

yields the desired unit $\bar{u}$. □

Recall $\mathcal{Z} \subset Z$ is a general linear section, where Z is a space of lines in $\mathbb{P}^n$ which are maximally tangent to our given hypersurface X. We have removed from Z any possible irreducible components consisting entirely of lines on X, so the subset $Z_X \subset Z$ of lines on X is nowhere dense. We define $\mathcal{Z}_X = \mathcal{Z} \cap Z_X$ and $\mathcal{Z}^0 = \mathcal{Z} \setminus \mathcal{Z}_X$. By generality, $\mathcal{Z}^0 \subset \mathcal{Z}$ is dense. Define $\mathcal{Q}^0 = \bar{p}^{-1}\mathcal{Z}^0$ (resp. $\mathcal{P}^0 = q^{-1}(\mathcal{Z}^0)$). The $\mathbb{P}^1$-bundle $\mathcal{Q}^0 \to \mathcal{Z}^0$ has a set-theoretic section $\bar{s}^0_\infty$ associating to a line ℓ the unique point in $\ell \cap X$.

Consider the diagram

(2.3)
$$\begin{tikzcd} \mathcal{P} \arrow[r, shift left, "s_0"] \arrow[d, "q"'] \arrow[dr, "g"] & \mathcal{P} \times_{\mathcal{Z}} \mathcal{Q} \arrow[l, shift left, "p"] \arrow[d, "p_2"] \\ \mathcal{Z} & \mathcal{Q}. \arrow[l, "\bar{p}"] \end{tikzcd}$$

Here the section s_0 corresponds to the map g. Similarly, the set-theoretic section $\bar{s}^0_\infty$ gives rise to a set-theoretic section $s^0_\infty\colon\ \mathcal{P}^0 \to \mathcal{P}^0 \times_{\mathcal{Z}} \mathcal{Q}$. By making a further blow-up of faces of Y, enlarging A and changing notation (cf. the last part of lemma 2.3), we may assume that the closure $\tilde{\mathcal{P}}$ of $s^0_\infty(\mathcal{P}^0)$ in $\mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}$ is finite over $\mathbb{P} \setminus A$, hence finite over $\mathcal{P} \setminus f^{-1}(A)$. Replacing $\mathcal{P}$ with $\tilde{\mathcal{P}}$ and changing notation, we may assume s^0_∞ gives rise to another section $s_\infty\colon\ \mathcal{P} \to \mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}$.

The picture is now

$$\begin{tikzcd} Y \arrow[d, hook] & & \\ \mathbb{P} & \mathcal{P} \arrow[l, "f"'] \arrow[r, shift left=3, "s_0"] \arrow[r, shift right=3, "s_\infty"'] & \mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}. \arrow[l, "p" description] \\ A \arrow[u, hook] & & \end{tikzcd} \tag{2.4}$$

Let $\overline{\mathcal{P}} \subset \mathcal{P}$ be the closed subscheme where $s_0 = s_\infty$. Then

$$(\mathcal{P}^0 \cap f^{-1}(Y))_{\mathrm{red}} \subset \overline{\mathcal{P}}. \tag{2.5}$$

Indeed, we can check this down on $\mathbb{P}^n$, i.e., we can ignore the alteration π. Points in $\mathcal{P}^0$ map to pairs consisting of a line ℓ maximally tangent to X but not lying on X, together with a point $y \in \ell$. The fibre $p^{-1}(\ell, y) = \{(\ell, y, z) \mid z \in \ell\}$. The sections s_0 and s_∞ are given respectively by $s_0(\ell, y) = (\ell, y, y)$ and $s_\infty(\ell, y) = (\ell, y, \ell \cap X)$. Since $Y = \pi^{-1}(X)$, we get the desire inclusion (2.5) after alteration.

LEMMA 2.6. *Possibly enlarging the divisor A (preserving the hypothesis that A meets faces of Y properly), there exists a rational function t on $\mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}$ such that*

$$(\mathrm{div}\ t) \cap (f \circ p)^{-1}(\mathbb{P} \setminus A) = (s_0 - s_\infty) \cap (f \circ p)^{-1}(\mathbb{P} \setminus A), \tag{2.6}$$

and such that further, $t|((f \circ p)^{-1}(Y))_{\mathrm{red}} \cap (\mathcal{P}^0 \times_{\mathcal{Z}} \mathcal{Q}) \equiv 1$.

Proof. By assumption, the map $\mathcal{P} \setminus f^{-1}(A) \to \mathbb{P} \setminus A$ is finite. There are thus a finite set of points of $\mathcal{P}$ lying over generic points of faces of Y.

Let $L = p_*(\mathcal{O}(s_\infty - s_0))$. As $R^i p_*(\mathcal{O}(s_\infty - s_0)) = 0$ for $i > 0$, L is an invertible sheaf on $\mathcal{P}$; adding divisors to A meeting faces properly, we can assume that L is trivial on $\mathcal{P} \setminus f^{-1}A$. A generating section of L thus gives a generating section $\tilde{t}$ of $\mathcal{O}(s_\infty - s_0)$ over $\mathcal{P} \times_{\mathcal{Z}} \mathcal{Q} \setminus (fp)^{-1}(A)$. We let t be the corresponding rational function on $\mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}$.

Clearly t satisfies (2.6). The fact that t can be taken to be $\equiv 1$ on the indicated divisor follows from (2.5) and Lemma 2.5. Indeed, the Lemma shows that the restriction of t comes from a unit on $f^{-1}Y \cap \mathcal{P}^0$. Enlarging A, this unit lifts to a unit on $\mathcal{P} \setminus f^{-1}A$. Normalizing t, we can assume this unit is 1. $\square$

Proof of Theorem 1.1. We use Lemma 2.6 to construct an effective cycle $\mathcal{D} \in \mathcal{Z}^n((\mathbb{P} \setminus A) \times U, 1)$ with

$$\begin{aligned} &\mathrm{Supp}(\mathcal{D}) \cap (Y \setminus A) \times U \times S^1 = \emptyset, \\ &\partial(\mathcal{D}) = N \cdot \Delta \in \mathcal{Z}^n((\mathbb{P} \setminus A) \times U, 0) \end{aligned} \tag{2.7}$$

for some integer $N \neq 0$. This suffices to prove the theorem.

To construct $\mathcal{D}$, we have the closed embedding i: $\mathcal{P} \times_{\mathcal{Z}} \mathcal{Q} \to \mathcal{P} \times \mathbb{P}^n$. Let

$$\mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}^{\#} = i^{-1}((\mathcal{P} \setminus f^{-1}(A)) \times U), \tag{2.8}$$

and let

$$\Gamma^{\#} \subset \mathcal{P} \times_{\mathcal{Z}} \mathcal{Q}^{\#} \times \mathbb{P}^1 \tag{2.9}$$

be the closure of the graph of t. Let

$$\overline{\Gamma} := i_* \Gamma^{\#} \subset (\mathcal{P} \setminus f^{-1}(A)) \times U \times \mathbb{P}^1 \tag{2.10}$$

be the image of $\Gamma^{\#}$, and let

$$\Gamma \subset (\mathcal{P} \setminus f^{-1}(A)) \times U \times S^1; \quad S^1 = \operatorname{Spec} k[t_0, t_1]/(t_0 + t_1 - 1) \tag{2.11}$$

be the pull-back of $\overline{\Gamma}$ via $(t_0, t_1) \mapsto (-t_0 \colon t_1)$. We let $\Gamma_* \subset \mathcal{P} \times \mathbb{P}^n \times \mathbb{P}^1$ be the closure of Γ.

By Lemma 2.6, we have

$$\begin{aligned} \Gamma_* \cap (f^{-1}(Y \setminus A) \times \mathbb{P}^n \times \mathbb{P}^1) \subset\ & (f^{-1}(Y \setminus A) \times X \times \mathbb{P}^1) \\ & \cup (f^{-1}(Y \setminus A) \times \mathbb{P}^n \times \{1\}). \end{aligned} \tag{2.12}$$

Thus

$$\Gamma_* \cap (f^{-1}(Y \setminus A) \times U \times S^1) = \emptyset. \tag{2.13}$$

Also we have

$$\partial(\Gamma) = (f \times \mathrm{id})^*(\Delta) \in \mathcal{Z}^n((\mathcal{P} \setminus f^{-1}(A)) \times U, 0). \tag{2.14}$$

Thus, setting

$$\mathcal{D} := (f \times \mathrm{id})_*(\Gamma) \in \mathcal{Z}^n((\mathbb{P} \setminus A) \times U, 1), \tag{2.15}$$

we have

$$\operatorname{rest}(\mathcal{D}) = 0, \quad \partial(\mathcal{D}) = (f \times \mathrm{id})_* \circ (f \times \mathrm{id})^*(\Delta) = N \cdot \Delta, \tag{2.16}$$

where $N = \deg(f) \neq 0$. This completes the proof. $\square$

3. The proof of Theorem 1.2. This section is devoted to the proof of Theorem 1.2.

In what follows, we write simply $H^a(X, b)$ to denote either geometric étale cohomology, viz. $X/\mathbb{F}_q$, $H^a(X, b) := H^a_{\text{ét}}(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell(b))$ for $(\ell, \operatorname{char}(\mathbb{F}_q)) = 1$, or de Rham cohomology $H^a_{DR}(X)$ for X over a field of characteristic 0 (with the Hodge filtration shifted by b), or rigid cohomology with the Frobenius action multiplied by p^{-b} for X over a perfect field of characteristic p. On de Rham cohomology we denote by F the Hodge filtration [6].

LEMMA 3.1. *Let $\mathbb{P}$ be smooth and let $A \subset \mathbb{P}$ be a divisor meeting faces of Y properly, where $Y \subset \mathbb{P}$ is a normal crossing divisor. Let s: $H^a_A(\mathbb{P}, Y; 0) \to H^a(\mathbb{P}, Y; 0)$ be the canonical map. Then*

(1) *In the de Rham case,* Image(s) $\subset F^1 H^a(\mathbb{P}, Y; 0)$.

(2) *In the étale case, the eigenvalues of Frobenius F_q on* Image(s) *are all divisible by q.*

(3) *In the rigid case, the slopes of F_a on* Image(s) *are all* ≥ 1.

Proof. We have a diagram

$$\begin{array}{ccccc} H^{a-1}(Y,0) & \longrightarrow & H^a(\mathbb{P},Y;0) & \longrightarrow & H^a(\mathbb{P},0) \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle s} \\ H^{a-1}_A(Y,0) & \longrightarrow & H^a_A(\mathbb{P},Y;0) & \longrightarrow & H^a_A(\mathbb{P},0). \end{array} \tag{3.1}$$

The assertion for the middle vertical arrow reduces to the comparable assertions for the left and right hand vertical arrows. (In the de Rham case, one must use the fact that gr_F is an exact functor.) Then the spectral sequences

$$\begin{aligned} E_1^{s,t} &= H^t_{A\cdot Y^{(s)}}(Y^{(s)},0) \Rightarrow H^{s+t}_A(Y,0) \\ E_1^{s,t} &= H^t(Y^{(s)},0) \Rightarrow H^{s+t}(Y,0) \end{aligned} \tag{3.2}$$

reduce the problem to the case where the relative divisor Y is smooth. Thus it suffices to consider the right-hand vertical arrow.

Suppose for a while we work with étale cohomology. We mimic Berthelot's method as in [13], Lemma 2.1. Let $\cdots \subset A_j \subset A_{j-1} \cdots \subset A_0 = A$ be a finite stratification by closed subsets such that $A_{i-1} \setminus A_i$ is smooth. The localization sequence

$$\cdots \to H^b_{A_j}(\overline{\mathbb{P}}, \mathbb{Q}_\ell) \to H^b_{A_{j-1}}(\overline{\mathbb{P}}, \mathbb{Q}_\ell) \to H^a_{(A_{j-1}\setminus A_j)}(\overline{\mathbb{P}\setminus A_j}, \mathbb{Q}_\ell) \to \cdots \tag{3.3}$$

commutes with the Frobenius action. Therefore we may assume that both A and $\mathbb{P}$ are smooth, but no longer projective. We consider an affine covering $\mathbb{P} = \cup_{i=0}^N U_i$. The spectral sequence

$$\begin{aligned} E_2^{ab} = H^a\big(\cdots \to H^b_A(\overline{U^{a-1}}, \mathbb{Q}_\ell) \to H^b_A(\overline{U^a}, \mathbb{Q}_\ell) \\ \to H^b_A(\overline{U^{a+1}}, \mathbb{Q}_\ell\big) \to \cdots) \Longrightarrow H^{a+b}_A(\overline{\mathbb{P}}, \mathbb{Q}_\ell) \end{aligned} \tag{3.4}$$

allows us to reduce to the case where $\mathbb{P}$ is smooth affine, $A \subset \mathbb{P}$ is smooth, where $r = \mathrm{codim}(A) \geq 1$. By purity, we have a functorial Gysin isomorphism

$$H^{a-2r}(\overline{A}, \mathbb{Q}_\ell) \overset{\text{Gysin}}{\to} H^a_A(\overline{\mathbb{P}}, \mathbb{Q}_\ell(r)) \tag{3.5}$$

By functoriality, this commutes with Frobenius, and we know that the eigenvalues of Frobenius acting on the left term are algebraic integers (use duality with H^*_c and e.g. Corollary 5.5.3(iii) in [8]). But this is equivalent to saying that the Frobenius eigenvalues on $H^a_A(\overline{\mathbb{P}}, \mathbb{Q}_\ell)$ are all divisible by q^r, finishing the proof when $k = \mathbb{F}_q$.

The same sort of argument in the de Rham case reduces us to showing

$$\mathrm{Image}(H^b_{DR}(A) \;\to\; H^{b+2r}_{DR}(\mathbb{P})) \subset F^r H^{b+2r}_{DR}(\mathbb{P}) \tag{3.6}$$

when A is a smooth codimension r subvariety of the smooth affine $\mathbb{P}$; we may even assume that A is a transverse intersection of r smooth divisors on $\mathbb{P}$. By induction on r, we may assume that A has codimension 1. We take a smooth compactification $\mathbb{P} \subset \mathbb{P}'$ such that A has a smooth compactification $A \subset A' \subset \mathbb{P}$ and the divisor $A' \cup W$, $W = (\mathbb{P}' \setminus \mathbb{P})$ is a normal crossing divisor. Then the Gysin map is the connecting homomorphism of the residue sequence

$$0 \to \Omega^\bullet_{\mathbb{P}'}(\log(W)) \to \Omega^\bullet_{\mathbb{P}'}(\log(W + A')) \to \Omega^{\bullet-1}_{A'}(\log(W \cap A')) \to 0. \tag{3.7}$$

Since one has the exact subsequence

$$\begin{aligned} 0 \;\to\; &\Omega^{\geq(p+1)}_{\mathbb{P}'}(\log(W)) \to \Omega^{\geq(p+1)}_{\mathbb{P}'}(\log(W + A')) \\ \to\; &\Omega^{\geq p}_{A'}(\log(W \cap A')) \to 0, \end{aligned} \tag{3.8}$$

one sees that $F^p H^b_{DR}(A)$ cobounds to $F^{p+1} H^{b+2}_{DR}(\mathbb{P})$. This finishes the proof when k has charateristic 0.

Finally, when k is perfect of characteristic p, we use a similar argument, replacing étale cohomology by rigid cohomology. This finishes the proof of the lemma. □

Proof of Theorem 1.2. $\mathbb{P}$ will be a smooth, projective variety of dimension n, $Y \subset \mathbb{P}$ is a normal crossings divisor, and $U := \mathbb{P} \setminus Y$. Consider the diagonal (1.10). Using the cycle class map from motivic cohomology (*cf.* section 4), we view the diagonal as being a class in our theory

$$\begin{aligned} [\Delta] \in H^{2n}((\mathbb{P}, Y) \times U, n) \;&\cong\; \bigoplus_{a+b=2n} H^b(U, n) \otimes H^a(\mathbb{P}, Y; 0) \\ &\cong \bigoplus_{a=0}^{2n} \mathrm{Hom}(H^a(\mathbb{P}, Y; 0), H^a(\mathbb{P}, Y; 0)). \end{aligned} \tag{3.9}$$

(The isomorphism on the right uses the existence of a good theory of compactly supported cohomology in our cohomology. Of course, the homomorphism on the right is the identity.) By the hypotheses of Theorem 1.2, $[\Delta]$ dies in $H^{2n}((\mathbb{P} \setminus A, Y \setminus Y \cap A) \times U, n)$, which means that the map s below is onto:

$$H^a_A(\mathbb{P}, Y; 0) \overset{s}{\twoheadrightarrow} H^a(\mathbb{P}, Y; 0) \to H^a(\mathbb{P} \setminus A, Y \setminus Y \cap A; 0). \tag{3.10}$$

Also, since $\mathbb{P} \to \mathbb{P}^n$ is an alteration, the pullback map $H^a(\mathbb{P}^n, X; 0) \to H^a(\mathbb{P}, Y; 0)$ is injective. The assertions of the theorem are now consequences of Lemma 3.1 above. □

4. Cycle maps. Let $H^*_?$ be one of the cohomology theories used above: étale cohomology with relevant twists and Galois action [8], de Rham cohomology with Hodge filtration [6] or rigid cohomology with Frobenius action [1]. We explain how to define cycle maps

$$\mathrm{cl}^n\colon\ H^{2n}_M(T, \mathcal{I}(A), \mathbb{Z}(n)) \to H^{2n}_?(T, A), \tag{4.1}$$

natural with respect to smooth pull-back and projective push-forward, where $H^{2n}_?(T, A)$ denotes the theory on T with compact supports relative to A; if $\bar{T}$ is a compactification of T, we have the usual compactly supported cohomology $H^*_{?,c}(T) := H^*_?(\bar{T}, \bar{T} \setminus T)$, which is canonically defined independent of the choice of $\bar{T}$.

For T smooth over k and W a closed subset, we have the cohomology with supports $H^*_{?,W}(T)$. We have as well the relatively compact version $H^*_{?,W}(T, A)$ and the natural commutative diagram

$$\begin{array}{ccc} H^*_{?,W}(T, A) & \longrightarrow & H^*_?(T, A) \\ \downarrow & & \downarrow \\ H^*_{?,W}(T) & \longrightarrow & H^*_?(T). \end{array} \tag{4.2}$$

If $W \cap A = \emptyset$, the map

$$H^*_{?,W}(T, A) \to H^*_{?,W}(T) \tag{4.3}$$

is an isomorphism. $H^*_?$ satisfies the *homotopy property*: the map

$$p^*\colon\ H^*_?(T, A) \to H^*_?(T \times \mathbb{A}^1, A \times \mathbb{A}^1) \tag{4.4}$$

is an isomorphism.

Let R be the coefficient ring $H^0_?(k)$. We have the group of codimension n cycles $\mathcal{Z}^n(T) = \mathcal{Z}^n(T, 0)$. For $W \subset T$ a closed subset, we let $\mathcal{Z}^n_W(T) \subset \mathcal{Z}^n(T)$ be the subgroup of cycles with support on W. If $\mathrm{codim}_T W \geq n$, we have the *purity isomorphism*

$$\mathrm{cl}^n_W\colon\ \mathcal{Z}^n_W(T) \otimes R \to H^{2n}_{?,W}(T), \tag{4.5}$$

which is natural with respect to maps $f\colon\ (T', W') \to (T, W)$, $f^{-1}(W) \subset W'$, with $\mathrm{codim}_{T'} W' \geq n$. Taking the limit over W and forgetting supports defines the map

$$\mathrm{cl}^n\colon\ \mathcal{Z}^n(T) \to H^{2n}_?(T) \tag{4.6}$$

with $\mathrm{cl}^n(f^*Z)=f^*\mathrm{cl}^n(Z)$ for $Z \in \mathcal{Z}^n(T)$, and $f\colon\ T' \to T$ with $\mathrm{codim}_{T'} f^{-1}(\mathrm{Supp}(Z)) \geq n$. The maps cl^n are also natural with respect to projective push-forward and products.

Now let A be a closed subset of some smooth T, and let W be a closed subset of T with $\mathrm{codim}_T W \geq n$ and $W \cap A = \emptyset$. Via the isomorphisms (4.3), (4.5), we have the isomorphism

$$\mathrm{cl}^n_W\colon\ \mathcal{Z}^n_W(T) \otimes R \to H^{2n}_{?,W}(T, A). \tag{4.7}$$

Taking the limit of such W and forgetting supports gives us the natural map

$$\mathrm{cl}^n_A(0)\colon\ \mathcal{Z}^n(T, \mathcal{I}(A), 0) \to H^{2n}_?(T, A). \tag{4.8}$$

Similarly, we have the natural map

$$\mathrm{cl}^n_A(1)\colon\ \mathcal{Z}^n(T, \mathcal{I}(A), 1) \to H^{2n}_?(T \times S^1, A \times S^1). \tag{4.9}$$

This yields the commutative diagram

$$\begin{array}{ccc} \mathcal{Z}^n(T, \mathcal{I}(A), 1) & \xrightarrow{\mathrm{cl}^n_A(1)} & H^{2n}_?(T \times S^1, A \times S^1) \\ {\scriptstyle \delta_1^* - \delta_0^*}\downarrow & & \downarrow{\scriptstyle \delta_1^* - \delta_0^*} \\ \mathcal{Z}^n(T, \mathcal{I}(A), 0) & \xrightarrow[\mathrm{cl}^n_A(0)]{} & H^{2n}_?(T, A). \end{array} \tag{4.10}$$

By the homotopy property (4.4), the right-hand vertical arrow is zero, so $\mathrm{cl}^n_A(0)$ descends to the desired map

$$\mathrm{cl}^n_A\colon\ H^{2n}_M(T, \mathcal{I}(A), \mathbb{Z}(n)) \to H^{2n}_?(T, A). \tag{4.11}$$

The naturality of cl^n_A with respect to flat pull-back, projective push-forward and products follows from that of the cycle-classes with support.

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