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# Appendix to "Congruences for rational points on varieties over finite fields" by N. Fakhruddin and C. S. Rajan

**Hélène Esnault**

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We give a positive answer to questions 4.2 and 4.4 of [5].

## 1. Hodge type

In this section we consider varieties $X$ defined over a field of characteristic 0, their de Rham cohomology $H^m(X)$ together with their Hodge filtration $F$.

**Theorem 1.1.** *Let $f : X \to Y$ be a proper surjective morphism between smooth varieties defined over a field of characteristic 0 with $Y$ connected. We assume $f$ is flat, e.g. $Y$ has dimension 1. If there is a closed point $y \in Y$ such that $f$ is smooth in $y$ and $gr_0^F H^m(f^{-1}(y)) = 0$ for all $m \geq 1$, then $gr_0^F H^m(f^{-1}(z)_{\rm red}) = 0$ for all closed points $z \in Y$ and all $m \geq 1$.*

*Proof.* Since $f^{-1}(y)$ is smooth, $gr_0^F H^m(f^{-1}(y)) = H^m(f^{-1}(y), \mathcal{O}) = 0$ for all $m \geq 1$. Since $R^m f_*\mathcal{O}_X$ is locally free on the smooth locus of $f$, then $R^m f_*\mathcal{O}_X$ has to be a torsion sheaf on $Y$. Since $X$ and $Y$ are smooth and $f$ is surjective, by Kollár's torsion-freeness theorem [6], one has

$$R^m f_*\mathcal{O}_X = 0. \tag{1.1}$$

Let $\mathfrak{m}_z$ be the maximal ideal of a closed point $z$. Since $Y$ is smooth, locally on $Y$, there are smooth divisors $D_1, \ldots, D_a$, $a = \dim(Y)$ with

$$\mathfrak{m}_z = (\mathcal{O}_Y(-D_1), \ldots, \mathcal{O}_Y(-D_a)).$$

By flatness one has a resolution

$$0 \to \mathcal{O}_X(-\sum_1^a f^*(D_i)) \to \ldots \to \oplus_{i_1<\ldots<i_r} \mathcal{O}_X(-f^*(D_{i_1} + \ldots + D_{i_r})) \to \ldots \to \mathcal{O}_X \to \mathcal{O}_{f^{-1}(z)} \to 0. \tag{1.2}$$

H. Esnault
Universität Essen, FB6, Mathematik, 45117 Essen, Germany
(e-mail: esnault@uni-essen.de)

Thus the vanishing (1.1) and projection formula imply

$$H^m(X, \mathcal{O}_{f^{-1}(z)}) = 0 \, \forall \, m \geq 1. \tag{1.3}$$

Let now $f' : X' \to Y'$ be a compactification of $f$, that is $X'$, $Y'$ are smooth proper, $X \subset X'$ and $Y \subset Y'$ are open dense, and $f' = f|_Y$. Let $\mathcal{I} = f^{'*}\mathfrak{m}_z$ be the ideal sheaf of the fiber $f^{'-1}(z)$. Then by [4], Proposition 1.2, the map $\sigma^*$

$$\begin{aligned} \sigma^* : H^n(X', \mathcal{I}) &\to gr_0^F H^n(\tilde{X}, (\sigma^{-1} f^{'-1}(z))_{\text{red}}) \\ &= gr_0^F H^n(X', (f^{'-1}(z))_{\text{red}}) \end{aligned} \tag{1.4}$$

is surjective for all $n$, where $\sigma : \tilde{X} \to X'$ is a birational map, isomorphic over $X' \setminus f^{'-1}(z)$, with $\tilde{X}$ smooth, and $\sigma^{-1} f^{'-1}(z)$ a normal crossing divisor. From the commutative diagram of exact sequences

$$\begin{array}{ccccc} H^m(X', \mathcal{O}_{X'}) & \longrightarrow & H^m(X', \mathcal{O}_{f'^{-1}(z)}) & \longrightarrow & H^{m+1}(X', \mathcal{I}) \\ = \downarrow & & \sigma^* \downarrow & & \downarrow \sigma^* \\ gr_0^F H^m(X') & \longrightarrow & gr_0^F H^m(f^{-1}(z)_{\text{red}}) & \longrightarrow & gr_0^F H^n(X', (f^{'-1}(z))_{\text{red}}) \end{array} \tag{1.5}$$

we conclude that the vertical map in the middle is surjective, thus by (1.3) that $gr_0^F H^m(f^{-1}(z)_{\text{red}}) = 0$. □

*Remark 1.2.* Because of the resolution (1.2) it is not possible to separate the $m$ involved in the proof of the Theorem.

**Corollary 1.3.** *Let $f : X \to Y$ be a proper morphism between smooth varieties defined over a field $k$ of characteristic 0 with $Y$ connected. We assume that $f$ is flat. If there is a closed point $y \in Y$ such that $f$ is smooth in $y$ and $CH_0(f^{-1}(y) \times_k \overline{k(f^{-1}(y))}) = \mathbb{Q}$, then $gr_0^F H^m(f^{-1}(z)_{\text{red}}) = 0$ for all closed points $z \in Y$ and all $m \geq 1$.*

*Proof.* By S. Bloch's theorem, [2], Appendix to Lecture 1, the assumption implies $H^m(f^{-1}(y), \mathcal{O}) = gr_0^F H^m(f^{-1}(y)) = 0$ for all $m \geq 1$. One applies Theorem 1.1. □

*Remark 1.4.* According to Bloch's conjecture, the assumptions in Theorem 1.1 and in its corollary should be equivalent. As we are far from knowing Bloch's conjecture, we have formulated Theorem 1.1 purely in the coherent category.

## 2. Eigenvalues of Frobenius

In this section, we consider varieties $X$ defined over a finite field $k$, their étale cohomology $H^m(\overline{X}, \mathbb{Q}_\ell)$, with $\overline{X} = X \times_k \overline{k}$, and the eigenvalues of the geometric Frobenius $F$ acting on it.

**Theorem 2.1.** *Let $f : X \to Y$ be a proper surjective morphism of smooth irreducible varieties defined over a finite field k with q elements. Assume $CH_0((X \times_Y \overline{k(X)})) = \mathbb{Q}$. Then for all closed points $y \in Y(k)$, the eigenvalues of the geometric Frobenius F acting on $H^m(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ are divisible by q as algebraic integers for all $m \geq 1$.*

*Proof.* We may assume that $Y$ is affine of dimension $a$, containing $y$. We set $Y' = Y \setminus \{y\}$, $X' = X \setminus \{f^{-1}(y)_{\mathrm{red}}\}$. Then

$$H^n_c(\overline{Y'}, \mathbb{Q}_\ell) \to H^n_c(\overline{Y}, \mathbb{Q}_\ell) \tag{2.1}$$

is surjective for $n \geq 1$ and an isomorphism for $n \geq 2$ . Thus via the exact sequence

$$\begin{aligned} \ldots \to H^n_c(\overline{X'}, \mathbb{Q}_\ell) &\to H^n_c(\overline{X}, \mathbb{Q}_\ell) \\ \to H^n(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell) &\to H^{n+1}_c(\overline{X'}, \mathbb{Q}_\ell) \to \ldots \end{aligned} \tag{2.2}$$

one concludes that the image of $H^n_c(\overline{X}, \mathbb{Q}_\ell)$ in $H^n(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ is a quotient of $H^n_c(\overline{X}, \mathbb{Q}_\ell)/H^n_c(\overline{Y}, \mathbb{Q}_\ell)$ thus by 3.1, Proof of Theorem 1.1 of [5], the Frobenius eigenvalues on it are divisible by $q$. Thus we just have to consider the image of $H^n(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ in $H^{n+1}_c(\overline{X'}, \mathbb{Q}_\ell)$ for $n \geq 1$. Again by 3.1, Proof of Theorem 1.1 of [5], the Frobenius eigenvalues on the image of $H^n(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ in $H^{n+1}_c(\overline{X'}, \mathbb{Q}_\ell)/H^{n+1}_c(\overline{Y'}, \mathbb{Q}_\ell)$ are $q$-divisible, so we just have to consider the subspace of $H^n(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ falling in Im $H^{n+1}_c(\overline{Y'}, \mathbb{Q}_\ell)$ in $H^{n+1}_c(\overline{X'}, \mathbb{Q}_\ell)$. Since $n + 1 \geq 2$, one has an isomorphism in (2.1) and since $Y$ was chosen to be affine, $H^{n+1}_c(\overline{Y}, \mathbb{Q}_\ell) = 0$ for $n+1 < a$ by Artin's vanishing theorem [1], Théorème 3.1. By Deligne's integrality theorem [3], Corollaire 5.5.3, the Frobenius eigenvalues on $H^{n+1}_c(\overline{Y}, \mathbb{Q}_\ell) = 0$ are $q$-divisible for $n + 1 > a$. we conclude from the discussion that the Frobenius eigenvalues on $H^n(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ are all $q$-divisible, except for $n = 0$ and possibly for $n = a - 1$. This finishes the proof for $a = 1$. If $a > 1$, we finish the proof as follows. The fiber $f^{-1}(y)_{\mathrm{red}}$ is geometrically connected. The Lefschetz trace formula implies then that the eigenvalues of Frobenius on $H^{a-1}(\overline{f^{-1}(y)_{\mathrm{red}}}, \mathbb{Q}_\ell)$ are divisible by $q$ if and only if $f^{-1}(y)_{\mathrm{red}}$ has one $k$-point modulo $q$. This is Theorem 1.1 of [5]. □

*Remark 2.2.* One can avoid the Euler characteristic argument via the Lefschetz trace formula in the proof of Theorem 2.1 by replacing $f : X \to Y$ by $f \times \mathrm{id} : X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$ for example, as observed by N. Fakhruddin. However, one has to say that the version of Bloch's decomposition of the diagonal used in 3.1, Proof of Theorem 1.1 of [5] is equivalent to the assumption of Theorem 1.1 so as to see that one may replace $\overline{k(X)}$ by $\overline{k(X \times \mathbb{A}^1)}$ in the assumption. This gives of course as well a motivic proof of Corollary 1.3, however not of Theorem 1.1 in absence of a solution to Bloch's conjecture.

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