A REMARK ON HIGHER CONGRUENCES FOR THE NUMBER OF RATIONAL POINTS OF VARIETIES DEFINED OVER A FINITE FIELD

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Abstract. We show that the ℓ-adic cohomology of the mod p reduction Y of a regular model of a smooth proper variety defined over a local field, the cohomology of which is supported in codimension κ, can't be Tate up to level $(\kappa - 1)$. As a consequence, the number of rational points of Y can't fulfill the natural relation $|Y(\mathbb{F}_q)| \equiv \sum_{i\geq 0} q^i \cdot b_{2i}(\bar{Y})$ modulo q^κ.

Une remarque sur les congruences d'ordre supérieur pour le nombre de points rationnels de variétés définies sur un corps fini .

Résumé: Nous montrons que la cohomologie ℓ-adique de la réduction Y modulo p d'un modèle régulier d'une variété propre et lisse définie sur un corps local, dont la cohomologie est supportée en codimension $\kappa \geq 1$, ne peut être de Tate jusqu'en niveau $(\kappa-1)$. En conséquence, le nombre de points rationnels de Y ne peut vérifier la formule naturelle $|Y(\mathbb{F}_q)| \equiv \sum_{i\geq 0} q^i \cdot b_{2i}(\bar{Y})$ modulo q^κ.

Version française abrégée. Dans [5], Theorem 1.1, nous montrons que si $\mathcal{X}$ est un modèle régulier d'une variété X propre et lisse définie sur un corps local de corps résiduel $\mathbb{F}_q$, alors si la cohomologie ℓ-adique $H^i(\bar{X})$ est supportée en codimension ≥ 1 pour $i \geq 1$, le nombre de points rationnels de sa réduction Y modulo p vérifie $|Y(\mathbb{F}_q)| \equiv 1$ modulo q. En fait, pour être plus précis, sous cette hypothèse, les valeurs propres du Frobenius géométrique agissant sur la cohomologie ℓ-adique $H^i(\bar{Y})$ de Y sont divisibles par q en tant qu'entiers algébriques. Le but de cette note est de discuter une formulation en coniveau supérieur. Une façon naturelle de généraliser la condition de coniveau ≥ 1 pour $i \geq 1$ est de supposer que $H^i(\bar{X})/H^i(\bar{X})_{\rm alg}$ est supportée en codimension κ, où $H^i(\bar{X})_{\rm alg}$ est nulle si i est impair et sinon est la partie algébrique. Nous montrons cependant que cela n'implique pas que

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les valeurs propres du Frobenius géométrique sont divisibles par q^κ en tant qu'entiers algébriques sur $H^i(\bar{Y})/H^i(\bar{Y})_{q^{\frac{i}{2}}}$, où $H^i(\bar{Y})_{q^{\frac{i}{2}}}$ est nulle si i est impair et sinon est la partie sur laquelle le Frobenius agit par multiplication par $q^{\frac{i}{2}}$. En particulier la formule naturelle $|Y(\mathbb{F}_q)| \equiv \sum_{i\geq 0} q^i \cdot b_{2i}(\bar{Y})$ modulo q^κ n'est pas valable en général. Cette formulation est proposée par N. Fakhruddin dans [6] qui la montre sous certaines hypothèses pour une famille géométrique en égale caractéristique $p > 0$. Nous montrons en quoi ces hypothèses sont très fortes.

1. Introduction

In [5], Theorem 1.1, we show that if $\mathcal{X}$ is a regular model of a smooth proper variety X defined over a local field with finite residue field $\mathbb{F}_q$, then if ℓ-adic cohomology $H^i(\bar{X})$ is supported in codimension ≥ 1 for $i \geq 1$, the number of rational points of its mod p reduction Y fulfills $|Y(\mathbb{F}_q)| \equiv 1$ modulo q. To be more precise, the assumption implies that the eigenvalues of the geometric Frobenius acting on ℓ-adic cohomology $H^i(\bar{Y})$ of Y are q-divisible algebraic integers. The proof relies on a version of Deligne's integrality theorem [2], Corollaire 5.5.3 over local fields [3], Corollary 0.4. The goal of this note is to discuss a formulation in higher coniveau level. A natural generalization of the coniveau ≥ 1 condition for $i \geq 1$ is to assume that $H^i(\bar{X})/H^i(\bar{X})_{\rm alg}$ is supported in codimension $\geq \kappa$, where $H^i(\bar{X})_{\rm alg}$ is equal to 0 if i is odd, else is the algebraic part of cohomology. This means that there is a codim $\geq \kappa$ subscheme $Z \subset X$ so that $H^i(\bar{X}) \xrightarrow{\text{rest}=0} H^i(\bar{X} \setminus \bar{Z})/{\rm Im}(H^i(\bar{X})_{\rm alg}$. Said differently, $H^i(\bar{X}) = H^i(\bar{X})_{\rm alg}$ for $i \leq 2\kappa$, and $H^i_{\bar{Z}}(\bar{X}) \twoheadrightarrow H^i(\bar{X})$ for $i \geq 2\kappa$.

However we show that this assumption does not imply that the eigenvalues of the geometric Frobenius acting on $H^i(\bar{Y})/H^i(\bar{Y})_{q^{\frac{i}{2}}}$ are divisible by q^κ-divisible algebraic integers, where $H^i(\bar{Y})_{q^{\frac{i}{2}}}$ is equal to 0 if i is odd, else is the part of cohomology on which Frobenius acts by multiplication by $q^{\frac{i}{2}}$. In particular, the formula $|Y(\mathbb{F}_q)| \equiv \sum_{i\geq 0} q^i \cdot b_{2i}(\bar{Y})$ modulo q^κ does not hold in general. This formulation was proposed in [6] by N. Fakhruddin, who shows it under certain assumptions in a geometric family in equal characteristic $p > 0$. We show how strong are those assumptions.

Our example consists of a Godeaux surface in characteristic 0. We take a reduction mod p which is a cone over a smooth curve C of higher degree. After desingularization of the mod p reduction, $H^1(\bar{C})(-1)$

enters $H^3(\bar{Y})$, and this destroys the possibility of the $|Y(\mathbb{F}_q)| \equiv 1 + q \cdot b_2(\bar{Y}) \bmod q^2$ congruence.

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2. The example

Let us consider the Godeaux surface $X_0/\mathbb{Q}_p$ defined as the quotient of the Fermat quintic $F \subset \mathbb{P}^3_{\mathbb{Q}_p}$ of homogeneous equation $px_0^5 + x_1^5 + x_2^5 + x_3^5$ by the group μ_5 acting via $\xi \cdot (x_i) = (\xi^i \cdot x_i)$. Here p is prime to 5, and ξ generates the group of 5-th roots of unity. As well known [1], V, 15 and VII, 11, $H^0(X_0, \Omega^1_{X_0}) = H^0(X_0, \Omega^2_{X_0})$ and by comparison of de Rham with étale cohomology, one obtains $H^1(\bar{X}_0) = H^3(\bar{X}_0) = 0$, $H^{2i}(\bar{X}_0) = H^{2i}_{\rm alg}(\bar{X}_0)$ for $i = 0, 1, 2$. Let us assume we have a regular model $\mathcal{X} \to \mathrm{Spec}(R)$ of X_0 over an extension $R \supset \mathbb{Z}_p$, with local field $K = \mathrm{Frac}(R)$ and residue field $\mathbb{F}_q$. Thus the general fiber is $X = X_0 \times_{\mathbb{Q}_p} K$, and we denote by Y the mod p reduction over $\mathbb{F}_q$.

We use the computation in [5], sections 2 and 3. One has an exact sequence

$$H^i_{\bar{Y}}(\mathcal{X}^u) \to H^i(\bar{Y}) \xrightarrow{\mathrm{sp}^u} H^i(X^u) \to H^{i+1}_{\bar{Y}}(\mathcal{X}^u) \tag{2.1}$$

where ^{u} means the pull back via the extension $K \subset K^u$ to the maximal unramified extension, and $^-$ means the pull back via the extension to the algebraic closure. The sequence is equivariant with respect to the action of the geometric Frobenius Frob $\in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on $H^*(\bar{Y}), H^*_{\bar{Y}}(\mathcal{X}^u), H^*(X^u)$. One also has the exact sequence

$$0 \to H^1(I, H^{i-1}(\bar{X})) \to H^i(X^u) \to H^i(\bar{X})^I \to 0 \tag{2.2}$$

where $I \subset \mathrm{Gal}(\bar{K}/K)$ is the inertia group, with quotient $\mathrm{Gal}(\bar{K}/K)/I = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. The sequence is equivariant with respect to the action of Frob. So using Gabber's purity theorem [7], Theorem 2.1.1. as in [5], section 2, one obtains

$$H^1(\bar{Y}) = 0, \tag{2.3}$$

and an equivariant exact sequence

$$0 \to H^0(\bar{Y}^0)(-1) \to H^2(\bar{Y}) \to H^2(\bar{X})^I, \tag{2.4}$$

where $Y^0 = Y\setminus$ singular locus. Thus in particular, Frob acts via multiplication by q on $H^2(\bar{Y})$. So via Grothendieck-Lefschetz trace formula [8] and the fact that $H^4(\bar{Y}) = \oplus_{\rm components}\mathbb{Q}_\ell(-2)$, we conclude

$$|Y(\mathbb{F}_q)| \equiv 1 + q \cdot b_2(\bar{Y}) - \mathrm{Tr}\ \mathrm{Frob}|H^3(\bar{Y}) \bmod q^2. \tag{2.5}$$

The question becomes whether $H^3(\bar{Y})$ dies or not.

We now construct $\mathcal{X}$ and show $H^3(\bar{Y}) \neq 0$ for this $\mathcal{X}$. The mod p reduction in $\mathbb{P}^3_{\mathbb{F}_p}$ of the model $\mathcal{F} \subset \mathbb{P}^3_{\mathbb{Z}_p}$ of F defined by the same equation $px_0^5 + x_1^5 + x_2^5 + x_3^5$ is the cone over the Fermat curve $Q_{\mathbb{F}_p} \subset \mathbb{P}^2_{\mathbb{F}_p}$ of equation $x_1^5 + x_2^5 + x_3^5$. Then μ_5 acquires one single fix point $(1 : 0 : 0 : 0) \in \mathbb{P}^3_{\mathbb{F}_p}$ which is the vertex of cone$(Q_{\mathbb{F}_p})$. We base change $\mathbb{Z}_p \subset R$ via $\pi^5 = p$ and denote by $k = \mathbb{F}_q$ the residue field and $K = \mathrm{Frac}(R) \supset \mathbb{Q}_p$ the local field. So $\mathcal{F} \times_{\mathbb{Z}_p} R \subset \mathbb{P}^3_R$ is defined by the equation $\pi^5 x_0^5 + x_1^5 + x_2^5 + x_3^5$. The μ_5 operation is still defined by $\xi \cdot x_i = \xi^i x_i$ and now the only fix point $x := (1 : 0 : 0 : 0) \in \mathbb{P}^3_{\mathbb{F}_q}$ is at the same time the only point in which $\mathcal{F} \times_{\mathbb{Z}_p} R$ is not regular. The affine equation of $\mathcal{F} \times_{\mathbb{Z}_p} R$ in $(\mathbb{A}^3_R, x_0 \neq 0)$ with coordinates $X_i = \frac{x_i}{x_0}$ on which μ_5 acts via $\xi \cdot X_i = \xi^i X_i$, is $\pi^5 + X_1^5 + X_2^5 + X_3^5$. We blow up the singularity x to obtain $\sigma : \mathcal{F}' \to \mathcal{F} \times_{\mathbb{Z}_p} R$. Then $\sigma^{-1}(x)$ is isomorphic to the Fermat quintic Z_2 in $\mathbb{P}^3_{\mathbb{F}_q}$ of equation $X_0^5 + X_1^5 + X_2^5 + X_3^5$ with action $\xi \cdot X_i = \xi^i X_i$. Consequently, μ_5 acts fix point free on $\mathcal{F}'$ and the quotient $\mathcal{X}$, which is defined over R, is a regular model of $X = X_0 \times_{\mathbb{Q}_p} K := (F/\mu_5) \times_{\mathbb{Q}_p} K$. Furthermore, $\sigma^{-1}(\mathcal{F} \times_{\mathbb{Z}_p} \mathbb{F}_q)$ is the union of two components, one being the blow up Z_1 in the vertex of cone$(Q_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{F}_q)$, the other one being the Fermat quintic Z_2. Thus the mod p fiber Y of $\mathcal{X}$ has two components $S_i = Z_i/\mu_5$. They meet along $C = (Q_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{F}_q)/\mu_5$. As $p \neq 5$, the covering $Q_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathbb{F}_q \to C$ is étale, and genus$(C) = 2$.

The normalization sequence for Y yields a Frob equivariant exact sequence

$$H^3(\bar{Y}) \to H^3(\bar{S}_1) \oplus H^3(\bar{S}_2) \to 0. \tag{2.6}$$

On the other hand, one has

$$0 \neq H^1(\bar{C})(-1) = H^1(\bar{Q}_{\mathbb{F}_p})^{\mu_5}(-1) = H^3_c(\bar{Z}_1 \setminus \bar{Q}_{\mathbb{F}_p})^{\mu_5} = H^3(\bar{Z}_1)^{\mu_5} = H^3(\bar{S}_1). \tag{2.7}$$

Thus

$$H^3(\bar{Y}) \twoheadrightarrow H^1(\bar{C})(-1) \neq 0 \tag{2.8}$$

which shows $H^3(\bar{Y}) \neq 0$.

3. Discussion

3.1. **Higher dimension.** One can produce examples as above in all dimensions by taking the product $\mathcal{X} \times_R \mathbb{P}^n$, which is still regular. Then $H^i(X \times_K \mathbb{P}^n)/H^i(X \times_K \mathbb{P}^n)_{\mathrm{alg}} = 0$ for all i, while $H^{3+2j}(Y \times_{\mathbb{F}_q} \mathbb{P}^n) \neq 0$ for all $j \geq 0$.

3.2. **Motivic condition.** From (2.1), using (2.2) and applying [3], Corollary 0.4 to the eigenvalues of $H^i(X^u)$, we see immediately that the eigenvalues of Frob on $H^i(\bar{Y})$ fulfill

$$\text{(3.1)} \quad \text{sp}^u \text{ injective } + N^\kappa(H^*(\bar{X})/H^*(\bar{X})_{\text{alg}}) = (H^*(\bar{X})/H^*(\bar{X})_{\text{alg}}) \\ \Longrightarrow \text{eigenvalues Frob}|H^i(\bar{Y}) \\ = \begin{cases} 0 & i < 2\kappa \ i \text{ odd} \\ q^{\frac{i}{2}} & i \leq 2\kappa \ i \text{ even} \\ \in q^\kappa \cdot \bar{\mathbb{Z}} & i \geq 2\kappa. \end{cases}$$

Here N^κ is the coniveau filtration as explained in the Introduction.

In [6], N. Fakhruddin analyzes the motivic conditions for a family $f : \mathbb{X} \to S$ defined over a finite field k, with $S, \mathbb{X}$ smooth, to have the property that a singular fiber Y over a closed point s with residue field $\mathbb{F}_q \supset k$ fulfills the property $|Y(\mathbb{F}_q)| = \sum_{i \geq 0}(-1)^i q^i \cdot b_{2i}(\bar{Y})$ modulo q^κ. More precisely, he studies the motivic conditions in a geometric family forcing the eigenvalue behavior described in (3.1). He singles out three conditions. We explain them and analyze the consequences they have on the completion $\mathcal{X} = \mathbb{X} \times_S R$ at s of the family f. Here R is the completion of the equal characteristic ring of functions at $s \in S$. Surely, as in [4], the first one is base change for the Chow groups $CH_i(\bar{X}), i \leq (\kappa - 1)$. We know by Bloch's type argument that this implies the coniveau condition in level κ on $H^*(\bar{X})/H^*(\bar{X})_{\text{alg}}$, but we are extremely far of understanding that this is equivalent to it, as predicted by the general Bloch-Beilinson conjectures. The second one is that $R^i f_* \mathbb{Q}_\ell$ are constant local systems. This is to say that the specialization map $H^i(\bar{Y}) \to H^i(\bar{X})$ is an isomorphism, which in particular forces sp^u to be injective, but is stronger than this. So we see that those two conditions imply the weaker cohomological conditions in (3.1) which already force the eigenvalue conclusion on $H^i(\bar{Y})$. The third condition says that the Chow groups $CH_i(\bar{Y}), i \leq (\kappa - 1)$, are hit by specialization. This should translate into the condition sp^u injective above, which is then a consequence of the cohomological consequence of the condition forcing $R^i f_* \mathbb{Q}_\ell$ being a constant local system.

At any rate, even if, as explained above, the conditions developed in [6] are far from sharpness, they tacitly raise the question of a finer formulation, and are a motivation for this note.

3.3. **Formula.** It is of course extremely rare that one can check motivic conditions. It is in the rule easier to control cohomological conditions, and (3.1) gives conditions for a good behavior of rational points on Y. However, the condition sp^u injective is very nongeometric and likely very nonnatural as well. It would be better to understand a finer condition on the contribution of $H^i_{\bar{Y}}(\mathcal{X}^u)$ in $H^i(\bar{Y})$ via the sequence (2.1).

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