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On Motives Associated to Graph Polynomials

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Abstract: The appearance of multiple zeta values in anomalous dimensions and β-functions of renormalizable quantum field theories has given evidence towards a motivic interpretation of these renormalization group functions. In this paper we start to hunt the motive, restricting our attention to a subclass of graphs in four dimensional scalar field theory which give scheme independent contributions to the above functions.

0. Introduction

Calculations of Feynman integrals arising in perturbative quantum field theory [4, 5] reveal interesting patterns of zeta and multiple zeta values. Clearly, these are motivic in origin, arising from the existence of Tate mixed Hodge structures with periods given by Feynman integrals. We are far from a detailed understanding of this phenomenon. An analysis of the problem leads via the technique of Feynman parameters [12] to the study of motives associated to graph polynomials. By the seminal work of Belkale and Brosnan [3], these motives are known to be quite general, so the question becomes under what conditions on the graph does one find mixed Tate Hodge structures and multiple zeta values.

The purpose of this paper is to give an expository account of some general mathematical aspects of these "Feynman motives" and to work out in detail the special case of wheel and spoke graphs. We consider only scalar field theory, and we focus on *primitively divergent* graphs. (A connected graph Γ is primitively divergent if $\#Edge(\Gamma) = 2h_1(\Gamma)$, where h_1 is the Betti number of the graph; and if further for any connected proper subgraph the number of edges is strictly greater than twice the first Betti number.) From a motivic point of view, these play the role of "Calabi-Yau" objects in the sense that they have unique periods. Physically, the corresponding periods are renormalization scheme independent.

Graph polynomials are introduced in Sects. 1 and 2 as special cases of discriminant polynomials associated to configurations. They are homogeneous polynomials written in a preferred coordinate system with variables corresponding to edges of the graph. The corresponding hypersurfaces in projective space are *graph hypersurfaces*. Section 3 studies coordinate linear spaces contained in the graph hypersurface. The normal cones to these linear spaces are linked to graph polynomials of sub and quotient graphs. Motivically, the chain of integration for our period meets the graph hypersurface along these linear spaces, so the combinatorics of their blowups is important. (It is curious that arithmetically interesting periods seem to arise frequently (cf. multiple zeta values [11] or the study of periods associated to Mahler measure in the non-expansive case [8]) in situations where the polar locus of the integrand meets the chain of integration in combinatorially interesting ways.)

Section 4 is not used in the sequel. It exhibits a natural resolution of singularities $\mathbb{P}(N) \to X$ for a graph hypersurface X. $\mathbb{P}(N)$ is a projective bundle over projective space, and the fibres $\mathbb{P}(N)/X$ are projective spaces.

Section 5 introduces Feynman quadrics. The period of interest is interpreted as an integral (5.3) over $\mathbb{P}^{2r-1}(\mathbb{R})$. The integrand has simple poles along r distinct quadrics. When these quadrics are associated to a graph Γ, the period is shown to be convergent precisely when Γ is primitively divergent as above.

Section 6 reinterprets the above period as a relative period (6.10) associated to the graph hypersurface. This is the Schwinger trick [12].

Section 7 presents the graph motive in detail. Let $X \subset \mathbb{P}^{2n-1}$ be the graph hypersurface associated to a primitive divergent graph. Let $\Delta \subset \mathbb{P}^{2n-1}$ be the coordinate simplex (union of $2n$ coordinate hyperplanes). An explicit sequence of blowups in $\mathbb{P}^{2n-1}$ of linear spaces is described. Write $P \to \mathbb{P}^{2n-1}$ for the resulting variety. Let $f : Y \subset P$ be the strict transform of X, and let $B := f^{-1}(\Delta)$ be the total inverse image. Then the motive is

$$H^{2n-1}(P \setminus Y, B \setminus B \cap Y). \tag{0.1}$$

Section 8 considers what can be said directly about the motive of a graph hypersurface X using elementary projection techniques. The main tool is a theorem of C. L. Dodgson about determinants, published in 1866.

Section 9 describes what the theory of motivic cohomology suggests about graph motives in cases [5] where the period is related to a zeta value.

Section 10 considers the Schwinger trick from a geometric point of view. The main result is that in middle degree, the primitive cohomology of the graph hypersurface is supported on the singular set.

Sections 11 and 12 deal with wheel and spoke graphs. Write $X_n \subset \mathbb{P}^{2n-1}$ for the hypersurface associated to the graph which is a wheel with n spokes. The main results are

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X_n) \cong \mathbb{Q}(-2), \tag{0.2}$$
$$H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_n) \cong \mathbb{Q}(-2n+3). \tag{0.3}$$

Further, the de Rham cohomology $H_{DR}^{2n-1}(\mathbb{P}^{2n-1} \setminus X_n)$ in this case is generated by the integrand of our graph period (7.1). Note that nonvanishing of the graph period, which is clear by considerations of positivity, only implies that the integrand gives a nonzero cohomology class in $H_{DR}^{2n-1}(P \setminus Y, B \setminus B \cap Y)$. It does not a priori imply nonvanishing in $H_{DR}^{2n-1}(P^{2n-1} \setminus X_n)$.

Finally, Sect. 13 discusses various issues which remain to be understood, including the question of when the motive (0.1) admits a *framing*, the curious role of triangles in graphs whose period is known to be related to a ζ value, and the possibility of constructing a Hopf algebra H of graphs such that assigning to a primitive divergent graph its motive would give rise to a Hopf algebra map from H to the Hopf algebra MZV of mixed zeta values.

From a physics viewpoint, our approach starts with a linear algebra analysis of the configurations given by a graph and its relations imposed by the edges on the vertices, illuminating the structure of the graph polynomial. An all important notion then is the one of a subgraph, and the clarification of the correspondence between linear subvarieties and subgraphs is our next achievement.

We then introduce the Feynman integral assigned to a Feynman graphs based on the usual quadrics provided by the scalar propagators of free field theory. The map from that Feynman integral to an integration over the inverse square of the graph polynomial proceeds via the Schwinger trick [12], which we discuss in detail.

We next discuss the motive using relating chains of coordinate linear subspaces of the graph hypersurfaces with chains of subgraphs. This allows for a rather systematic stratification of the graph hypersurface which can be carried through for the wheel graphs, but fails in general. We give an example of such a failure. The wheels are then subjected to a formidable computation of their middle dimensional cohomology, a feat which we are at the time of writing unable to repeat for even the next most simple class of graphs, the zig-zag graphs of [4], which, at each loop order, evaluate indeed to a rational multiple of the wheel at the same loop order. After collecting our results for the de Rham class in the wheels case, we finish the paper with some outlook how to improve the situation.

1. Polynomials Associated with Configurations

Let K be a field and let E be a finite set. Write $K[E]$ for the K-vector space spanned by E. A *configuration* is simply a linear subspace $i_V : V \hookrightarrow K[E]$. The space $K[E]$ is self-dual in an evident way, so for $e \in E$ we may consider the functional $e^\vee \circ i_V : V \to K$. Fix a basis $v_1, \dots, v_d$ for V, and let M_e be the $d \times d$ symmetric matrix associated to the rank 1 quadratic form $(e^\vee \circ i_V)^2$ on V. Define a polynomial

$$\Psi_V(A) = \det\left(\sum_{e\in E} A_e M_e\right). \tag{1.1}$$

Ψ_V is homogeneous of degree d. Note that changing the basis of V only changes Ψ_V by a unit in $K^\times$.

Remark 1.1. Write $\iota_V : \mathbb{P}(V) \hookrightarrow \mathbb{P}^{\#E-1}$ for the evident embedding on projective spaces of lines. View the quadratic forms $(e^\vee \circ i_V)^2$ as sections in $\Gamma(\mathbb{P}(V), \mathcal{O}(2))$. Then ι_V is defined by the possibly incomplete linear series spanned by these sections, and Ψ_V is naturally interpreted as defining the *dual hypersurface* in $\mathbb{P}^{\#E-1,\vee}$ of sections of this linear system which define singular hypersurfaces in $\mathbb{P}(V)$, cf. Sect. 4.

Lemma 1.2. *Each A_e appears with degree ≤ 1 in Ψ_V.*

Proof. The matrix M_e has rank ≤ 1. If $M_e = 0$ then of course A_e doesn't appear and there is nothing to prove. If rank M_e is 1, then multiplying on the left and right by invertible matrices (which only changes Ψ_V by an element in $K^\times$) we may assume M_e is the matrix with 1 in position (1, 1) and zeroes elsewhere. In this case

$$\Psi_V = \det\begin{pmatrix} A_e + m_{ee} & \dots \\ \vdots & \vdots \end{pmatrix}, \tag{1.2}$$

where A_e appears only in entry $(1,1)$. The assertion of the lemma follows by expanding the determinant along the first row. □

As a consequence, we can write

$$\Psi_V(A) = \sum_{\{e_1,\ldots,e_d\}} c_{e_1,\ldots,e_d} A_{e_1} A_{e_2} \cdots A_{e_d}. \tag{1.3}$$

Lemma 1.3. *With notation as above, write $M_{e_1,\ldots,e_d}$ for the matrix (with respect to the chosen basis of V) of the composition*

$$V \to K[E] \xrightarrow{e' \mapsto 0,\ e' \neq e_i} Ke_1 \oplus \ldots \oplus Ke_d. \tag{1.4}$$

Then $c_{e_1,\ldots,e_d} = \det M^2_{e_1,\ldots,e_d}$.

Proof. As a consequence of Lemma 1.2, $c_{e_1,\ldots,e_d}$ is obtained from Ψ_V by setting $A_{e_i} = 1,\ 1 \leq i \leq d$ and $A_{e'} = 0$ otherwise, i.e. $c_{e_1,\ldots,e_d} = \det\left(\sum_i M_{e_i}\right)$. With respect to the chosen basis of V we may write $e^\vee \circ i_V = \sum a_{e,i} v_i^\vee : V \to K$. Then $M_e = (a_{e,i} a_{e,j})_{ij}$ so

$$M_{e_1,\ldots,e_d} = (a_{e,i}); \quad \sum_e M_e = (a_{e,i})(a_{j,e})^t = M_{e_1,\ldots,e_d} M^t_{e_1,\ldots,e_d}. \tag{1.5}$$

□

Corollary 1.4. *The coefficients of Ψ_V are the squares of the Plücker coordinates for $K[E] \twoheadrightarrow W$. More precisely, the coefficient of $\prod_{e \notin T} A_e$ is $\text{Plücker}_T(W)^2$.*

Remark 1.5. Let G denote the Grassmann of all $V_d \subset K[E]$. G carries a line bundle $\mathcal{O}_G(1) \cong \det(\mathcal{V})^\vee$, where $\mathcal{V} \subset K[E] \otimes_K \mathcal{O}_G$ is the universal subbundle. Sections of $\mathcal{O}_G(1)$ arise from the dual map $\bigwedge^d K[E] \cong \Gamma(G, \det \mathcal{V}^\vee)$. Lemma 1.3 can be interpreted universally as defining a section

$$\Psi \in \Gamma(G \times \mathbb{P}(K[E]), \mathcal{O}_G(2) \boxtimes \mathcal{O}_{\mathbb{P}}(1)). \tag{1.6}$$

Define $W = K[E]/V$ to be the cokernel of i_V. Dualizing yields an exact sequence

$$0 \to W^\vee \xrightarrow{i_{W^\vee}} K[E] \to V^\vee \to 0 \tag{1.7}$$

and hence a polynomial $\Psi_{W^\vee}(A)$ which is homogeneous of degree $\#E - d$.

Proposition 1.6. *We have the functional equation*

$$\Psi_V(A) = c \cdot \left(\prod_{e \in E} A_e\right) \Psi_{W^\vee}\left(A^{-1}\right); \quad c \in K^\times. \tag{1.8}$$

Proof. For $T \subset E$ with $\#T = \#E - d$, consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & K[T] \arrow[r, "\beta_T"] \arrow[d] & W \arrow[d, equal] & \\
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & K[E] \arrow[r] \arrow[d] & W \arrow[r] & 0 \\
 & V \arrow[r, "\alpha_{E-T}"] & K[E-T] \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \tag{1.9}$$

Fix bases for V and W so the isomorphism $\det K[E] \cong \det V \otimes \det W$ (canonical up to ± 1) is given by $c \in K^\times$. Then $c = \det \alpha_{E\setminus T} \det \beta_T^{-1}$. By the above, the coefficient in Ψ_V of $\prod_{e \notin T} A_e$ is $\det \alpha^2_{E\setminus T}$ while the coefficient of $\prod_{e\in T} A_e$ in $\Psi_{W^\vee}$ is $(\det \beta_T^t)^2$. The proposition follows immediately. □

Remark 1.7. Despite the simple relation between Ψ_V and $\Psi_{W^\vee}$ it is useful to have both. When we apply this machinery in the case of graphs, $\Psi_{W^\vee}$ admits a much more concrete description. On the other hand, Ψ_V is more closely related to the Feynman integrals and periods of motives.

Remark 1.8. Let $K[E] \twoheadrightarrow W$ be as above, and suppose W is given with a basis. Then the matrix $\sum_e A_e M_e$ associated to $i_{W^\vee} : W^\vee \hookrightarrow K[E]$ is canonical as well. In fact, a situation which arises in the study of graph polynomials is an exact sequence $K[E] \to W \to K \to 0$. In this case, the matrix $\sum A_e M_e$ has zero determinant. Define $W^0 := \mathrm{Image}(K[E] \to W)$. It is easy to check that the graph polynomial for $i_{W^{0\vee}} : W^{0\vee} \hookrightarrow K[E]$ is obtained from $\sum A_e M_e$ by removing the first row and column and taking the determinant.

2. Graph Polynomials

A finite graph Γ is given with edges E and vertices V. We orient the edges. Thus each vertex of Γ has entering edges and exiting edges. For a given vertex v and a given edge e, we define $\mathrm{sign}(v, e)$ to be -1 if e enters v and +1 if e exists v. We associate to Γ a configuration (defined over $\mathbb{Z}$) via the homology sequence

$$0 \to H_1(\Gamma, \mathbb{Z}) \to \mathbb{Z}[E] \xrightarrow{\partial} \mathbb{Z}[V] \to H_0(\Gamma, \mathbb{Z}) \to 0, \tag{2.1}$$

where the bounday map is $\mathbb{Z}$-linear and defined by $\partial(e) = \sum_{v\in V} \mathrm{sign}(v, e) \cdot v$. Then ∂ depends on the chosen orientation but $H_i(\Gamma, \mathbb{Z})$ do not.

When Γ is connected, we write $\mathbb{Z}[V]^0 := \ker(\mathbb{Z}[V] \xrightarrow{\deg} \mathbb{Z})$. We define the *graph polynomial* of Γ,

$$\Psi_\Gamma := \Psi_{H_1(\Gamma,\mathbb{Z})}. \tag{2.2}$$

Recall a *tree* is a connected and simply connected graph. A tree $T \subset \Gamma$ is said to be a *spanning tree* for the connected graph Γ if every vertex of Γ lies in T. (If Γ is not connected, we can extend the notion of spanning tree $T \subset \Gamma$ by simply requiring that $T \cap \Gamma_i$ be a spanning tree in Γ_i for each connected component $\Gamma_i \subset \Gamma$.)

Lemma 2.1. *Let T be a subgraph of a connected graph Γ. Let $E = E_\Gamma$ be the set of edges of Γ and let $E_T \subset E$ be the edges of T. Then T is a spanning tree if and only if one has an exact homology diagram as indicated:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathbb{Z}[E_T] \arrow[r, "\beta", "\cong"'] \arrow[d] & \mathbb{Z}[V]^0 \arrow[d] & & \\
0 \arrow[r] & H_1(\Gamma) \arrow[r] \arrow[d, equal] & \mathbb{Z}[E] \arrow[r, "\partial"] \arrow[d] & \mathbb{Z}[V] \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H_1(\Gamma) \arrow[r, "\alpha", "\cong"'] & \mathbb{Z}[E \setminus E_T] \arrow[r, "0"'] \arrow[d] & \mathbb{Z} \arrow[r, "\cong"'] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd} \tag{2.3}$$

Proof. Straightforward. □

Proposition 2.2. *With notation as above, we have*

$$\Psi_\Gamma(A) = \sum_{T\, span\ tr.} \prod_{e \notin T} A_e. \tag{2.4}$$

Proof. Fix a basis h_j for $H_1(\Gamma)$. Then

$$\Psi_\Gamma(A) = \det\Big(\sum_e A_e e^\vee(h_j) e^\vee(h_k)\Big). \tag{2.5}$$

Let $B \subset E$ have b elements, and let $E' = E \setminus B$. The coefficient of the monomial $\prod_{e \in B} A_e$ in $\Psi_\Gamma(A)$ is computed by setting $A_{e'} = 0$ for $e' \in E'$. The coefficient is non-zero iff the determinant (1.1) is non-zero under this specialization, and this is true iff we get a diagram as in (2.3), i.e. iff $E' = E_T$ for a spanning tree T. The coefficient of this monomial is $1 = \det(\alpha\alpha^t)$, where α is as in the bottom row of (2.3). □

Remark 2.3. If $\Gamma = \coprod \Gamma_i$ with Γ_i connected, then

$$\Psi_\Gamma = \prod_i \Psi_{\Gamma_i}, \tag{2.6}$$

as both the free abelian group on edges and H_1 are additive in i. If we define spanning "trees" in disconnected graphs as suggested above, Proposition 2.2 carries over to the disconnected case.

Corollary 2.4. *The coefficients of* Ψ_Γ *are all either* 0 *or* +1.

Definition 2.5. *The graph hypersurface* $X_\Gamma \subset \mathbb{P}^{\#(E_\Gamma)-1}$ *is the hypersurface cut out by* $\Psi_\Gamma = 0$.

Properties 2.6. We list certain evident properties of Ψ_Γ:

1. Ψ_Γ is a sum of monomials with coefficient +1.
2. No variable A_i appears with degree > 1 in any monomial.
3. Let Γ_1 and Γ_2 be graphs, and fix vertices $v_i \in \Gamma_i$. Define $\Gamma := \coprod \Gamma_i/\{v_1 \sim v_2\}$. Thus, $E_\Gamma = E_{\Gamma_1} \amalg E_{\Gamma_1}$ and $H_1(\Gamma) = H_1(\Gamma_1) \oplus H_1(\Gamma_2)$. Writing $A^{(i)}$ for the variables associated to edges of Γ_i, we see that $\Psi_\Gamma = \Psi_{\Gamma_1}(A^{(1)})\Psi_{\Gamma_2}(A^{(2)})$. Geometrically, the graph hypersurface $X_\Gamma : \Psi_\Gamma = 0$ is simply the join of the graph hypersurfaces X_{Γ_i}. (Recall, if $P_i \subset \mathbb{P}^N$ are linear subsets of projective space such that $P_1 \cap P_2 = \emptyset$ and $\dim P_1 + \dim P_2 = N - 1$, and $X_i \subset P_i$ are closed subvarieties, then the join $X_1 * X_2$ is simply the union of all lines joining points of X_1 to points of X_2.) In particular, if Γ_2 is a tree, so $\Psi_{\Gamma_2} = 0$, then X_Γ is a cone over X_{Γ_2}.
4. Defining Ψ_Γ via spanning trees (2.4) can lead to confusion in degenerate cases. For example, if Γ has only a single vertex (*tadpole graph*) and n edges, then $H_1(\Gamma) \cong \mathbb{Z}[E_\Gamma] \cong \mathbb{Z}^n$. Thus $\Psi_\Gamma = \prod_1^n A_i$, but there are no spanning trees.

3. Linear Subvarieties of Graph Hypersurfaces

Let Γ be a graph with $n = \#E_\Gamma$ edges. For convenience we take Γ to be connected. It will be convenient to use the notation $h_1(\Gamma) := \operatorname{rank} H_1(\Gamma)$. In talking about subgraphs of a given graph Γ, we will frequently not distinguish between the subgraph and the collection of its edges. (In particular, we will not permit isolated vertices.)

Recall we have associated to Γ a hypersurface $X_\Gamma \subset \mathbb{P}^{n-1}$. Our projective space has a distinguished set of homogeneous coordinates $A_e \leftrightarrow e \in E_\Gamma$, so we get a dictionary:

$$\begin{gathered}\text{Subgraphs } G \subset \Gamma \leftrightarrow \text{coordinate linear subspaces } L \subset \mathbb{P}^{n-1} \\ G \mapsto L(G) : A_e = 0, e \in G \\ L : A_e = 0, e \in S \subset E_\Gamma \mapsto G(L) = \bigcup_{e \in S} e \subset \Gamma.\end{gathered} \tag{3.1}$$

The Feynman period is the integral of a differential form on $\mathbb{P}^{n-1}$ with poles along X_Γ over a chain which meets X_Γ along the non-negative real loci of coordinate linear spaces contained in X_Γ. To give motivic meaning to this integral, it will be necessary to blow up such linear spaces. The basic combinatorial observation is

Proposition 3.1. *With notation as above, a coordinate linear space* L *is contained in* X_Γ *if and only if* $h_1(G(L)) > 0$.

Proof. Suppose $L : A_e = 0, e \in S$. Then $L \subset X_\Gamma$ if and only if every monomial in Ψ_Γ is divisible by A_e for some $e \in S$. In other words, iff no spanning tree of Γ contains S. The assertion now follows from

Lemma 3.2. *Let* $S \subset \Gamma$ *be a (not necessarily connected) subgraph. Then* S *is contained in some spanning tree for* Γ *iff* $h_1(S) = 0$.

Proof of Lemma . Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(S) & \longrightarrow & \mathbb{Z}[E_S] & \xrightarrow{c} & \mathbb{Z}[V_\Gamma]^0 & & \\
 & & \Big\downarrow i & & \Big\downarrow b & & \Big\| & & \\
0 & \longrightarrow & H_1(\Gamma) & \xrightarrow{a} & \mathbb{Z}[E_\Gamma] & \longrightarrow & \mathbb{Z}[V_\Gamma]^0 & \longrightarrow & 0.
\end{array} \tag{3.2}$$

Note that the map i is always injective. S is itself a spanning tree iff c is surjective and a and b have disjoint images. If we simply assume disjoint images with c not surjective, we can find $e \in E_\Gamma$ such that $e \notin \mathrm{im}(a) + \mathrm{im}(b)$. Then $S' = S \cup \{e\}$ still satisfies $h_1(S') = 0$. Continuing in this way, eventually c must be surjective. Since the images of a and b remain disjoint, c will be an isomorphism, and the resulting subgraph of Γ will be a spanning tree. $\square$

This completes the proof of the proposition. $\square$

Let Γ be a connected graph as above, and let $G \subset \Gamma$ be a subgraph. It will be convenient not to assume G connected. In particular, Ψ_G and X_G will be defined as in Remark 2.3. We define a modified quotient graph

$$\Gamma \twoheadrightarrow \Gamma//G \tag{3.3}$$

by identifying the connected components G_i of G to vertices $v_i \in \Gamma//G$ (but not identifying $v_i \sim v_j$). If G is connected, this is the standard quotient in topology. One gets a diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & H_1(G) & \longrightarrow & \mathbb{Z}[E_G] & \longrightarrow & \mathbb{Z}[V_G]^0 & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & H_1(\Gamma) & \longrightarrow & \mathbb{Z}[E_\Gamma] & \longrightarrow & \mathbb{Z}[V_\Gamma]^0 & \longrightarrow & 0 \\
 & & \Big\downarrow \pi & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & H_1(\Gamma//G) & \longrightarrow & \mathbb{Z}[E_{\Gamma//G}] & \longrightarrow & \mathbb{Z}[V_{\Gamma//G}]^0 & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{3.4}$$

Note with this modified quotient the map labeled π is surjective.

Our objective now is to relate the graph hypersurfaces X_Γ, X_G, $X_{\Gamma//G}$. To this end, we first consider the relation between spanning trees for the three graphs. If $T \subset \Gamma$ is a spanning tree, then $h_1(T \cap G) = 0$, but $T \cap G$ is not necessarily connected. In particular it is not necessarily a spanning tree for G.

There is an evident lifting from subgraphs $V \subset \Gamma//G$ to subgraphs $\widetilde{V} \subset \Gamma$ such that $\widetilde{V}$ and G have no common edges.

Lemma 3.3. *Let $U \subset G$ be a spanning tree (cf. Remark 2.3). Then the association*

$$V \mapsto T := \widetilde{V} \amalg U \tag{3.5}$$

induces a 1 to 1 correspondence between spanning trees V of $\Gamma//G$ and spanning trees T of Γ such that $U \subset T$.

Proof. Let T be a spanning tree for Γ and assume $U \subset T$. Necessarily, $G \cap T = U$. Indeed, $U \subset G \cap T$ and $h_1(G \cap T) = 0$. Since U is already a spanning tree, it follows from (2.3) that $G \cap T$ cannot be strictly larger than U.

By (3.4), $\pi(T) \cong T//U \subset \Gamma//G$ is connected and $h_1(\pi(T)) = 0$. It follows that $\pi(T)$ is a spanning tree for $\Gamma//G$. We have $T = \widetilde{\pi(T)} \amalg U$, so the association $T \mapsto \pi(T)$ is injective.

Finally, if $V \subset \Gamma//G$ is a spanning tree, then since

$$V \cong (\widetilde{V} \amalg U)//U,$$

it follows from (3.4) that $h_1(\widetilde{V} \amalg U) = 0$. One easily checks that this subgraph is connected and contains all the vertices of Γ, so it is a spanning tree. □

Proposition 3.4. *Let Γ be a connected graph, and let $G \subset \Gamma$ be a subgraph. Assume $h_1(G) = 0$. Let $X_\Gamma \subset \mathbb{P}(E_\Gamma)$ be the graph hypersurface, and let $L(G) : A_e = 0, e \in G$ be the linear subspace of $\mathbb{P}(E_\Gamma)$ corresponding to G. Then $L(G)$ is naturally identified with $\mathbb{P}(E_{\Gamma//G})$, and under this identification,*

$$X_{\Gamma//G} = X_\Gamma \cap L(G).$$

Proof. In this case, Lemma 3.3 implies that spanning trees for $\Gamma//G$ are in 1 to 1 correspondence with spanning trees for Γ containing G. It follows from Proposition 2.2 that

$$\Psi_{\Gamma//G} = \Psi_\Gamma|_{A_e=0, e\in G}.$$

□

Proposition 3.5. *Let $G \subset \Gamma$ be a subgraph, and suppose $h_1(G) > 0$. Then $L(G)$: $A_e = 0, e \in G$ is contained in X_Γ. Let $P \to \mathbb{P}(E_\Gamma)$ be the blowup of $L(G) \subset \mathbb{P}(E_\Gamma)$, and let $F \subset P$ be the exceptional locus. Let $Y \subset P$ be the strict transform of X_Γ in P. Then we have canonical identifications*

$$F \cong \mathbb{P}(E_G) \times \mathbb{P}(E_{\Gamma//G}), \tag{3.6}$$

$$Y \cap F = \Big(X_G \times \mathbb{P}(E_{\Gamma//G})\Big) \cup \Big(\mathbb{P}(E_G) \times X_{\Gamma//G}\Big). \tag{3.7}$$

Proof. Let $T \subset \Gamma$ be a spanning tree. We have $h_1(T \cap G) = 0$ so $T \cap G$ is contained in a spanning tree for G by Lemma 3.2. In particular, $\#(T \cap G) \geq \#E_G - h_1(G)$, with equality if and only if $T \cap G$ is a spanning tree for G.

The normal bundle for $L(G) \subset \mathbb{P}(E_\Gamma)$ is $\bigoplus_{e\in G} \mathcal{O}(1)$, from which it follows that $F \cong L(G) \times \mathbb{P}(E_G)$. Also, of course, $L(G) \cong \mathbb{P}(E_\Gamma \setminus E_G) \cong \mathbb{P}(E_{\Gamma//G})$.

We have $L(G) \subset X_\Gamma$ by Proposition 3.1. The intersection $F \cap Y$ is the projectivized normal cone of this inclusion. Algebraically, we identify

$$K[A_e]_{e\in\Gamma//G} \otimes K[A_e]_{e\in G} \tag{3.8}$$

with the tensor of the homogeneous coordinate rings for $\mathbb{P}(E_{\Gamma//G})$ and $\mathbb{P}(E_G)$. Our cone is the hypersurface in this product defined by the sum of terms in $\Psi_\Gamma = \sum_{T\subset\Gamma}\prod_{e\notin T} A_e$ of minimal degree in the normal variables $A_e, e \in G$. These correspond to spanning trees T with $\#G\cap T$ maximal. By the above discussion, these are the T such that $T\cap G$ is a spanning tree for G. It now follows from Lemma 3.3 that in fact the cone is defined by

$$\Psi_{\Gamma//G}(A_e)_{e\in\Gamma//G}\cdot\Psi_G(A_e)_{e\in G}\in K[A_e]_{e\in\Gamma//G}\otimes K[A_e]_{e\in G}. \tag{3.9}$$

The proposition is now immediate. □

Remark 3.6. The set $F\cap Y$ above can also be interpreted as the exceptional fibre for the blowup of $L(G)\subset X_\Gamma$.

Example 3.7. Fix an edge $e_0\in\Gamma$ and take $G=\Gamma\setminus e_0$. Then $L(G)=:p$ is a single point. If $p\notin X_\Gamma$, then $h_1(G)=0$ and Proposition 3.4 implies that $X_{\Gamma//G}=\emptyset$. If $p\in X_\Gamma$, then $F\cong\mathbb{P}(E_\Gamma\setminus e_0)$ and the exceptional divisor for the blowup of $p\in X_\Gamma$ is $X_{\Gamma\setminus e_0}$.

Algebraically, this all amounts to the identity

$$\Psi_\Gamma = A_{e_0}\Psi_{\Gamma\setminus e_0}+\Psi_{\Gamma/e_0}, \tag{3.10}$$

where the two graph polynomials on the right do not involve A_{e_0}.

4. Global Geometry

In this section, for a vector bundle E over a variety X we write $\mathbb{P}(E)$ for the projective bundle of hyperplane sections, so $a_*\mathcal{O}_{\mathbb{P}(E)}(1)=E$, with $a:\mathbb{P}(E)\to X$. In particular, a surjection of vector bundles $E\twoheadrightarrow F$ gives rise to a closed immersion $\mathbb{P}(F)\hookrightarrow\mathbb{P}(E)$.

Consider projective space $\mathbb{P}^r$ and its dual $(\mathbb{P}^r)^\vee$. One has the Euler sequence

$$0\to\mathcal{O}_{\mathbb{P}^r}\xrightarrow{e}\mathcal{O}_{\mathbb{P}^r}(1)\otimes\Gamma((\mathbb{P}^r)^\vee,\mathcal{O}(1))\to T_{\mathbb{P}^r}\to 0, \tag{4.1}$$

where T is the tangent bundle. Writing $T_0,\ldots,T_r$ for a basis of $\Gamma(\mathbb{P}^r,\mathcal{O}(1))$ and $\frac{\partial}{\partial T_i}\in\Gamma((\mathbb{P}^r)^\vee,\mathcal{O}(1))$ for the dual basis, we have

$$e(1)=\sum T_i\otimes\frac{\partial}{\partial T_i}\in\Gamma\Big(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}^r}(1)\otimes\Gamma((\mathbb{P}^r)^\vee,\mathcal{O}(1))\Big). \tag{4.2}$$

Geometrically, we can think of $e(1)$ as a homogeneous form of degree $(1,1)$ on $\mathbb{P}^r\times(\mathbb{P}^r)^\vee$ whose zeroes define $\mathbb{P}(T_{\mathbb{P}^r})\hookrightarrow\mathbb{P}^r\times(\mathbb{P}^r)^\vee$. The fibre in $\mathbb{P}(T_{\mathbb{P}^r})$ over a point $\frac{\partial}{\partial T_i}=a_i$ in $(\mathbb{P}^r)^\vee$ is the hyperplane cut out by $\sum a_iT_i$ in $\mathbb{P}^r$.

For $V\hookrightarrow\mathbb{P}^r$ a closed subvariety, define p_V to be the composition $p_V:\mathbb{P}(T_{\mathbb{P}^r}|_V)\hookrightarrow\mathbb{P}(T_{\mathbb{P}^r})\to(\mathbb{P}^r)^\vee$, and the fibre over $\frac{\partial}{\partial T_i}=a_i$ is $V\cap\{\sum a_iT_i=0\}$. Assuming V smooth, we have the normal bundle sequence

$$0\to T_V\to T_{\mathbb{P}^r}|_V\to N_{V/\mathbb{P}^r}\to 0. \tag{4.3}$$

Proposition 4.1. *Assume $V \hookrightarrow \mathbb{P}^r$ is a smooth, closed subvariety. Consider the diagram*

$$\begin{array}{ccc} \mathbb{P}(N_{V/\mathbb{P}^r}) & \xrightarrow{\hookrightarrow} & \mathbb{P}(T_{\mathbb{P}^r}|_V) \\ \downarrow & & \downarrow{\scriptstyle p_V} \\ (\mathbb{P}^r)^\vee & = & (\mathbb{P}^r)^\vee \end{array} . \tag{4.4}$$

We have

$$\mathbb{P}(N_{V/\mathbb{P}^r}) \cap p_V^{-1}(a) = \left(V \cap \left\{\sum a_i T_i = 0\right\}\right)_{\text{sing}}, \tag{4.5}$$

the singular points of the corresponding hypersurface section.

Proof. Let $x \in V \subset \mathbb{P}^r$ be a point. To avoid confusion we write dT_i for the dual basis to $\frac{\partial}{\partial T_i}$. To a sum $\sum a_i dT_i$ and a point $x \in V$ we can associate a point of $\mathbb{P}(T_{\mathbb{P}^r}|_V)$. Suppose $x \in p_V^{-1}(a)$. Then x is singular in this fibre if and only if $\sum a_i dT_i$ kills $T_{V,x} \subset T_{\mathbb{P}^r,x}$, and this is true if and only if $\sum a_i dT_i \in \mathbb{P}(N_{V/\mathbb{P}^r})$. □

Suppose now $V = \mathbb{P}^k$ and the embedding $\mathbb{P}^k \hookrightarrow \mathbb{P}^r$ is defined by a sublinear system in $\Gamma(\mathbb{P}^k, \mathcal{O}(2))$ spanned by quadrics $q_0, \ldots, q_k$. The fibres of the map $p : T_{\mathbb{P}^r/\mathbb{P}^k} \to (\mathbb{P}^r)^\vee$ are the degree 2 hypersurfaces $\left\{\sum a_i q_i = 0\right\} \subset \mathbb{P}^k$. Note that the singular set in such a hypersurface is a projective space of dimension $= k - \text{rank}(\sum a_i M_i)$, where the M_i are $(k+1) \times (k+1)$ symmetric matrices associated to the quadrics q_i. We conclude

Proposition 4.2. *With notation as above, define*

$$X = \left\{a \in (\mathbb{P}^r)^\vee \,|\, \text{rank}\left(\sum a_i M_i\right) < k+1\right\}. \tag{4.6}$$

Then writing $N = N_{\mathbb{P}^k/\mathbb{P}^r}$, the map $\mathbb{P}(N) \to X$ is a resolution of singularities of X. The fibres of this map are projective spaces, with general fibre $\mathbb{P}^0 =$ point.

5. Quadrics

Let $K \subset \mathbb{R}$ be a real field. (For the application to Feynman quadrics, $K = \mathbb{Q}$.) We will be interested in homogeneous quadrics

$$Q_i : q_i(Z_1, \ldots, Z_{2r}) = 0, \quad 1 \le i \le r \tag{5.1}$$

in $\mathbb{P}^{2r-1}$ with homogeneous coordinates $Z_1, \ldots, Z_{2r}$. The union $\cup_i^r Q_i$ of the quadrics has then degree $2r$. It implies that $\Gamma\left(\mathbb{P}^{2r-1}, \omega\left(\sum_1^r Q_i\right)\right) = K[\eta]$ for a generator η which, on the affine open $Z_{2r} \neq 0$ with affine coordinates $z_i = \frac{Z_i}{Z_r}$, $i = 1, \ldots, (2r-1)$, is $\eta|_{Z_{2r-1}\neq 0} = \frac{dz_1\wedge\ldots\wedge dz_{2r-1}}{\tilde{q}_1\cdots\tilde{q}_r}$, with $\tilde{q}_i = \frac{q_i}{Z_{2r}^2}$. By (standard) abuse of notations, we write

$$\eta = \frac{\Omega_{2r-1}}{q_1\cdots q_r}; \quad \Omega_{2r-1} := \sum_{i=1}^{2r} (-1)^i Z_i dZ_1 \wedge \cdots \widehat{dZ_i} \cdots \wedge dZ_{2r}. \tag{5.2}$$

The transcendental quantity of interest is the *period*

$$P(Q) := \int_{\mathbb{P}^{2r-1}(\mathbb{R})} \eta = \int_{z_1,\ldots,z_{2r-1}=-\infty}^{\infty} \frac{dz_1 \wedge \cdots \wedge dz_{2r-1}}{\tilde{q}_1 \cdots \tilde{q}_r}. \tag{5.3}$$

The integral is convergent and the period well defined, e.g. when the quadrics are all positive definite.

Suppose now $r = 2n$ above, so we consider quadrics in $\mathbb{P}^{4n-1}$. Let $H \cong K^n$ be a vector space of dimension n, and identify $\mathbb{P}^{4n-1} = \mathbb{P}(H^4)$. For $\ell : H \to K$ a linear functional, ℓ^2 gives a rank 1 quadratic form on H. A *Feynman quadric* is a rank 4 positive semi-definite form on $\mathbb{P}^{4n-1}$ of the form $q = q_\ell = (\ell^2, \ell^2, \ell^2, \ell^2)$. We will be interested in quadrics Q_i of this form (for a fixed decomposition $K^{4n} = H^4$). In other words, we suppose given linear forms ℓ_i on H, $1 \le i \le 2n$, and we consider the corresponding period $P(Q)$, where $q_i = (q_{\ell_i}, q_{\ell_i}, q_{\ell_i}, q_{\ell_i})$.

For $\ell : H \to K$ a linear form, write $\lambda = \ker(\ell)$, $\Lambda = \mathbb{P}(\lambda, \lambda, \lambda, \lambda) \subset \mathbb{P}(H^4) = \mathbb{P}^{4n-1}$. The Feynman quadric q_ℓ associated to ℓ is then a cone over the codimension 4 linear space Λ. For a suitable choice of homogeneous coordinates $Z_1, \ldots, Z_{4n}$ we have $q_\ell = Z_1^2 + \cdots + Z_4^2$.

Let $q_1, \ldots, q_{2n}$ be Feynman quadrics, and let Λ_i be the linear space associated to q_i as above. As K is a real field, $\mathbb{P}^{4n-1}(\mathbb{R})$ meets $Q_i(\mathbb{C})$ only on $\Lambda_i(\mathbb{R})$.

Lemma 5.1. *With notation as above, for $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, 2n\}$, write $r(I) = \mathrm{codim}_H(\lambda_{i_1} \cap \ldots \cap \lambda_{i_p})$. The integral (5.2) converges if and only if $\sup_I\{p(I) - 2r(I)\} < 0$. Here the sup is taken over all $I \subset \{1, \ldots, 2n\}$ and $p(I) = \#I$.*

Proof. Suppose $\lambda_1 \cap \ldots \cap \lambda_p$ has codimension r, with $2r \le p$. We can choose local coordinates x_j so that $\bigcap_{i=1}^p \Lambda_i : x_1 = \cdots = x_{4r} = 0$, and then make the blowup $y_j = \frac{x_j}{x_{4r}}$, $1 \le j \le 4r - 1$, $y_j = x_j$, $j \ge 4r$. Then

$$\frac{d^{4n-1}x}{q_1(x) \cdots q_{2n}(x)} = \frac{x_{4r}^{4r-1} d^{4n-1}y}{x_{4r}^{2p} \tilde{q}_1(y) \cdots \tilde{q}_{2n}(y)} \tag{5.4}$$

for suitable $\tilde{q}_i(y)$ which are regular in the y-coordinates. Since $|\prod \tilde{q}_i^{-1}| \ge C > 0$, it follows that the integral over a neighborhood of $0 \in \mathbb{R}^{4n-1}$ diverges if $(4r - 2p) \le 0$.

Suppose conversely that $\sup_I\{p(I) - 2r(I)\} < 0$. Note if $n = 1$, the quadrics are smooth and positive definite so the integrand has no pole along the integration chain and convergence is automatic. Assume $n > 1$. The above argument shows that blowing up an intersection of the Λ_i does not introduce a pole in the integrand along the exceptional divisor. Further, the strict transforms of the quadrics continue to have degree ≤ 2 in the natural local coordinates and to be cones over the strict transforms of the Λ_i. One knows that after a finite number of such blowups, the strict transforms of the Λ_i will meet transversally (see [10] for a minimal way to do it). All blowups and coordinates will be defined over $K \subset \mathbb{R}$, and one is reduced to checking convergence for an integral of the form

$$\int_U \frac{d^{4n-1}x}{(x_1^2 + \cdots + x_4^2) \cdots (x_{4n-7}^2 + \cdots + x_{4n-4}^2)} \tag{5.5}$$

with U a neighborhood of $0 \in \mathbb{R}^{2n-1}$. The change of variables $x_i = ty_i$, $i \le (4n - 4)$ introduces a $t^{4n-5-2n+2} = t^{2n-3}$ factor. Since $n \ge 2$, convergence is clear. □

Let Γ be a graph with N edges and n loops. Associated to Γ we have the configuration of N hyperplanes in the n-dimensional vector space $H = H_1(\Gamma)$, (2.1). As above, we map the Feynman quadrics $q_i = (\ell_i^2, \ell_i^2, \ell_i^2, \ell_i^2)$ on $\mathbb{P}^{4n-1}$, $1 \le i \le N$. The graph Γ

is said to be *convergent* (resp. *logarithmically divergent*) if $N > 2n$ (resp. $N = 2n$). When Γ is logarithmically divergent, the form

$$\omega_\Gamma := \frac{d^{4n-1}x}{q_1 \cdots q_{2n}} \tag{5.6}$$

has poles only along $\bigcup Q_i$, and we define the period

$$P(\Gamma) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \omega_\Gamma \tag{5.7}$$

as in (5.3).

Proposition 5.2. *Let Γ be a logarithmically divergent graph with n loops and $2n$ edges. The period $P(\Gamma)$ converges if and only if every subgraph $G \subsetneq \Gamma$ is convergent, i.e. if and only if Γ is primitive log divergent in the sense discussed in Sect. 0.*

Proof. Let $G \subset \Gamma$ be a subgraph with m loops and M edges, and assume $M \le 2m$. Let $I \subset \{1, \dots, 2n\}$ be the edges not in G. Note $H_1(G) \subset H_1(\Gamma)$ has codimension $n - m$ and is defined by the $2n - M$ linear functionals corresponding to edges in I. By Lemma 5.1, the fact that $2(n - m) \le 2n - M$ implies that the period integral $P(\Gamma)$ is divergent. Conversely, if the period integral is divergent, there will exist an I with $p(I) - 2r(I) \ge 0$. Let $G \subset \Gamma$ be the union of the edges not in I. Then G has $2n - p(I)$ edges. Also $H_1(G) \subset H_1(\Gamma)$ is defined by the vanishing of functionals associated to edges in I, so G has $n - r(I)$ loops. It follows that G is not convergent. □

6. The Schwinger Trick

Let $Q_i : q_i(Z_1, \dots, Z_{4n}) = 0$, $1 \le i \le 2n$ be quadrics in $\mathbb{P}^{4n-1}$. We assume the period integral (5.3) converges. Let M_i be the $4n \times 4n$ symmetric matrix corresponding to q_i, and write

$$\Phi(A_1, \dots, A_{2n}) := \det(A_1 M_1 + \cdots + A_{2n} M_{2n}). \tag{6.1}$$

The Schwinger trick relates the period integral $P(Q)$ (5.3) to an integral on $\mathbb{P}^{2n-1}$,

$$\int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 \cdots q_{2n}} = C \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\sqrt{\Phi}}. \tag{6.2}$$

Here $\sigma^{2n-1}(\mathbb{R}) \subset \mathbb{P}^{2n-1}(\mathbb{R})$ is the locus of all points $s = [s_1, \dots, s_{2n}]$ such that the projective coordinates $s_i \ge 0$. C is an elementary constant, and the Ω's are as in (5.2). Note the homogeneity is such that the integrands make sense.

Lemma 6.1. *With notation as above, define*

$$g(A) = \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}}{(A_1 q_1 + \cdots + A_{2n} q_{2n})^{2n}}. \tag{6.3}$$

Then

$$g(A)\sqrt{\Phi} = c\pi^{-2n}; \quad c \in \overline{\mathbb{Q}}^\times, \ [\mathbb{Q}(c) : \mathbb{Q}] \le 2. \tag{6.4}$$

If $\Phi = \Xi^2$ for a polynomial $\Xi \in \mathbb{Q}[A_1, \dots, A_{2n}]$, then $c \in \mathbb{Q}^\times$.

Proof. By analytic continuation, we may suppose that $Q_a : \sum A_i q_i = 0$ is smooth. The integral is then the period associated to $H^{4n-1}(\mathbb{P}^{4n-1} \setminus Q_a)$. As generator for the homology we may either take $\mathbb{P}^{4n-1}(\mathbb{R})$ or the tube $\tau \subset \mathbb{P}^{4n-1} \setminus Q_a$ lying over the difference of two rulings $\ell_1 - \ell_2$ in the even dimensional smooth quadric Q_a. (More precisely, let $S \subset N \xrightarrow{p} X$ be the sphere bundle for some metric on the normal bundle N of X, where $X \subset \mathbb{P}^{2n-1}$ is defined by $\Phi = 0$. Take $\tau = p^{-1}(\ell_1 - \ell_2)$.) The two generators differ by a rational scale factor c. Integrating over τ shows that $g(A)$ is defined up to a scale factor ± 1 on $\mathbb{P}^{2n-1} \setminus X$. The monodromy arises because the rulings ℓ_i on Q_a can be interchanged as a winds around X. It follows easily that the left-hand side in (6.4) is homogeneous of degree 0 and single-valued on $\mathbb{P}^{2n-1} \setminus X$. To study its behavior near X we restrict to a general line in $\mathbb{P}^{2n-1}$. In affine coordinates, we can then assume the family of quadrics looks like $\left(\sum_1^{4n-1} x_i^2\right) - t = 0$, where t is a parameter on the line. The integral then becomes

$$\int_\gamma \frac{dx_1 \wedge \ldots \wedge dx_{4n-1}}{\left(\sum x_i^2 - t\right)^{2n}} = \text{const} \cdot t^{-\frac{1}{2}} \tag{6.5}$$

for a suitable cycle γ. The change of variable $x_i = y_i t^{\frac{1}{2}}$ gives the value const $\cdot\, t^{-\frac{1}{2}}$ from which one sees that $g(A)\sqrt{\Phi}$ is constant. Since $H^{4n-1}(\mathbb{P}^{4n-1} \setminus Q_a) \cong \mathbb{Q}(-2n)$ as Hodge structure, $g(A) = c_0 \pi^{-2n}$ for some $c_0 \in \mathbb{Q}^\times$, and the lemma follows. □

With notation as above, define

$$f(A) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{(A_1 q_1 + \cdots + A_{2n} q_{2n}) q_2 q_3 \cdots q_{2n}}. \tag{6.6}$$

Note that $f(A)$ is defined for q_i positive definite and $A_j \geq 0$ but not all $A_j = 0$. We have

$$g(A) = \frac{-1}{(2n-1)!} \frac{\partial^{2n-1}}{\partial A_2 \ldots \partial A_{2n}} f(A). \tag{6.7}$$

Write $a_i = \frac{A_i}{A_1}, 2 \leq i \leq 2n$, and define $F(a_2, \ldots, a_{2n}) := A_1 f(A)$. Note the various partials $\partial^{i-1}/\partial a_2 \ldots \partial a_i F(a)$ vanish as $a_i \to +\infty$ with $a_j \geq 0, \forall j$. Also $\frac{\Omega_{2n-1}}{A_1^{2n}} = -da_2 \wedge \ldots \wedge da_{2n}$. Thus

$$\begin{aligned} \int_{\sigma^{2n-1}(\mathbb{R})} g(A)\Omega_{2n-1}(A) = -\int_{\sigma^{2n-1}(\mathbb{R})} A_1^{2n} g(A) da_2 \wedge \ldots \wedge da_{2n} = \\ \frac{1}{(2n-1)!} \int_{a_2,\ldots,a_{2n}=0}^{+\infty} \frac{\partial^{2n-1}}{\partial a_2 \ldots \partial a_{2n}} F(a) da_2 \wedge \ldots \wedge da_{2n} = \\ \frac{-1}{(2n-1)!} F(0, \ldots, 0) = \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 q_2 \cdots q_{2n}} = P(Q). \end{aligned} \tag{6.8}$$

This identity holds by analytic extension in the q's where both integrals are defined. Combining (6.8) with Lemma 6.1 we conclude

Proposition 6.2. *With notation as above, assuming the integral defining $P(Q)$ is convergent, we have*

$$P(Q) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 q_2 \cdots q_{2n}} = \frac{c}{\pi^{2n}} \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\sqrt{\Phi}}. \tag{6.9}$$

Corollary 6.3. *Let Γ be a graph with n loops and $2n$ edges. Assume every proper subgraph of Γ is convergent, and let $q_1, \ldots, q_{2n}$ be the Feynman quadrics associated to Γ (cf. Sect. 5). The symmetric matrices M_i* (6.1) *in this case are block diagonal*

$$M_i = \begin{pmatrix} N_i & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 \\ 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & N_i \end{pmatrix}$$

and $\Phi = \Psi_\Gamma^4$, where $\Psi_\Gamma = \det(A_1 N_1 + \ldots + A_{2n} M_{2n})$ is the graph polynomial (2.2). *The Schwinger trick yields* (*cf.* (5.7))

$$P(\Gamma) := \int_{\mathbb{P}^{4n-1}(\mathbb{R})} \frac{\Omega_{4n-1}(Z)}{q_1 q_2 \cdots q_{2n}} = \frac{c}{\pi^{2n}} \int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2} \tag{6.10}$$

for $c \in \mathbb{Q}^\times$.

7. The Motive

We assume as in Sect. 5 that the ground field $K \subset \mathbb{R}$ is real. Let Γ be a graph with n loops and $2n$ edges and assume every proper subgraph of Γ is convergent. Our objective in this section is to consider the motive with period

$$\int_{\sigma^{2n-1}(\mathbb{R})} \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2}. \tag{7.1}$$

We consider $\mathbb{P}^{2n-1}$ with fixed homogeneous coordinates $A_1, \ldots, A_{2n}$ associated with the edges of Γ. Linear spaces $L \subset \mathbb{P}^{2n-1}$ defined by vanishing of subsets of the A_i will be referred to as coordinate linear spaces. For such an L, we write $L(\mathbb{R}^{\geq 0})$ for the subset of real points with non-negative coordinates.

Lemma 7.1. $X_\Gamma(\mathbb{C}) \cap \sigma^{2n-1}(\mathbb{R}) = \bigcup_{L \subset X_\Gamma} L(\mathbb{R}^{\geq 0})$, *where the union is taken over all coordinate linear spaces $L \subset X_\Gamma$.*

Proof. We know by Corollary 2.4 that Ψ_Γ is a sum of monomials with coefficients +1. The lemma is clear for the zero set of any polynomial with coefficients > 0. □

Remark 7.2. (i) The assertion of the lemma is true for any graph polynomial. We do not need hypotheses about numbers of edges or loops.
(ii) By Proposition 3.1, coordinate linear spaces $L \subset X_\Gamma$ correspond to subgraphs $G \subset \Gamma$ such that $h_1(G) > 0$.

Proposition 7.3. *Let* Γ *be as above. Define*

$$\eta = \eta_\Gamma = \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2} \tag{7.2}$$

as in (5.2). *There exists a tower*

$$\begin{aligned} &P = P_r \xrightarrow{\pi_{r,r-1}} P_{r-1} \xrightarrow{\pi_{r-1,r-2}} \dots \xrightarrow{\pi_{2,1}} P_1 \xrightarrow{\pi_{1,0}} \mathbb{P}^{2n-1}, \\ &\pi = \pi_{1,0} \circ \dots \circ \pi_{r,r-1}, \end{aligned} \tag{7.3}$$

where P_i *is obtained from* P_{i-1} *by blowing up the strict transform of a coordinate linear space* $L_i \subset X_\Gamma$ *and such that*
(i) $\pi^*\eta_\Gamma$ *has no poles along the exceptional divisors associated to the blowups.*
(ii) Let $B \subset P$ *be the total transform in* P *of the union of coordinate hyperplanes* $\Delta^{2n-2} : A_1 A_2 \cdot A_{2n} = 0$ *in* $\mathbb{P}^{2n-1}$. *Then* B *is a normal crossings divisor in* P. *No face (= non-empty intersection of components) of* B *is contained in the strict transform* Y *of* X_Γ *in* P.
(iii) the strict transform of $\sigma^{2n-1}(\mathbb{R})$ *in* P *does not meet* Y.

Proof. Our algorithm to construct the blowups will be the following. Let S denote the set of coordinate linear spaces $L \subset \mathbb{P}^{2n-1}$ which are maximal, i.e. $L \in S, L \subset L' \subset X_\Gamma \Rightarrow L = L'$. Define

$$\mathcal{F} = \{L \subset X_\Gamma \text{ coordinate linear space} \mid L = \bigcap L^{(i)},\ L^{(i)} \in S\}. \tag{7.4}$$

Let $\mathcal{F}_{\min} \subset \mathcal{F}$ be the set of minimal elements in $\mathcal{F}$. Note that elements of $\mathcal{F}_{\min}$ are disjoint. Define $P_1 \xrightarrow{\pi_{1,0}} \mathbb{P}^{2n-1}$ to be the blowup of elements of $\mathcal{F}_{\min}$. Now define $\mathcal{F}_1$ to be the collection of strict transforms in P_1 of elements in $\mathcal{F} \setminus \mathcal{F}_{\min}$. Again elements in $\mathcal{F}_{1,\min}$ are disjoint, and we define P_2 by blowing up elements in $\mathcal{F}_{1,\min}$. Then $\mathcal{F}_2$ is the set of strict transforms in P_2 of $\mathcal{F}_1 \setminus \mathcal{F}_{1,\min}$, etc. This process clearly terminates.

Note that to pass from P_i to P_{i+1} we blow up strict transforms of coordinate linear spaces L contained in X_Γ. There will exist an open set $U \subset \mathbb{P}^{2n-1}$ such that $P_i \times_{\mathbb{P}^{2n-1}} U \cong U$ and such that $L \cap U \neq \emptyset$. It follows that to calculate the pole orders of $\pi^*\eta_\Gamma$ along exceptional divisors arising in the course of our algorithm it suffices to consider the simple blowup of a coordinate linear space $L \subset X_\Gamma$ on $\mathbb{P}^{2n-1}$. Suppose $L : A_1 = \dots A_p = 0$. By assumption, the subgraph $G = \{e_1, \dots, e_p\} \subset \Gamma$ is convergent, i.e. $p > 2h_1(G)$. As in Proposition 3.5, if $I = (A_1, \dots, A_p) \subset K[A_1, \dots, A_{2n}]$, then $\Psi_\Gamma \in I^{h_1(G)} - I^{h_1(G)+1}$ so the denominator of η_Γ contributes a pole of order $2h_1(G)$ along the exceptional divisor. On the other hand, writing $a_i = \frac{A_i}{A_{2n}}$, a typical open in the blowup will have coordinates $a_i' = \frac{a_i}{a_p}, i < p$ together with $a_p, \dots, a_{2n-1}$ and the exceptional divisor will be defined by $a_p = 0$. Thus

$$\begin{aligned} da_1 \wedge \dots \wedge da_{2n-1} &= d(a_p a_1') \wedge \dots \wedge d(a_p a_{p-1}') \wedge da_p \wedge \dots \\ &= a_p^{p-1} da_1' \wedge \dots \wedge da_{p-1}' \wedge da_p \dots . \end{aligned} \tag{7.5}$$

Finally, $\pi^*\eta$ will vanish to order $p - 1 - 2h_1(G) \geq 0$ on the exceptional divisor, so the algorithm will imply (i). Here we observe that at least on the strata for which p is even, $\pi^*\eta$ not only is regular along the exceptional divisor, but indeed really vanishes to order ≥ 1.

Recall the dictionary (3.1) between subgraphs $G = G(L) \subset \Gamma$ and coordinate linear spaces $L = L(G)$.

Lemma 7.4. *Let $\mathcal{F}$ be as above, and let $\emptyset \neq L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_r$ be a chain of faces in $\mathcal{F}$ which is saturated in the sense that it cannot be made longer using elements of $\mathcal{F}$. Let $G_r \subsetneq G_{r-1} \ldots \subsetneq G_1 \subsetneq G_0 := \Gamma$ be the chain of subgraphs. Then $h_1(G_j) = r+1-j$. In particular, $n = h_1(\Gamma) = r+1$. For $j \geq 1$ and any $e \in G_j \setminus G_{j+1}$ we have $h_1(G_j \setminus e) = h_1(G_j) - 1 = h_1(G_{j+1})$.*

Proof of Lemma 2. Let $G \subset \Gamma$ be a (not necessarily connected) subgraph. Consider the property

$$\forall e \in G,\ h_1(G \setminus e) < h_1(G). \tag{7.6}$$

I claim we can write $G = \bigcup G^{(i)}$, where the $G^{(i)}$ have the same minimality property and in addition $h_1(G^{(i)}) = 1$. We argue by induction on $h = h_1(G)$. If $h = 1$ we can just take G. If $h > 1$, then for every $e \in G$ we can find a $G_e \subset G$ such that $e \in G_e$, $h_1(G_e) = 1$, and G_e is minimal. Indeed, since $h_1(G \setminus e) < h_1(G)$, we can find a connected subgraph $G' \subset G$ such that $e \in G'$, $h_1(G') = 1$, and $h_1(G' \setminus e) = 0$. Now just remove $e' \neq e$ from G' until the resulting subgraph is minimal.

Since $e \in G_e$ we have $G = \bigcup G_e$ as desired. Applying our dictionary, $L(G) = \bigcap L(G_e)$. Note the $L(G_e) \subset X$ are maximal. We conclude that $L(G) \in \mathcal{F}$ for any $G \subset \Gamma$ satisfying (7.6). Conversely, if $G = \bigcup G^{(i)}$ with $L(G^{(i)})$ maximal in X, then every vertex in G lies on at least 2 edges (because this holds for the $G^{(i)}$). If for some $e \in G$ we had $h_1(G) = h_1(G \setminus e)$, we would then necessarily have that $G \setminus e$ was disconnected. If $e \in G^{(1)} \subset G$, then since $G^{(1)}$ has no external edges, it would follow that $G^{(1)} \setminus e$ was disconnected. This would imply $h_1(G^{(1)} \setminus e) = h_1(G^{(1)})$, a contradiction.

We conclude that $L \in \mathcal{F}$ iff $G(L)$ satisfies (7.6). The lemma now is purely graph-theoretic, concerning the existence of chains of subgraphs satisfying (7.6). Basically the condition is that the G_i have no external edges and are "1-particle irreducible" in the physicist's sense. (Note of course that we cannot assume this for $G_0 = \Gamma$, which is given.) To construct such a chain one simply takes $G_r \subset \Gamma$ minimal such that $h_1(G_r) = 1$ and G_{r-i} minimal such that $G_{r-i+1} \subset G_{r-i}$ and $h_1(G_{r-i+1}) > h_1(G_{r-i})$. Note the G_j are not necessarily connected. □

We now prove (ii). Let $\pi : P \to \mathbb{P}^{2n-1}$ be constructed as above, using the $\mathcal{F}_{i,\min}$. 0-faces of $B \subset P$ will be referred to as *vertices* (not to be confused with vertices of the graph). It will suffice to show that no vertex lies in the strict transform Y. Let $v \in P$ be a vertex. The question of whether $v \in Y$ is local around v, so we may localize our tower (7.3), replacing P_i with Spec $(\mathcal{O}_{P_i,v_i})$, where $v_i \in P_i$ is the image of v. In particular, $\mathbb{P}^{2n-1}$ is replaced by Spec $(\mathcal{O}_{\mathbb{P}^{2n-1},v_0})$, where $v_0 \in \mathbb{P}^{2n-1}$ is the image of v. Note the image v_i of v in P_i is always a vertex.

We modify the tower by throwing out the steps for which Spec $(\mathcal{O}_{P_i,v_i}) \to$ Spec $(\mathcal{O}_{P_{i-1},v_{i-1}})$ are isomorphisms. For convenience, we don't change notation. All our P_i are now local. Let $E_1, \ldots, E_r \subset P$ be the exceptional divisors, where E_i comes by pullback from P_i. Write $L_i := \pi(E_i) \subset P_0 :=$ Spec $(\mathcal{O}_{\mathbb{P}^{2n-1},v_0})$. We claim that $v_0 \in L_1$, and $L_1 \subsetneq L_2 \subsetneq \ldots \subsetneq L_r$ is precisely the sort of saturated chain in $\mathcal{F}$ considered in Lemma 7.4 above. Indeed, at each stage, v maps to the exceptional divisor from the stage before. (If v does not map to the exceptional divisor in P_i, then the local rings at the image of v in P_i and P_{i-1} are isomorphic, and this arrow is dropped under localization.)

Our task now will be to compute $Y \cap \bigcap_{i=1}^r E_i$. We will do this step by step. (We drop the assumption that our chain is saturated.) Suppose first $r = 1$, i.e. there is only one blowup. Let $L_1 \subset \mathbb{P}^{2n-1}$ be the linear space being blown and suppose L_1 has codimension p_1. Then by Proposition 3.5 if we write $G_1 = G(L_1) \subset \Gamma$ and $\Gamma//G_1$ for the quotient identifying each connected component of G to a point, we have $E_1 \cong L_1 \times \mathbb{P}^{p_1-1}$ and

$$Y_1 \cap E_1 = (X_{\Gamma//G_1} \times \mathbb{P}^{p_1-1}) \cup (L_1 \times X_{G_1}). \tag{7.7}$$

Now suppose we have $L_1 \subset L_2$ and we want to compute $Y_2 \cap E_1 \cap E_2 \subset P_2$. (We write abusively E_1 for the pullback to P_2 of E_1. $Y_i \subset P_i$ is the strict transform of X.). Locally at v_0 let $L_i : a_1 = \ldots = a_{p_i} = 0$ with $p_1 > p_2$. Let f be a local defining equation for X near v_0 and write

$$f = \sum c_{I,J}(a_1, \ldots, a_{p_2})^I (a_{p_2+1}, \ldots, a_{p_1})^J \tag{7.8}$$

with evident multi-index notation. Write $|I|, |J|$ for the total degree of a multi-index. We are interested in points of P_1 where the strict transform of L_2 meets E_1. Typical local coordinates at such points look like

$$a'_i := a_i/a_{p_1},\ 1 \le i < p_1,\ a'_{p_1} = a_{p_1}, \ldots (\text{ coords. not involving the a's}). \tag{7.9}$$

To compute the intersection of the strict transform with the two exceptional divisors on P_2, we let $\nu := \min(|I| + |J|)$ in (7.8), and write

$$f_1 = \sum (a'_{p_1})^{|I|+|J|-\nu} c_{I,J}(a'_1, \ldots, a'_{p_2})^I (a'_{p_2+1}, \ldots, a'_{p_1-1})^J. \tag{7.10}$$

This is the equation for $Y_1 \subset P_1$. We then take the image in the cone for the second blowup by taking the sum only over those terms with $|I| = |I|_{min}$ minimal:

$$\tilde{f}_1 = \sum_{\substack{I,J \\ |I|=|I|_{min}}} (a'_{p_1})^{|I|+|J|-\nu} c_{I,J}(a'_1, \ldots, a'_{p_2})^I (a'_{p_2+1}, \ldots, a'_{p_1-1})^J. \tag{7.11}$$

Notice that *a priori* a'_{p_1} might divide $\tilde{f}_1$. We claim in fact that it does not, i.e. that there exists I, J such that $c_{I,J} \neq 0$ and both $|I|$ and $|I| + |J|$ are minimum. To see this, note

$$|I|_{\min} = h_1(G_2); \quad \min(|I| + |J|) = h_1(G_1). \tag{7.12}$$

Assuming $L_1 \subset L_2$ is part of a saturated tower, we have as in Lemma 7.4 that $h_1(G_1) = h_1(G_2) + 1$. If no nonzero term in f has both $|I|$ and $|I| + |J|$ minimal, then every term with $|I| + |J|$ minimal must have $|I| = |I|_{\min} + 1$ and $|J| = 0$. But this would mean that the graph polynomial for G_1 would not involve the variables $A_{p_2+1}, \ldots, A_{p_1}$. Since the G_i have no external edges and $h_1(G_i \setminus e) < h_1(G_i)$, there are spanning trees (disjoint unions of spanning trees if G_i is not connected) avoiding any given edge, so this is a contradiction.

In general, if we have $L_1 \subset \ldots \subset L_r$ saturated we write

$$f = \sum_{I_1,\ldots,I_r} c_{I_q,\ldots,I_r}(a_1, \ldots, a_{p_r})^{I_r} (a_{p_r+1}, \ldots, a_{p_{r-1}})^{I_{r-1}} \cdots (a_{p_2+1}, \ldots, a_{p_1})^{I_1}. \tag{7.13}$$

We have

$$\begin{aligned} \min(|I_r|) = \min(|I_{r-1}| + |I_r|) - 1 = \\ \ldots = \min(|I_r| + \cdots + |I_1|) - r + 1. \end{aligned} \tag{7.14}$$

We claim there exist spanning trees T for G_1 such that T does not contain any $G_i \setminus G_{i+1}$. This will mean there exist $c_{I_q,\dots,I_r} \neq 0$ such that $\sum_1^r |I_j|$ is minimum but $|I_j| \neq 0$ for any j. By (7.14), this in turn implies for such a monomial that $\sum_{i=q}^r |I_i|$ is minimal for all q. To show the existence of T, choose $e_i \in G_i \setminus G_{i+1}$ for $1 \le i \le r-1$ and $e_r \in G_r$. It suffices to show that $h_0(G_1 \setminus \{e_1,\dots,e_r\}) = h_0(G_1)$. We have $h_0(G_1 \setminus e_1) = h_0(G_1)$ (since h_1 drops). Mayer Vietoris yields an exact sequence

$$\begin{aligned} \dots \to H_1(G_1 \setminus e_1) \to H_0(G_2 \setminus \{e_2,\dots,e_r\}) \to \\ H_0(G_2) \oplus H_0(G_1 \setminus \{e_1,\dots,e_r\}) \to H_0(G_1 \setminus e_1) \to 0. \end{aligned} \tag{7.15}$$

We have inductively $H_0(G_2 \setminus \{e_2,\dots,e_r\}) \cong H_0(G_2)$ and we deduce

$$H_0(G_1 \setminus \{e_1,\dots,e_r\}) \cong H_0(G_1 \setminus e_1) \cong H_0(G_1). \tag{7.16}$$

Let f be as in (7.13) and assume there exists $c_{I_q,\dots,I_r} \neq 0$ as above. We claim that $Y \cap E_1 \cap \dots \cap E_r$ can be computed as follows. For clarity, it is convenient to change notation a bit and write $D_i \subset P_i$ for the exceptional divisor. Abusively, E_i will denote any pullback of D_i to a P_j for $j > i$. Take the strict transform Y_1 to P_1 and intersect with D_1. Now take the strict transform $\mathrm{st}_{2,1}(Y_1 \cap D_1)$ of $Y_1 \cap D_1$ to P_2 and intersect with D_2. Continue in this fashion. The assertion is

$$Y \cap \bigcap_1^r E_i = E_r \cap st_{r,r-1}\Big(D_{r-1} \cap st_{r-1,r-2}(D_{r-2} \cap \dots st_{2,1}(D_1 \cap Y_1)\dots)\Big). \tag{7.17}$$

This is just an elaboration on (7.11), (7.13). The left hand-side amounts to taking the terms with $|I_1| + \dots + |I_r|$ minimal, removing appropriate powers of defining equations for the exceptional divisors, and then restricting; while the right-hand side takes those terms with $\sum_q^r |I_j|$ minimum for $q = 1,\dots,r-1$. By what we have seen, these yield the same answer.

It remains to see that the intersection (7.17) doesn't contain the vertex v. We have seen (7.7) that $D_1 \cap Y_1$ is a union of the pullbacks of graph hypersurfaces for G_1 and $\Gamma//G_1$. We have a cartesian diagram

$$\begin{array}{ccccccc} E_1 \cap D_2 \cong L_1 \times \mathbb{P}^{p_2-1} \times \mathbb{P}^{p_1-p_2-1} & \longrightarrow & P_2 & \longrightarrow & BL(\lambda_2 \subset \mathbb{P}^{p_1-1}) & \longrightarrow & \mathbb{P}^{p_2-1} \\ \downarrow & & \downarrow & & \downarrow & & \\ D_1 \cong L_1 \times \mathbb{P}^{p_1-1} & \longrightarrow & P_1 & \xrightarrow{\rho_1} & \mathbb{P}^{p_1-1} & & \end{array}, \tag{7.18}$$

where $\lambda_2 \cong \mathbb{P}^{p_2-1}$ corresponds to $L_2 \supset L_1$, and the strict transform in P_1 is the pullback $\widetilde{L}_2 = \rho_1^{-1}(\lambda_2)$. Of course the picture continues in this fashion all the way up. In the end, we get

$$L_1 \times \mathbb{P}^{p_r-1} \times \mathbb{P}^{p_{r-1}-p_r} \times \dots \times \mathbb{P}^{p_1-p_2-1}. \tag{7.19}$$

The strict transform of X here, by (7.17), is the union of pullbacks of graph hypersurfaces

$$pr_{L_1}^{-1} X_{\Gamma//G_1} \cup pr_r^{-1} X_{G_r} \cup pr_{r-1}^{-1} X_{G_{r-1}//G_r} \cup \dots \cup pr_1^{-1} X_{G_1//G_2}. \tag{7.20}$$

Now each of the graphs involved has $h_1 = 1$, so each of the graph hypersurfaces is linear. As we have seen, they involve all the edge variables so they do not vanish at any of the vertices. This completes the proof of Proposition 7.3(ii). Finally, the proof of (iii) is straightforward from (ii). One uses the existence of local coordinates as in (7.13) with respect to which the defining equation of the strict transform is a sum of monomials with coefficients > 0, and elements in the strict transform $\tilde{\sigma}$ of $\sigma^{2n-1}(\mathbb{R})$ have coordinates ≥ 0. (Points in $Y \cap \tilde{\sigma}$ could be specialized to vertices.) $\square$

We are now in a position to make explicit the motive (0.1) associated to a primitive divergent graph $\Gamma \subset \mathbb{P}^{2n-1}$. Let $P \xrightarrow{\pi} \mathbb{P}^{2n-1}$ be as in Proposition 7.3. Let $\Delta \subset \mathbb{P}^{2n-1}$ be the union of the $2n$ coordinate hyperplanes. Let $B := \pi^*\Delta$ and let $Y \subset P$ be the strict transform of the graph hypersurface $X = X_\Gamma$. Consider the motive (0.1):

$$H := H^{2n-1}(P \setminus Y, B \setminus B \cap Y). \tag{7.21}$$

By construction,

Proposition 7.5. *The divisor $B \subset P$ has normal crossings. The Hodge structure on the Betti realization H_B has the following properties:*
(i) H_B has weights in $[0, 4n-2]$. $W_0 H_B \cong \mathbb{Q}(0)$.
(ii) The strict transform $\tilde{\sigma}$ of the chain $\sigma^{2n-1}(\mathbb{R})$ in Proposition 7.3(iii) represents an homology class in $H_{2n-1}(P \setminus Y, B \setminus B \cap Y)$. The composition

$$W_0 H_B \hookrightarrow H_B \xrightarrow{\int_{\tilde{\sigma}}} \mathbb{Q}$$

is a vector space isomorphism.

Proof. We have the exact sequence

$$0 \to H^{2n-2}(B \setminus Y \cap B)/H^{2n-2}(P \setminus Y) \to H \to H^{2n-1}(P - Y). \tag{7.22}$$

Write $B = \bigcup B_i,\ B^{(r)} = \coprod B_{i_1} \cap \ldots \cap B_{i_r}$. We have a spectral sequence of Hodge structures

$$E_1^{p,q} = H^q(B^{(p+1)} \setminus B^{(p+1)} \cap Y) \Rightarrow H^{p+q}(B \setminus B \cap Y). \tag{7.23}$$

From known properties of weights for open smooth varieties, we get an exact sequence

$$H^0(B^{(2n-2)}) \to H^0(B^{(2n-1)}) \to W_0 H \to 0. \tag{7.24}$$

An analogous calculation with B replaced by $\Delta \subset \mathbb{P}^{2n-1}$ yields $\mathbb{Q}(0)$ as cokernel. It is easy to see that blowing up strict transforms of linear spaces doesn't change this cokernel. This proves (i). Assertion (ii) is straightforward. $\square$

An optimist might hope for a bit more. Whether for all primitive divergent graphs, or for an identifiable subset of them, one would like that the maximal weight piece of H_B should be Tate,

$$gr^W_{max} H_B = \mathbb{Q}(-p)^{\oplus r}. \tag{7.25}$$

Further one would like that there should be a rank 1 sub-Hodge structure $\iota : \mathbb{Q}(-p) \hookrightarrow gr^W_{max} H_B$ such that the image of $\eta_\Gamma \in H_{DR}$ in $gr^W_{max} H_{DR}$ spans $\iota(\mathbb{Q}(-p))_{DR}$. Our main result is that this is true for wheel and spoke graphs, (Sects. 11, 12).

8. The Motive II

In this section we consider the class of the graph hypersurface $[X_\Gamma]$ in the Grothendieck group K_{mot} of quasi-projective varieties over k with the relation $[X] = [Y] + [X \setminus Y]$ for Y closed in X. We assume Γ has N edges and n loops. The basic result of [3] is that $[X_\Gamma]$ can be quite general. In particular, the motive of X_Γ is not in general mixed Tate.

From the physicists' point of view, of course, one is primarily interested in the period (6.10). Results in [3] do not exclude the possibility of some mixed Tate submotive yielding this period. The methods of [3] seem to require graphs with physically unrealistic numbers of edges, so it is worth looking more closely at $[X_\Gamma]$. In this section we pursue a naive projection technique based on the fact that graph and related polynomials have degree ≤ 1 in each variable. We stratify X_Γ and examine whether the strata are mixed Tate. For $N = 2n \geq 12$, we identify a possible non-mixed Tate stratum.

Curiously, the stratum we consider turns out to be mixed Tate in "most" cases, but with a computer it is not difficult to generate cases where it may not be. We give such an example with 12 edges. Note however that Stembridge [13] has shown that all graphs with $\leq$ 12 edges are mixed Tate, so the particular example we give must in fact be mixed Tate. Techniques and results in this section should be compared with [13], which predates our work.

The basic observation of Kontsevich is that for X_Γ mixed Tate, there will exist a polynomial P_Γ with $\mathbb{Z}$-coefficients such that for any finite field $\mathbb{F}_q$ we have $\#X(\mathbb{F}_q) = P_\Gamma(q)$. Stembridge has implemented a computer algorithm for checking this. It might be of interest to try some of our examples to see if they satisfy Kontsevich's condition.

If we fix an edge e, by (3.10) we can write the graph polynomial

$$\Psi_\Gamma = A_e \cdot \Psi_{\Gamma\setminus e} + \Psi_{\Gamma/e}. \tag{8.1}$$

Projecting from the point v_e defined by $A_e(v_e) = 1, \;\; A_{e'}(v_e) = 0, e' \neq e$ yields $pr_e : \mathbb{P}^{N_1} \setminus \{v_e\} \to \mathbb{P}^{N-2}$ and

$$X_\Gamma \setminus pr_e^{-1}(X_{\Gamma\setminus e}) \cap X_\Gamma \xrightarrow{\cong} \mathbb{P}^{N-2} \setminus X_{\Gamma\setminus e}. \tag{8.2}$$

One might hope to stratify X_Γ and try to analyse its motive in this way. We know, however, by [3] that in general this motive is very rich, and such elementary techniques will not suffice to understand it. Indeed, we have

$$pr_e^{-1}(X_{\Gamma\setminus e}) \cap X_\Gamma = pr_e^{-1}(X_{\Gamma\setminus e} \cap X_{\Gamma/e}), \tag{8.3}$$

so already at the second step we must analyse an intersection of two graph hypersurfaces. What is amusing is that, in fact, one can continue a bit further, and the process gives some indication of where motivic complications might first arise.

Lemma 8.1. *Assume Γ has n loops and $2n$ edges. Enumerate the edge variables $A_1, \ldots,$ A_{2n} in such a way that $A_1 A_2 \cdots A_n$ appears with coefficient 1 in Ψ_Γ. Then we can write*

$$\Psi_\Gamma = \det(m_{ij} + \delta_{ij} A_i)_{1\leq i,j\leq n}; \quad m_{ij} = m_{ij}(A_{n+1}, \ldots, A_{2n}). \tag{8.4}$$

In other words, the first n variables appear only on the diagonal.

Proof. Let $T \subset \Gamma$ be the subgraph with edges $e_{n+1}, \ldots, e_{2n}$. Our assumption implies that T is a spanning tree, so $\mathbb{Z}[E_\Gamma] \cong H_1(\Gamma) \oplus \mathbb{Z}e_{n+1} \oplus \ldots \oplus \mathbb{Z}e_{2n}$. The linear functionals $e_i^\vee$ thus induce an isomorphism

$$(e_1^\vee, \ldots, e_n^\vee) : H_1(\Gamma) \cong \mathbb{Z}^n. \tag{8.5}$$

With respect to this basis of $H_1(\Gamma)$ the rank 1 quadratic forms $(e_i^\vee)^2$ correspond to the matrices with 1 in position (i,i) and zeroes elsewhere, for $1 \le i \le n$. Define (m_{ij}) to be the symmetric matrix associated to the quadratic form $\sum_{n+1}^{2n} A_i (e_i^\vee)^2$. The assertion of the lemma is now clear. $\square$

Lemma 8.2 ([9]). *Let $\psi = \det(m_{ij} + \delta_{ij} A_i)_{1\le i,j\le n}$, where the m_{ij} are independent of $A_1, \ldots, A_n$. For $1 \le k \le n$ write $\psi^k := \frac{\partial}{\partial A_k}\psi$ and $\psi_k := \psi|_{A_k=0}$. For $I, J \subset \{1, \ldots, n\}$ with $\#I = \#J$, define $\psi(I,J)$ to be the determinant as above with the rows in I and the columns in J removed. Let $1 \le k, \ell \le n$ be distinct integers and assume $k, \ell \notin I \cup J$. Then*

$$\begin{aligned}\psi(I,J)^{k\ell}\psi(I,J)_{kl} - \psi(I,J)_k^\ell \psi(I,J)_l^k &= \pm\psi(I \cup \{k\}, J \cup \{\ell\}) \\ &\quad \times \psi(I \cup \{\ell\}, J \cup \{k\}).\end{aligned} \tag{8.6}$$

The two factors on the right have degrees ≤ 1 in A_i for $i \le n$.

Proof. We can drop the rows in I and the columns in J to begin with and ignore the A_ν for $\nu \notin \{k, \ell\}$. In this way, we reduce to the following assertion. Let M be an $n \times n$ matrix with coefficients in a commutative ring. Assume $n \ge 2$. Write $M(S,T)$ for the matrix with rows in S and columns in T deleted. Then

$$\begin{aligned}&\det M(\{1,2\},\{1,2\}) \cdot \det M - \det M(\{1\},\{1\}) \cdot \det M(\{2\},\{2\}) \\ &\quad = -\det M(\{1\},\{2\}) \cdot \det M(\{2\},\{1\})\end{aligned} \tag{8.7}$$

(By convention, the determinant of a 0×0-matrix is 1.) This is a straightforward exercise.

We attempt to stratify our graph hypersurface X_Γ using the above lemmas. To fix ideas, we assume Γ has $2n$ edges and n loops.

Step 1. We order the edges so Ψ_Γ admits a description as in Lemma 8.1.

Step 2. Project as in (8.2) with $e = e_1$, to conclude

$$\begin{aligned}[X_\Gamma] &= [\mathbb{P}^{2n-2}] + [\mathrm{Cone}(X_{\Gamma\backslash e_1} \cap X_{\Gamma/e_1})] - [X_{\Gamma\backslash e_1} \cap X_{\Gamma/e_1}] \\ &= [\mathbb{P}^{2n-2}] + 1 + ([\mathbb{A}^1] - 1)[X_{\Gamma\backslash e_1} \cap X_{\Gamma/e_1}].\end{aligned} \tag{8.8}$$

Step 3. Using (3.10), we can write (with notation as in Lemma 8.2 and $\Psi = \Psi_\Gamma$)

$$\begin{aligned}\Psi_{\Gamma\backslash e_1} &= \frac{\partial}{\partial A_1}\Psi_\Gamma = A_2 \Psi_{\Gamma\backslash\{e_1,e_2\}} + \Psi_{(\Gamma\backslash e_1)/e_2} = A_2\Psi^{12} + \Psi_2^1, \\ \Psi_{\Gamma/e_1} &= \Psi_\Gamma|_{A_1=0} = A_2 \Psi_{(\Gamma/e_1)\backslash e_2} + \Psi_{\Gamma/\{e_1,e_2\}} = A_2\Psi_1^2 + \Psi_{12}.\end{aligned} \tag{8.9}$$

Eliminating A_2, we conclude that projection from $\mathbb{P}^{2n-2}$ onto $\mathbb{P}^{2n-3}$ with coordinates $A_3, \ldots, A_{2n}$ carries $X_{\Gamma-e_1} \cap X_{\Gamma/e_1}$ onto the hypersurface defined by $\Psi_2^1\Psi_1^2 - \Psi^{12}\Psi_{12} = 0$. By Lemma 8.2,

$$\Psi_2^1\Psi_1^2 - \Psi^{12}\Psi_{12} = \Psi(1,2)\Psi(2,1) = \Psi(1,2)^2. \tag{8.10}$$

(The right-hand identity holds because $\Psi = \Psi_\Gamma$ is the determinant of a symmetric matrix.)

Step 4. Write $\mathcal{V}(I)$ for the locus of zeroes of a homogeneous ideal I. The projection in Step 3 blows up on $\mathcal{V}(\Psi_2^1, \Psi_1^2, \Psi^{12}, \Psi_{12})$, and we conclude

$$\begin{aligned}
&[X_{\Gamma\backslash e_1} \cap X_{\Gamma/e_1}] \\
&\quad = [X(1,2)] + [\text{Cone } \mathcal{V}(\Psi_2^1, \Psi_1^2, \Psi^{12}, \Psi_{12})] - [\mathcal{V}(\Psi_2^1, \Psi_1^2, \Psi^{12}, \Psi_{12})] \\
&\quad = [X(1,2)] + 1 + ([\mathbb{A}^1] - 1)[\mathcal{V}(\Psi_2^1, \Psi_1^2, \Psi^{12}, \Psi_{12})]. \qquad (8.11)
\end{aligned}$$

Step 5. One could try to study the motive of $\mathcal{V}(\Psi_2^1, \Psi_1^2, \Psi^{12}, \Psi_{12})$, but the elimination theory gets complicated, so instead we focus on $[X(1,2)]$. Since $\Psi(1,2)$ has degree ≤ 1 in A_3 we may project onto $\mathbb{P}^{2n-4}$ with coordinates $A_4, \ldots, A_{2n}$. It might seem that we could repeat the argument starting from Step 2 above, but there is a problem. Writing $\Psi = \det M$ with M symmetric, we have $\Psi(1,2) = \det M(1,2)$, where $M(1,2)$ is obtained from M by deleting the first row and the second column. This matrix is no longer symmetric. Just as in (8.2), the projection $X(1,2) \to \mathbb{P}^{2n-4}$ blows up over $\mathcal{V}(\Psi(1,2)^3, \Psi(1,2)_3)$.

Step 6. Just as in Step 3, we project $\mathcal{V}(\Psi(1,2)^3, \Psi(1,2)_3)$ to $\mathbb{P}^{2n-5}$ with coordinates $A_5, \ldots, A_{2n}$. When we eliminate A_4 we find the image of the projection is given by the zeroes of

$$\begin{aligned}
&\Psi(1,2)^{34}\Psi(1,2)_{34} - \Psi(1,2)_4^3\Psi(1,2)_3^4 \qquad (8.12)\\
&\overset{\text{Lemma 8.2}}{=} \Psi(\{1,3\},\{2,4\}) \cdot \Psi(\{1,4\},\{2,3\}).
\end{aligned}$$

Step 7. At this point something new has happened. The right-hand side in (8.12) is not a square. Although both factors have degree ≤ 1 in A_5, we will at the next stage in our motivic stratification have to deal with

$$\mathcal{V}(\Psi(\{1,3\},\{2,4\}), \Psi(\{1,4\},\{2,3\})). \qquad (8.13)$$

Here Lemma 8.2 no longer applies. We find by example that eliminating A_5, the resulting hypersurface in $\mathbb{P}^{2n-6}$ in general no longer factors into factors with degrees ≤ 1 in A_6. Projection then is no longer an isomorphism at the generic point, and the argument is blocked.

Example 8.3. The computer yields the following example of a graph with 6 loops and 12 edges for which the projection (8.13) has an irreducible factor with degree 2 in A_6. Take 7 vertices labeled $1, 2, \ldots, 7$ and connect them with edges as indicated:

$$\begin{aligned}
&(1,2), (2,3), (3,4), (4,5), (5,6), (6,7), (7,2), (7,3), \\
&\qquad (6,4), (5,1), (5,3), (4,1). \qquad (8.14)
\end{aligned}$$

Note that this graph is mixed Tate though by explicit computation, which finds it $\sim \zeta(3)\zeta(5)$.

9. General Remarks

Let Γ be a graph with n loops and $2n$ edges. We assume all subgraphs of Γ are convergent so the period $P(\Gamma)$ is defined (Proposition 5.2). The Schwinger trick (Corollary 6.3) relates $P(\Gamma)$ to an integral computed in Schwinger coordinates in $\mathbb{P}^{2n-1}$. To avoid confusion, we write $P_{\text{quadric}}(\Gamma)$ for the period (5.7) of the configuration of Feynman quadrics associated to Γ and $P_{\text{graph}}(\Gamma)$ for the graph period. We have by (6.10),

$$P_{\text{quadric}}(\Gamma) \in \mathbb{Q}^{\times}\pi^{-2n} P_{\text{graph}}(\Gamma). \tag{9.1}$$

Proposition 7.3 shows that there is a suitable birational transformation $\pi : P \to \mathbb{P}^{2n-1}$ defined over $\mathbb{Q}$, such that the integrand $\eta \in \Gamma(\mathbb{P}^{2n-1}, \omega(2X))$ keeps poles only along the strict transform Y of the discriminant hypersurface X, that is $\pi^*(\eta) \in \Gamma(P, \omega(2Y))$. Thus, denoting by B the total transform of the union Δ of coordinate hyperplanes $A_i = 0$, the form η yields a class

$$\pi^*\eta \in \Gamma(P, \omega(2Y)) \to H^{2n-1}_{DR}(P \setminus Y, B \setminus B \cap Y) \tag{9.2}$$

in relative de Rham cohomology. On the other hand, Proposition 7.3 shows that the strict transform $\tilde{\sigma}^{2n-1}(\mathbb{R})$ of the cycle of integration is disjoint from Y. Thus it yields a relative homology class

$$\tilde{\sigma}^{2n-1}(\mathbb{R}) \in H_{2n-1}(P \setminus Y, B \setminus B \cap Y) = H^{2n-1}_{\text{Betti}}(P \setminus Y, B \setminus B \cap Y)^{\vee} \tag{9.3}$$

in Betti cohomology. More precisely

Claim 9.1. *The period integral* (5.3) $P_{\text{quadric}}(\Gamma) \in \pi^{-2n}\mathbb{Q}^{\times} \cdot P_{\text{graph}}(\Gamma)$*, where* $P_{\text{graph}}(\Gamma)$ *is a period of the cohomology* $H^{2n-1}(P \setminus Y, B \setminus B \cap Y)$*. By period here we mean the integral of an algebraic de Rham form* $\pi^*\eta$ *defined over* $\mathbb{Q}$ *against a* $\mathbb{Q}$*-homology chain* $\tilde{\sigma}^{2n-1}$.

Suppose now, as has been established in a number of cases [5], that the period is related to a zeta value: $P_{\text{quadric}}(\Gamma) \in \pi^{\mathbb{Z}}\mathbb{Q}^{\times}\zeta(p)$. Then the general guideline for what we wish to understand is the following.

One has now a good candidate for a triangulated category of mixed motives over $\mathbb{Q}$, defined by Voevodsky, Levine and Hanamura ([6], Sect. 1 and references there for the discussion here). One further considers the triangulated subcategory spanned by $\mathbb{Q}(n)$, $n \in \mathbb{Z}$. In this category, one has

$$\mathrm{Hom}^j(\mathbb{Q}(0), \mathbb{Q}(p)) = \begin{cases} \mathbb{Q} & p = j = 0 \\ K_{2p-1}(\mathbb{Q}) \otimes \mathbb{Q} & p \geq 1,\ j = 1 \\ 0 & \text{else} \end{cases}. \tag{9.4}$$

The iterated extensions of $\mathbb{Q}(n)$ form an abelian subcategory which is the heart of a t-structure.

Borel's work on the K-theory of number fields [2, 14] tells us that $K_{2p-1}(\mathbb{Q}) \otimes \mathbb{Q} \cong \mathbb{Q}$ for $p = 2n - 3$, $n \geq 2$, so there is a one dimensional space of motivic extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(p)$. We want to understand their periods. Let E be a nontrivial such extension. We write $E_{DR} = \mathbb{Q} \cdot e_0 \oplus \mathbb{Q} \cdot e_p$, with $F^0 E_{DR} = \mathbb{Q}e_0$. The Betti realization is $E_{\mathbb{C}} = \mathbb{C} \cdot e_0 \oplus \mathbb{C} \cdot e_p$ and $E_{\mathbb{Q}} = \mathbb{Q} \cdot (2\pi i)^p e_p \oplus \mathbb{Q} \cdot (e_0 + \beta e_p)$ for a suitable β. The corresponding Hodge structures on the $\mathbb{Q}(i)$ are

$$(\mathbb{Q}(0)_{DR} = \mathbb{Q} \cdot \epsilon_0, \mathbb{Q}(0)_{\mathbb{Q}} = \mathbb{Q} \cdot \epsilon_0), \quad (\mathbb{Q}(p)_{DR} = \mathbb{Q} \cdot \epsilon_p, \mathbb{Q}(p)_{\mathbb{Q}} = \mathbb{Q} \cdot (2\pi i)^p \epsilon_p). \tag{9.5}$$

We have an exact sequence

$$0 \to \mathbb{Q}(p) \to E \to \mathbb{Q}(0) \to 0 \tag{9.6}$$

given by $\epsilon_p \mapsto e_p,\ e_0 \mapsto \epsilon_0$. The ambiguity here is that we can replace $e_0 + \beta e_p$ by $e_0 + (\beta + c(2\pi i)^p)e_p$ for $c \in \mathbb{Q}$ as a basis element for $E_{\mathbb{Q}}$, so $\beta \in \mathbb{C}/(2\pi i)^p\mathbb{Q}$ is well defined. In fact, $\mathrm{Ext}^1_{MHS}(\mathbb{Q}(0), \mathbb{Q}(p)) = \mathbb{C}/(2\pi i)^p\mathbb{Q}$ and β is the class of E.

To compute the period, consider the dual object $E^\vee$, with $E^\vee_{DR} = \mathbb{Q}e_0^\vee \oplus \mathbb{Q}e_p^\vee$ and $E^\vee_{\mathbb{Q}} = \mathbb{Q}e_0^\vee \oplus \mathbb{Q}(2\pi i)^{-p}(e_p^\vee - \beta e_0^\vee)$. By definition, the period is obtained by pairing $F^0 E_{DR}$ against a lifting in $E^\vee_{\mathbb{Q}}$ of the generator $(2\pi i)^p e_p^\vee \in \mathbb{Q}(-p)_{\mathbb{Q}} = E^\vee_{\mathbb{Q}}/\mathbb{Q}(0)_{\mathbb{Q}}$. This yields

$$\langle e_0, (2\pi i)^{-p}(e_p^\vee - \beta e_0^\vee)\rangle = -(2\pi i)^{-p}\beta. \tag{9.7}$$

It is better from the period viewpoint to dualize and consider the period of $E^\vee$, which is an extension of $\mathbb{Q}(-p)$ by $\mathbb{Q}(0)$. This yields

$$\langle e_p^\vee, e_0 + \beta e_p\rangle = \beta. \tag{9.8}$$

For E a non-split motivic extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(p)$, p odd, ≥ 3, let $\beta \in \mathbb{C}/(2\pi i)^p\mathbb{Q}$ be the extension class. Note $\mathrm{Im}(\beta) \in \mathbb{R}$ is well defined. One knows by the Borel regulator theory [2, 14] that $\zeta(p) \in \mathrm{Im}(\beta)\mathbb{Q}^\times$.

Now consider our graph Γ with period related to $\zeta(p)$. The motive $H^{2n-1}(P \setminus Y, B \setminus B \cap Y)$ has lowest weight piece $\mathbb{Q}(0)$, so we might expect to find inside it a subquotient motive of rank 2 which is an extension of $\mathbb{Q}(-p)$ by $\mathbb{Q}(0)$. By the above discussion, we would then hope

$$P_{\mathrm{graph}}(\Gamma) \in \zeta(p)\mathbb{Q}^\times. \tag{9.9}$$

By (6.10) this would yield $P_{\mathrm{quadric}}(\Gamma) \in \pi^{-2n}\zeta(p)\mathbb{Q}^\times$. For example, take $\Gamma = \Gamma_n$ to be the wheel with n spokes. Then $p = 2n - 3$ and we expect, if indeed the ζ-values computed in [5] are motivic, to find

$$P_{\mathrm{graph}}(\Gamma_n) \in \zeta(2n-3)\mathbb{Q}^\times; \quad P_{\mathrm{quadric}}(\Gamma_n) \in \pi^{-2n}\zeta(2n-3)\mathbb{Q}^\times. \tag{9.10}$$

The aim of the next sections is to show for the wheel and spoke family of examples what can be done motivically. We will show in particular

$$H^{2n-1}(\mathbb{P}^{2n-1} \setminus X) = \mathbb{Q}(-2n+3). \tag{9.11}$$

Moreover, $H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus X)$ is spanned by η. Even in this special case, we are not able to find a suitable rank 2 subquotient motive of $H^{2n-1}(P \setminus Y, B \setminus B \cap Y)$.

10. Correspondences

We will assume in this section that Γ has n loops and $2n$ edges. So one has $2n$ Feynman quadrics which we denote by q_e, of Eq. Q_e, see Sect. 5. Recall concretely that to an edge e, one associates coordinates $x_e(i), i = 1, \ldots, 4 = j$. Given an orientation of Γ, to a vertex v, one associates the relation $\sum_e \mathrm{sign}(v, e)x_e(i) = 0$ for all $i = 1, \ldots, j = 4$. Then $q_e =: q_e^j$ is defined by $Q_e^j := \sum_{a=1}^{4=j} x_e(a)^2 = 0$ in $\mathbb{P}^{jn-1}$. One defines $\mathcal{Q} = \mathcal{Q}_j \subset \mathbb{P}^{jn-1} \times \mathbb{P}^{2n-1}$ by the equation $\sum_e A_e Q_e^j = 0$. This defines a correspondence

$$\begin{array}{ccc} \mathbb{P}^{2n-1} \times \mathbb{P}^{jn-1} \setminus \mathcal{Q}_j & \xrightarrow{\mathbb{A}^{2n-1}-\text{fibration}} & \mathbb{P}^{jn-1} \setminus \cap_{e=1}^{2n} q_e^j \\ {\scriptstyle \pi_j}\downarrow & & \\ \mathbb{P}^{2n-1} & & \end{array} . \tag{10.1}$$

We discuss now this correspondence for the Feynman quadrics, i.e. $j = 4$. On the other hand, we can consider all the definitions above for other j, and we discuss the resulting correspondence (10.1) for $j = 1$ and $j = 2$ as well.

For $j = 1$, we rather consider the projection $\mathrm{proj} : \mathcal{Q}_1 \to \mathbb{P}^{2n-1}$. Let us denote by $\Sigma \subset \mathbb{Q}_1$ the closed subscheme with $\mathrm{proj}^{-1}(x) \cap \Sigma = \mathrm{Sing}(\mathrm{proj}^{-1}(x))$. Then $\Sigma \to X$ is the desingularization $\mathbb{P}(N) \to X$ studied in Proposition 4.2.

We assume now $j = 2$. Recall that if $Z \subset \mathbb{P}^{2N+1}$ is a smooth even dimensional quadric, then

$$H_c^j(\mathbb{P}^{2N+1} \setminus Z) = \begin{cases} 0 & j \neq 2N \\ \mathbb{Q}(-N)[\ell_1 - \ell_2] & j = 2N \end{cases}, \tag{10.2}$$

where ℓ_i are the 2 rulings of Z. We define

$$X_i = \left\{(A) \in \mathbb{P}^{2n-1}, \mathrm{rk}\left(\sum_e A_e Q_e^1\right) < n - i\right\}. \tag{10.3}$$

So $X = X_0$, and X_{i+1} is the singular locus of X_i. We denote by $j = j_0 : \mathbb{P}^{2n-1} \setminus X \to \mathbb{P}^{2n-1}$, $j_i : \overline{X_{i-1} \setminus X_i \to X_{i-1}}$. Over X_i, the quadric $\sum_e A_e q_e^j$ is a cone over a smooth quadric $\sum_e A_e q_e^j \subset \mathbb{P}^{j(n-i)-1}$, thus by homotopy invariance and base change for $R(\pi_j)_!$ ([7]), one obtains

Proposition 10.1.

$$R^i(\pi_4)_!\mathbb{Q} = \begin{cases} j_!\mathbb{Q}(-2n+1) & i = 4n-1 \\ (j_1)_!\mathbb{Q}(-2n-1) & i = 4n+3 \\ \cdots & \cdots \\ (j_a)_!\mathbb{Q}(-2n+1-2a) & i = 4n+4a \end{cases}, \tag{10.4}$$

$$R^i(\pi_2)_!\mathbb{Q} = \begin{cases} j_!\mathbb{Q}(-n+1) & i = 2n-1 \\ (j_1)_!\mathbb{Q}(-n) & i = 4n+1 \\ \cdots & \cdots \\ (j_a)_!\mathbb{Q}(-2n+1-a) & i = 2n+2a \end{cases}. \tag{10.5}$$

We draw now two consequences from this computation.

Proposition 10.2. *One has maps*

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X) \to H_c^{2n}(\mathbb{P}^{4n-1} \setminus \cap_{e=1}^{2n} q_e^4) \to \quad H_c^{4n-1}(\mathbb{P}^{4n-1} \setminus \cup_{e=1}^{2n} q_e^4)$$

in particular dually

$$H^{4n-1}(\mathbb{P}^{4n-1} \setminus \cup_{e=1}^{2n} q_e^4)(2n) \to H^{2n-1}(\mathbb{P}^{2n-1} \setminus X). \tag{10.6}$$

Proof. By (10.4), the term $E_2^{2n-1,4n-1} = H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X)(-2n+1)$ of the Leray spectral sequence for π_4 maps to $H_c^{2n-1+4n-1}(\mathbb{P}^{2n-1} \times \mathbb{P}^{4n-1} \setminus \mathcal{Q}_4)$, which in turn is equal to $H_c^{2n}(\mathbb{P}^{4n-1} \setminus \cap_{e=1}^{2n} q_e^4)(-2n+1)$ by homotopy invariance. The second map comes from the Mayer-Vietoris spectral sequence for $\cup_{e=1}^{2n} q_e^4$. □

Remark 10.3. We will see in Sect. 11 on the wheel with n spokes that for $n = 3$, the first map is an isomorphism, but in general, we do not control it.

Proposition 10.4. *Assume $\cap_{e=1}^{2n} q_e^2 \neq \emptyset$, for example for the wheel with n spokes (see Sect. 11). Then*

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X) = H^{2n-2}(X)/H^{2n-2}(\mathbb{P}^{2n-1})$$

is supported along X_a for some $a \geq 1$.

Proof. By homotopy invariance again and by assumption, we have

$$H_c^{2n-1+2n-1}(\mathbb{P}^{2n-1} \times \mathbb{P}^{2n-1} \setminus \mathcal{Q}_2) = H_c^0(\mathbb{P}^{2n-1} \setminus \cap_{e=1}^{2n} q_e^2) = 0. \tag{10.7}$$

So the Leray spectral sequence for π_2 together with (10.4) imply that $E_\infty^{2n-1,2n-1} = 0$, with $E_2^{2n-1,2n-1} = H_c^{2n-1}(\mathbb{P}^{2n-1} \setminus X)(-n+1)$. So, since $R^i(\pi_2)_!$ is supported in lower strata of X, this shows the proposition. □

Remark 10.5. We will see in Sect. 11 on the wheel with n spokes that for $n = 3$, the Leray spectral sequence will equate $H^0(X_1)(-1) \xrightarrow{\cong} H^4(X)/H^4(\mathbb{P}^5)$.

11. Wheel and Spokes

The purpose of this section is to compute the middle dimensional cohomology for a graph polynomial in a non-trivial case. The geometry we will be using involves only projections, homotopy invariance and Artin vanishing theorem. Consequently, our cohomology computation holds for Betti or étale cohomology, and would for motivic cohomology if one had Artin vanishing. To unify notations, we denote this cohomology as $H(?, \mathbb{Q})$ rather than $\mathbb{Q}_\ell$ in the ℓ-adic case.

Fix $n \geq 3$ and let $\Gamma = WS_n$ be the graph which is a wheel with n spokes. WS_n has vertices $\{0, 1, \ldots, n\}$ and edges $e_i = (0, i)$, $1 \leq i \leq n$ and $e_j = (j-n, j-n+1 \mod n)$, $n+1 \leq j \leq 2n$. Suitably oriented, $\ell_i = e_i + e_{i+n} - e_{i+1 \mod n}$, $1 \leq i \leq n$ form a basis for the loops. The following is straightforward.

Lemma 11.1. *Γ has n loops and $2n$ edges. Every proper subgraph $\Gamma' \subsetneq \Gamma$ is convergent so the period $P(\Gamma)$ is defined (see Proposition 5.2).*

Proof. Omitted. □

Let T_i, $1 \le i \le 2n$ be variables. The graph polynomial of Γ can be written

$$\Psi_\Gamma(T) = \det\left(\sum_{i=1}^{2n} T_i M^{(i)}\right), \tag{11.1}$$

where

$$M^{(i)} = (M^{(i)}_{pq})_{1\le p,q\le n}; \quad M^{(i)}_{pq} = e_i^\vee(\ell_p)e_i^\vee(\ell_q). \tag{11.2}$$

It follows easily that

$$\Psi_\Gamma = \det\begin{pmatrix} T_1+T_2+T_{n+1} & -T_2 & 0 & \dots & 0 & -T_1 \\ -T_2 & T_2+T_3+T_{n+2} & -T_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -T_1 & 0 & 0 & \dots & -T_n & T_n+T_1+T_{2n} \end{pmatrix}. \tag{11.3}$$

It will be convenient to make the change of variables

$$B_i = T_{i+1} + T_{i+2} + T_{i+1+n}, \ A_i = -T_{i-2}, \tag{11.4}$$

where all the indices are counted modulo n and taken in $[0, \dots, n]$. Write

$$\Psi_n = \Psi_n(A,B) = \det\begin{pmatrix} B_0 & A_0 & 0 & \dots & \dots & A_{n-1} \\ A_0 & B_1 & A_1 & \dots & \dots & 0 \\ 0 & A_1 & B_2 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ A_{n-1} & 0 & \dots & \dots & A_{n-2} & B_{n-1} \end{pmatrix}. \tag{11.5}$$

The graph hypersurface in the A, B-coordinates is given by

$$\mathbb{P}^{2n-1} \supset X_n : \Psi_n(A,B) = 0. \tag{11.6}$$

Define $H^*(X_n,\mathbb{Q})_{\mathrm{prim}} := \mathrm{coker}(H^*(\mathbb{P}^{2n-1},\mathbb{Q}) \to H^*(X_n,\mathbb{Q}))$. We formulate now our main theorem.

Theorem 11.2. *Let $X_n \subset \mathbb{P}^{2n-1}$ be the graph polynomial hypersurface for the wheel with $n \ge 3$ spokes. Then one has*

$$H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_n) \cong \mathbb{Q}(-2n+3)$$

or equivalently, via duality

$$H^{2n-2}(X_n,\mathbb{Q})_{\mathrm{prim}} \cong \mathbb{Q}(-2).$$

In particular, $H^{2n-1}(X_n,\mathbb{Q})_{\mathrm{prim}}$ is independent of $n \ge 3$.

Proof. The proof is quite long and involves several geometric steps. We first define homogeneous polynomials Q_{n-1} and K_n as indicated:

$$\begin{aligned}\Psi_n = & B_0 Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}) \\ & + K_n(B_1, \ldots, B_{n-1}, A_0, \ldots, A_{n-1}).\end{aligned} \tag{11.7}$$

Here

$$\begin{aligned}& Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}) \\ & = \det \begin{pmatrix} B_1 & A_1 & 0 & \ldots & \ldots & 0 \\ A_1 & B_2 & A_2 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & \ldots & \ldots & \ldots & A_{n-2} & B_{n-1} \end{pmatrix}.\end{aligned} \tag{11.8}$$

Lemma 11.3. *One has inductive formulae:*

$$\begin{aligned}Q_{n-1} & (B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}) \\ & = B_1 Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}) \\ & \quad - A_1^2 Q_{n-3}(B_3, \ldots, B_{n-1}, A_3, \ldots, A_{n-2}) \\ & = B_{n-1} Q_{n-2}(B_1, \ldots, B_{n-2}, A_1, \ldots, A_{n-3}) \\ & \quad - A_{n-2}^2 Q_{n-3}(B_1, \ldots, B_{n-3}, A_1, \ldots, A_{n-4});\end{aligned} \tag{11.9}$$

and

$$\begin{aligned}K_n & (B_1, \ldots, B_{n-1}, A_0, \ldots, A_{n-1}) \\ & = -A_0^2 Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}) \\ & \quad - A_{n-1}^2 Q_{n-2}(B_1, \ldots, B_{n-2}, A_1, \ldots, A_{n-3}) + 2(-1)^{n-1} A_0 \cdots A_{n-1}.\end{aligned} \tag{11.10}$$

Proof. Straightforward. □

The following lemma is a direct application of Artin's vanishing theorem [1], Théorème 3.1, and homotopy invariance, and will be the key ingredient to the computation.

Lemma 11.4. *Let $V \subset \mathbb{P}^N$ be a hypersurface which is a cone over the hypersurface $W \subset \mathbb{P}^a$. Then one has*

$$H^i(\mathbb{P}^N \setminus V) = 0 \text{ for } i > a$$

or equivalently

$$H_c^j(\mathbb{P}^N \setminus V) = 0 \text{ for } j < 2N - a.$$

Proof. The projection $\mathbb{P}^N \setminus V \to \mathbb{P}^a \setminus W$ is a $\mathbb{A}^{N-a}$-fibration. By homotopy invariance, $H_c^j(\mathbb{P}^N \setminus V) = H_c^{j-2(N-a)}(\mathbb{P}^a \setminus W)(-(N-a))$ and by Artin's vanishing $H_c^{j-2(N-a)}(\mathbb{P}^a \setminus W) = 0$ for $j - 2(N-a) < a$, i.e. for $j < 2N - a$. □

For a homogeneous ideal I or a finite set $F_1, F_2, \ldots$ of homogeneous polynomials, we write $\mathcal{V}(I)$ or $\mathcal{V}(F_1, F_2, \ldots)$ for the corresponding projective scheme. We will need to pass back and forth via various projections. In confusing situations we will try to specify the ambiant projective space. A superscript (i) will mean the ambient projective space is $\mathbb{P}^i$. In the following lemma, $\mathbb{P}^{2n-1}$ has coordinates $(B_0 : \ldots : B_{n-1} : A_0 : \ldots : A_{n-1})$ and $\mathbb{P}^{2n-2}$ drops the B_0.

Lemma 11.5. *We have*

$$H^{2n-2}(X_n, \mathbb{Q}) \cong H^{2n-4}\Big(\mathcal{V}(Q_{n-1}, K_n)^{(2n-2)}, \mathbb{Q}(-1)\Big). \tag{11.11}$$

Proof. By (11.7), one has

$$X_n \cap \mathcal{V}(Q_{n-1}) = \mathcal{V}(Q_{n-1}, K_n)^{(2n-1)}. \tag{11.12}$$

Let $p = (1, 0, \ldots, 0) \in \mathbb{P}^{2n-1}$. Projection from p gives an isomorphism (use (11.7) to solve for B_0)

$$\pi_p : X_n \setminus X_n \cap \mathcal{V}(Q_{n-1}) \cong \mathbb{P}^{2n-2} \setminus \mathcal{V}(Q_{n-1}). \tag{11.13}$$

We get a long exact sequence

$$\begin{aligned} &H_c^{2n-2}\,(\mathbb{P}^{2n-2} \setminus \mathcal{V}(Q_{n-1})) \to H^{2n-2}(X_n) \to \\ &\quad H^{2n-2}(\mathcal{V}(K_n, Q_{n-1})^{(2n-1)}) \to H_c^{2n-1}(\mathbb{P}^{2n-2} \setminus \mathcal{V}(Q_{n-1})). \end{aligned} \tag{11.14}$$

Since the polynomial Q_{n-1} does not involve A_0 or A_{n-1}, we can apply Lemma 11.4 with $N = 2n - 2$ and $a = 2n - 4$ to deduce

$$H_c^i(\mathbb{P}^{2n-2} \setminus \mathcal{V}(Q_{n-1})) = (0), \quad i < 2n. \tag{11.15}$$

We conclude

$$H^{2n-2}(X_n) \cong H^{2n-2}(\mathcal{V}(K_n, Q_{n-1})^{(2n-1)}). \tag{11.16}$$

The projection π_p is an $\mathbb{A}^1$-fibration,

$$\mathcal{V}(K_n, Q_{n-1})^{(2n-1)} - p \to \mathcal{V}(K_n, Q_{n-1})^{(2n-2)},$$

and we obtain

$$\begin{aligned} &H^{2n-2}(\mathcal{V}(K_n, Q_{n-1})^{(2n-1)}) \\ &\cong\ (2n-2 > 0)_c^{2n-2}(\mathcal{V}(K_n, Q_{n-1})^{(2n-1)} - p) \\ &\quad\cong H^{2n-4}(\mathcal{V}(K_n, Q_{n-1})^{(2n-2)})(-1). \end{aligned} \tag{11.17}$$

□

We now consider the line ℓ with coordinate functions A_0, A_{n-1},

$$\begin{aligned} &\ell \subset \mathbb{P}^{2n-2}(B_1 : \ldots : B_{n-1} : A_0 : \ldots : A_{n-1}), \\ &\ell : B_1 = \ldots = B_{n-1} = A_1 = \ldots = A_{n-2} = 0. \end{aligned} \tag{11.18}$$

One has $\ell \subset \mathcal{V}(Q_{n-1}, K_n)^{(2n-2)}$. The sequence

$$\begin{aligned} 0 \to\ &H_c^{2n-4}(\mathcal{V}(Q_{n-1}, K_n)^{(2n-2)} \setminus \ell) \\ &\to H^{2n-4}(\mathcal{V}(Q_{n-1}, K_n)^{(2n-2)}) \to H^{2n-4}(\ell) \end{aligned} \tag{11.19}$$

together with the previous lemma implies

$$H_c^{2n-4}(\mathcal{V}(Q_{n-1}, K_n)^{(2n-2)} \setminus \ell)(-1) \cong H^{2n-2}(\tilde{X}_n, \mathbb{Q}), \tag{11.20}$$

where

$$H^{2n-2}(X_n) = H^{2n-2}(\tilde{X}_n) \text{ for } n > 3,$$

and for $n = 3$,

$$H^4(\tilde{X}_3) = \ker(H^4(X_3) \to H^2(\ell)(-1)) \cong H^4(X_3)_{\mathrm{prim}}.$$

The next step is now motivated by the shape of the matrix (11.5). If we wish to induct on n, we have to find the geometry which gets rid of the corner term A_{n-1} in the matrix. We project further to $\mathbb{P}^{2n-4} = \mathbb{P}^{2n-4}(B_1 : \ldots : B_{n-1} : A_1 : \ldots : A_{n-2})$. Let

$$r : \mathcal{V}(Q_{n-1}, K_n)^{(2n-2)} \setminus \ell \to \mathcal{V}(Q_{n-1})^{(2n-4)} \tag{11.21}$$

be the projection with center ℓ. It is clear from (11.10) that the fibres of r are conics in the variables A_0, A_{n-1} with discriminant

$$\begin{aligned}
&\delta_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}) \\
&\quad := Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}) \cdot Q_{n-2}(B_1, \ldots, B_{n-2}, A_1, \ldots, A_{n-3}) \\
&\qquad -(A_1 \cdots A_{n-2})^2.
\end{aligned} \tag{11.22}$$

We show that in fact the situation is degenerated:

Lemma 11.6. *One has*

$$\begin{aligned}
&\delta_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}) \\
&\quad = Q_{n-3}(B_2, \ldots, B_{n-2}, A_2, \ldots, B_{n-3}) \cdot Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}).
\end{aligned} \tag{11.23}$$

In particular, the general fibre of r in (11.21) *is a double line (so* $\{Q_{n-1} = K_n = 0\}$ *is non-reduced).*

Proof. We compute in the ring

$$K\Big[B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}, \frac{1}{Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}}\Big]. \tag{11.24}$$

One has

$$\begin{aligned}
B_1 = {} & A_1^2 Q_{n-3}(B_3, \ldots, B_{n-1}, A_3, \ldots, A_{n-2})/Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}) \\
& + Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2})/Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}).
\end{aligned} \tag{11.25}$$

This yields

$$\begin{aligned}
\delta_{n-1} = A_1^2 \Big(& \delta_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}) \\
& - Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2}) \cdot Q_{n-4}(B_3, \ldots, B_{n-2}, A_3, \ldots, A_{n-3}) \Big) \\
& + Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}) \cdot Q_{n-3}(B_2, \ldots, B_{n-2}, A_2, \ldots A_{n-3}).
\end{aligned} \tag{11.26}$$

We now argue by induction starting with $n = 3$:

$$\delta_{3-1} = B_1 B_2 - A_1^2 = Q_2(B_1, B_2, A_1) \cdot 1. \tag{11.27}$$

□

From Lemma 11.6 we see that the reduced scheme $\mathcal{V}(Q_{n-1}, K_n)_{\mathrm{red}} \setminus \ell$ is fibred over $\mathcal{V}(Q_{n-1})^{(2n-4)} \subset \mathbb{P}^{2n-4}$ with general fibre $\mathbb{A}^1$. The fibres jump to $\mathbb{A}^2$ over the closed set

$$Z_{n-1} : \mathcal{V}\Big(Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}),$$
$$Q_{n-2}(B_1, \ldots, B_{n-2} A_1, \ldots, A_{n-3}), Q_{n-2}(B_2, \ldots, B_{n-1}, A_2, \ldots, A_{n-2})\Big). \tag{11.28}$$

As a consequence, we get an exact sequence

$$H^{2n-9}(Z_{n-1})(-3) \to H_c^{2n-6}(\mathcal{V}(Q_{n-1})^{(2n-4)} \setminus Z_{n-1})(-2) \to$$
$$H^{2n-2}(\tilde{X}_n) \to H^{2n-8}(Z_{n-1})(-3) \tag{11.29}$$

with the tilde as in (11.20).

Lemma 11.7. (i) *The restriction map* $H^i(\mathbb{P}^{2n-4}) \to H^i(Z_{n-1})$ *is surjective for* $i < 2n-7$.
(ii) $Z_2 = \emptyset$.
(iii) *For* $n \geq 4$ *we have*

$$H^{2n-7}(Z_{n-1}) \cong H_c^{2n-6}(\{Q_{n-1} = 0\}^{(2n-4)} \setminus Z_{n-1}).$$

Proof. (i) Z_{n-1} is defined by 3 equations, thus by Artin's vanishing theorem $H_c^i(\mathbb{P}^{2n-4} \setminus Z_{n-1}) = 0$ vanishes for $i < 2n-6$.
(ii) One has $Z_2 : B_1 B_2 - A_1^2 = B_1 = B_2 = 0$ in $\mathbb{P}^2(B_1 : B_2 : A_1)$, so $Z_2 = \emptyset$.
(iii) For $n \geq 4$ we have

$$H^{2n-7}(\mathcal{V}(Q_{n-1})^{(2n-4)}) \to H^{2n-7}(Z_{n-1}) \to$$
$$H_c^{2n-6}(\mathcal{V}(Q_{n-1})^{(2n-4)} \setminus Z_{n-1}) \to H^{2n-6}(\mathcal{V}(Q_{n-1})^{(2n-4)}) \to$$
$$H^{2n-6}(Z_{n-1}). \tag{11.30}$$

Since $H^i(\mathbb{P}^{2n-4}) \twoheadrightarrow H^i(\mathcal{V}(Q_{n-1})^{(2n-4)})$ for $i \leq 2n-6$, the lemma follows. □

Now we may put together Lemma 11.7 and (11.29) to deduce

Lemma 11.8. *We have*

$$H^{2n-7}(Z_{n-1})(-2) \cong H^{2n-2}(X_n)/H^{2n-2}(\mathbb{P}^{2n-1}); \quad n \geq 4$$
$$H^2(X_3)/H^2(\mathbb{P}^5) \cong H^0(\mathcal{V}(Q_2)^{(2)})(-2) = \mathbb{Q}(-2). \tag{11.31}$$

In order to prove Theorem 11.2 it will therefore suffice to prove

Theorem 11.9. *Let*

$$Z_n := \mathcal{V}\Big(Q_n(B_1, \ldots, B_n, A_1, \ldots, A_{n-1}),$$
$$Q_{n-1}(B_1, \ldots, B_{n-1}, A_1, \ldots, A_{n-2}), Q_{n-1}(B_2, \ldots, B_n, A_2, \ldots, A_{n-1})\Big). \tag{11.32}$$

Then, for $n \geq 3$ *we have* $H^{2n-5}(Z_n, \mathbb{Q})) \cong \mathbb{Q}(0)$.

$$Q_p(i) := Q_p(B_i, \ldots, B_{i+p-1}, A_i, \ldots, A_{i+p-2}). \tag{11.33}$$

Given a closed subvariety $V \subset \mathbb{P}^N$, write $\ell(V) \geq r$ if the restriction maps $H^i(\mathbb{P}^N) \to H^i(V)$ are surjective for all $i \leq r$. (It is equivalent to require these maps to be an isomorphism for $i \leq \min(2\dim V, r)$.) For $V = \mathcal{V}(I)$ it is convenient to write $\ell(I) := \ell(\mathcal{V}(I))$. For example a linear subspace has $\ell = \infty$. A disjoint union of 2 points has $\ell = -1$.

In what follows, the term *variety* is used loosely to mean a reduced (but not necessarily irreducible) algebraic scheme over a field. We begin with some elementary properties of ℓ.

Lemma 11.10. *Let $L \subset \mathbb{P}^N$ be a linear subspace of dimension p. Let $\pi: \mathbb{P}^N \setminus L \to \mathbb{P}^{N-p-1}$ be the projection with center L. For $V \subset \mathbb{P}^{N-p-1}$ a closed subvariety, write (abusively) $\pi^{-1}(V) \subset \mathbb{P}^N$ for the cone over V. Then $\ell(\pi^{-1}(V)) = \ell(V) + 2(p+1)$.*

Proof. $\pi: \mathbb{P}^N \setminus L \to \mathbb{P}^{N-p-1}$ is an $\mathbb{A}^{p+1}$-bundle. By homotopy invariance, we have a commutative diagram

$$\begin{array}{ccc} H_c^{i+2(p+1)}(\mathbb{P}^N \backslash L) & \longrightarrow & H_c^{i+2(p+1)}(\pi^{-1}(V)\backslash L) \\ \downarrow \cong & & \downarrow \cong \\ H^i(\mathbb{P}^{N-p-1})(-p-1) & \xrightarrow{\text{surj.}} & H^i(V)(-p-1). \end{array} \tag{11.34}$$

The bottom horizontal map is surjective for $i \leq \ell(V)$, so the top map is surjective in that range as well. Now consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_c^j(\mathbb{P}^N \backslash L) & \longrightarrow & H^j(\mathbb{P}^N) & \xrightarrow{a} & H^j(L) & \longrightarrow & 0 \\ & & \downarrow \text{surj.} & & \downarrow c & & \| & & \\ 0 & \longrightarrow & H_c^j(\pi^{-1}(V)\backslash L) & \longrightarrow & H^j(\pi^{-1}V) & \xrightarrow{b} & H^j(L) & \longrightarrow & 0 \end{array} \tag{11.35}$$

Note the maps a, b are surjective in all degrees, so we get short-exact sequences for all j. The left-hand vertical map is surjective if and only if the central map c is surjective. Since the left-hand map is surjective for $j \leq \ell(V) + 2(p+1)$ by (11.34), the lemma follows. $\square$

Lemma 11.11. *Let $V, W \subset \mathbb{P}^N$ be closed subvarieties. If $V \cap W \neq \emptyset$, then*

$$\ell(V \cup W) \geq \min\Big(\ell(V), \ell(W), 2\dim(V \cap W), \ell(V \cap W) + 1\Big). \tag{11.36}$$

Proof. We use Mayer-Vietoris,

$$\begin{aligned} H^{i-1}(V) \oplus H^{i-1}(W) \to H^{i-1}(V \cap W) \to H^i(V \cup W) \\ \to H^i(V) \oplus H^i(W) \xrightarrow{g} H^i(V \cap W). \end{aligned} \tag{11.37}$$

Note in general if we have $A \subset B \subset \mathbb{P}^N$, then $H^i(B) \twoheadrightarrow H^i(A)$ for $i \leq \ell(A)$. Thus, for $i \leq \ell(V \cap W) + 1$ we get

$$0 \to H^i(V \cup W) \to H^i(V) \oplus H^i(W) \xrightarrow{g} H^i(V \cap W). \tag{11.38}$$

For $i \leq \min(\ell(W), 2\dim(V \cap W))$ the map g above is injective on $0 \oplus H^i(W)$, so $\dim H^i(V \cup W) \leq \dim H^i(V)$ and the lemma follows. $\square$

The proof of Theorem 11.9 proceeds by writing

$$Z_n = \mathcal{V}(Q_n(1), Q_{n-1}(1)) \cap \mathcal{V}(Q_n(1), Q_{n-1}(2)) \tag{11.39}$$

from (11.32). We remark that the automorphism of projective space given by

$$B_1 \mapsto B_n, B_2 \mapsto B_{n-1}, \ldots A_1 \mapsto A_{n-1}, \ldots A_{n-1} \mapsto A_1 \tag{11.40}$$

carries $Q_n(1) \mapsto Q_n(1)$ and $Q_{n-1}(1) \mapsto Q_{n-1}(2)$ so the varieties on the right in (11.39) are isomorphic.

Lemma 11.12. *We have*

$$\ell(Q_2(1), Q_1(2)) = \ell(Q_2(1), Q_1(1)) = \ell(Q_2(1)) = \infty. \tag{11.41}$$

For $n \geq 3$,

$$\ell(Q_n(1), Q_{n-1}(2)),\ \ell(Q_n(1), Q_{n-1}(1)),\ \ell(Q_n(1)) \geq 2n-3. \tag{11.42}$$

Proof. We write

$$a_n := \ell(Q_n(1)),\ b_n := \ell(Q_n(1), Q_{n-1}(2)). \tag{11.43}$$

(Using the automorphism (11.40), we need only consider these.) We have

$$Q_2(1) = B_1B_2 - A_1^2,\ Q_1(i) = B_i \tag{11.44}$$

from which the lemma is immediate in the case $n = 2$. For $n = 3$ we have the exact sequence

$$H_c^i(\mathbb{P}^3 \setminus \mathcal{V}(Q_2(2))^{(3)}) \to H^i(\mathcal{V}(Q_3(1))) \to H^i(\mathcal{V}(Q_3(1), Q_2(2))) \tag{11.45}$$

(cf. (11.48) below). Since $\ell(\mathcal{V}(Q_2(2))) = \infty$, the group on the left vanishes for $i < 6$. On the other hand

$$\begin{aligned}\mathcal{V}(Q_3(1), Q_2(2)) = \{B_3 = A_2 = 0\} \cup \{A_1 = B_2B_3 - A_2^2 = 0\}\\ \subset \mathbb{P}^4(B_1, B_2, B_3, A_1, A_2).\end{aligned} \tag{11.46}$$

Each of the two pieces on the right has $\ell = \infty$. Their intersection is the linear space $L := \{A_2 = A_1 = B_3 = 0\}$ which is a line. Lemma 11.11 gives $b_3 := \ell(Q_3(1), Q_2(2)) \geq 2$, but we can consider directly the situation for H^3,

$$\ldots \twoheadrightarrow H^2(L) \to H^3(\mathcal{V}(Q_3(1), Q_2(2))) \to 0 \oplus 0, \tag{11.47}$$

and conclude $a_3 \geq b_3 \geq 3 = \max(3, 2 \cdot 4 - 5)$.

The proof of the lemma for $n \geq 4$ is recursive. We have, projecting from the point $B_1 = 1$, $B_i = A_j = 0$ using (11.9),

$$\begin{array}{ccccc} H_c^i\Big(\mathcal{V}(Q_n(1))\setminus\mathcal{V}(Q_n(1), Q_{n-1}(2))\Big) & \longrightarrow & H^i\Big(\mathcal{V}(Q_n(1))\Big) & \longrightarrow & H^i\Big(\mathcal{V}(Q_n(1), Q_{n-1}(2))\Big) \\ \big\downarrow\cong & & & & \\ H_c^i(\mathbb{P}^{2n-3}\setminus\mathcal{V}(Q_{n-1}(2))^{(2n-3)}). & & & & \end{array} \tag{11.48}$$

Dropping the variable A_1, $\mathbb{P}^{2n-3} \setminus \mathcal{V}(Q_{n-1}(2))^{(2n-3)}$ becomes an $\mathbb{A}^1$-bundle over $\mathbb{P}^{2n-4} \setminus \mathcal{V}(Q_{n-1}(2))^{(2n-4)}$, so

$$H_c^i\Big(\mathcal{V}(Q_n(1)) \setminus \mathcal{V}(Q_n(1), Q_{n-1}(2))\Big) = 0 \tag{11.49}$$

for $i \leq a_{n-1} + 3$. We conclude from (11.36) that

$$a_n \geq \min(a_{n-1} + 3, b_n). \tag{11.50}$$

As a consequence of (11.9),

$$\begin{aligned}(Q_n(1), Q_{n-1}(2)) &= \Big(B_1 Q_{n-1}(2) - A_1^2 Q_{n-2}(3), Q_{n-1}(2)\Big) \\ &= (A_1^2 Q_{n-2}(3), Q_{n-1}(2)).\end{aligned} \tag{11.51}$$

In terms of $\mathcal{V}$ this reads

$$\mathcal{V}(Q_n(1), Q_{n-1}(2)) = \mathcal{V}(Q_{n-2}(3), Q_{n-1}(2))^{(2n-2)} \cup \mathcal{V}(Q_{n-1}(2), A_1)^{(2n-2)}. \tag{11.52}$$

The varieties on the right are cones with fibres of dimensions 2 and 1 respectively. From Lemmas 11.10 and 11.11 we conclude

$$\begin{aligned}b_n &\geq \min(b_{n-1} + 4, a_{n-1} + 2, 2\dim \mathcal{V}(Q_{n-1}(2), Q_{n-2}(3)) + 2, b_{n-1} + 3) \\ &= \min(a_{n-1} + 2, b_{n-1} + 3, 4n - 10).\end{aligned} \tag{11.53}$$

Starting with $a_3, b_3 \geq 3$ and plugging recursively into (11.53) and (11.50), the inequalities of the lemma, $a_n, b_n \geq 2n - 3$, follow. □

We return now to the proof of Theorem 11.9.

Lemma 11.13. *We have the decompositions*

$$\begin{aligned}\mathcal{V}(Q_n(1), Q_{n-1}(2)) = \mathcal{V}(A_1, Q_{n-1}(2)) \cup \mathcal{V}(A_2, Q_{n-2}(3)) \cup \ldots \\ \cup \mathcal{V}(A_{n-1}, B_n),\end{aligned} \tag{11.54}$$

$$\begin{aligned}\mathcal{V}(Q_n(1), Q_{n-1}(1)) = \mathcal{V}(A_{n-1}, Q_{n-1}(1)) \cup \mathcal{V}(A_{n-2}, Q_{n-2}(1)) \cup \ldots \\ \cup \mathcal{V}(A_1, B_1),\end{aligned} \tag{11.55}$$

$$\begin{aligned}&\mathcal{V}(Q_n(1), Q_{n-1}(1)) \cup \mathcal{V}(Q_n(1), Q_{n-1}(2)) \\ &= \mathcal{V}(A_1, Q_n(1)) \cup \mathcal{V}(A_2, Q_n(1)) \cup \ldots \cup \mathcal{V}(A_{n-1}, Q_n(1)) = \mathcal{V}\left(\prod_{i=1}^{n-1} A_i, Q_n(1)\right).\end{aligned} \tag{11.56}$$

Proof. For (11.54), we appeal repeatedly to (11.9),

$$\mathcal{V}(Q_n(1), Q_{n-1}(2)) = \mathcal{V}(A_1, Q_{n-1}(2)) \cup \mathcal{V}(Q_{n-1}(2), Q_{n-2}(3)) = \ldots. \tag{11.57}$$

To prove (11.55), we apply the automorphism (11.40) to (11.54). Finally, from the determinant formula (11.8) one sees the congruences

$$Q_n(1) \equiv Q_p(1) \cdot Q_{n-p}(p+1) \mod A_p; \quad 1 \leq p \leq n-1. \tag{11.58}$$

We can use these to combine the $\mathcal{V}(A_i, *)$ from (11.54) and (11.55). □

The idea now is to use Mayer-Vietoris on (11.39) and (11.56). We get

$$\begin{gathered}H^{2n-5}\Big(\mathcal{V}(Q_n(1), Q_{n-1}(2))\Big) \oplus H^{2n-5}\Big(\mathcal{V}(Q_n(1), Q_{n-1}(1))\Big)\\ \to H^{2n-5}(Z_n) \to H^{2n-4}\Big(\mathcal{V}\left(\prod_{i=1}^{n-1} A_i, Q_n(1)\right)\Big)\\ \to H^{2n-4}\Big(\mathcal{V}(Q_n(1), Q_{n-1}(2))\Big) \oplus H^{2n-4}\Big(\mathcal{V}(Q_n(1), Q_{n-1}(1))\Big) \to H^{2n-4}(Z_n).\end{gathered} \tag{11.59}$$

The vanishing results from Lemma 11.12 now yield

$$H^{2n-5}(Z_n) \cong H^{2n-4}\left(\mathcal{V}\left(\prod_{i=1}^{n-1} A_i, Q_n(1)\right)\right)\Big/ H^{2n-4}(\mathbb{P}^{2n-2}). \tag{11.60}$$

The final step in the proof of Theorem 11.9 will be to analyse the spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0,\ldots,i_p} H^q\Big(\mathcal{V}(A_{i_0}, \ldots, A_{i_p}, Q_n(1))\Big) \Rightarrow H^{p+q}\left(\mathcal{V}\left(\prod_{i=1}^{n-1} A_i, Q_n(1)\right)\right). \tag{11.61}$$

We can calculate $H^q\Big(\mathcal{V}(A_{i_0}, \ldots, A_{i_p}, Q_n(1))\Big)$ as follows. Write $n_0 = i_0, n_1 = i_1 - i_0, \ldots, n_p = i_p - i_{p-1}, n_{p+1} = n - i_p$. Thus we have a partition $n = \sum_0^{p+1} n_j$. As in (11.58) we may factor

$$Q_n(1)|_{A_{i_0}=\ldots=A_{i_p}=0} = Q_{n_0}(1)Q_{n_1}(i_0+1) \cdot Q_{n_{p+1}}(i_p+1)|_{A_{i_0}=\ldots=A_{i_p}=0}. \tag{11.62}$$

Each $Q_{n_j}(i_{j-1}+1)$ is a homogeneous function on $\mathbb{P}^{2n_j-2}$. Note if $n_j = 1$, $Q_1(i) = B_i$ is a homogeneous function on $\mathbb{P}^0$. (The homogeneous coordinate ring of $\mathbb{P}^0$ is a polynomial ring in one variable.)

We have linear spaces

$$L_j \subset \mathbb{P}^{2n-p-3}(A_1, \ldots, \widehat{A}_{i_0}, \ldots, \widehat{A}_{i_p}, \ldots, A_{n-1}, B_1, \ldots, B_n)$$

and cone maps $\pi_j : \mathbb{P}^{2n-p-2} \setminus L_j \to \mathbb{P}^{2n_j-2}$. (When $n_j = 1$, L_j is a hyperplane.) Then $\mathcal{V}(A_{i_0}, \ldots, A_{i_p}, Q_n(1))$ is the union of the cones $\pi_j^{-1}(\mathcal{V}(Q_{n_j}(i_j)))$. (When $n_j = 1$, the cone is just L_j.) Write

$$U_j = \mathbb{P}^{2n_j-2} \setminus \mathcal{V}(Q_{n_j}(i_j))$$

($U_j = \text{pt}$ when $n_j = 1$) and

$$U = \mathbb{P}^{2n-p-3} \setminus \bigcup_{j=0}^{p+1} \pi_j^{-1}(\mathcal{V}(Q_{n_j}(i_j))).$$

The map $\prod \pi_j : U \to \prod U_j$ is a $\mathbb{G}_m^{p+1}$-bundle. Thus

$$H_c^*\Big(\mathbb{P}^{2n-p-3} \setminus \mathcal{V}(A_{i_0}, \ldots, A_{i_p}, Q_n(1))\Big)$$
$$= H^*(U) \cong H_c^*(\mathbb{G}_m^{p+1}) \otimes \bigotimes_{j=0}^{p+1} H_c^*(U_j). \tag{11.63}$$

Suppose now that some $n_j > 1$. Then, by Lemma 11.12, these cohomology groups vanish in degrees less than or equal to

$$p + 1 + \sum_{j=0}^{p+1}(2n_j - 2) = 2n - p - 3. \tag{11.64}$$

It follows that we have surjections

$$H^i(\mathbb{P}^{2n-2}) \twoheadrightarrow H^i(\mathcal{V}(A_{i_0}, \ldots, A_{i_p}, Q_n(1))); \quad i \leq 2n - p - 4. \tag{11.65}$$

Note this includes the middle dimensional cohomology.

The exceptional case is when all the $n_j = 1$. Then $p = n-2$. Formula (11.64) would suggest $H_c^*(U) = (0), * < n$, but in fact $U \cong \mathbb{G}_m^{n-1}$ has $H_c^{n-1}(U) \neq 0$. We have

$$E_1^{n-2,q} = H^q\Big(\mathcal{V}(A_1, \ldots, A_{n-1}, Q_n(1))\Big) = H^q\left(\mathcal{V}\left(\prod_{i=1}^n B_i\right)\right). \tag{11.66}$$

It follows that $E_2^{n-2,n-2} = \mathbb{Q}$, and $E_2^{p,q} = (0)$ for $p+q = 2n-4$, if $p \neq 0, n-2$. One has

$$E_2^{0,2n-4} = \ker\Big(\bigoplus_{i=1}^{n-1} H^{2n-4}(\mathcal{V}(A_i, Q_n(1)))$$
$$\to \bigoplus_{I=\{i_1,i_2\}} H^{2n-4}(\mathcal{V}(A_{i_1}, A_{i_2}, Q_n(1)))\Big). \tag{11.67}$$

Again by (11.65) $E_2^{0,2n-4} = \mathbb{Q}$ is generated by the class of the hyperplane section. Finally, the differential d_r reads

$$E_r^{p-r,q+r-1} \to E_r^{p,q} \to E_r^{p+r,q-r+1}. \tag{11.68}$$

We have $r \geq 2$. In the case $p + q = 2n - 4$, the group on the left vanishes by (11.65), the group in the middle vanishes for $p \neq 0, n-2$, and the group on the right vanishes for $p = n-2$ because we have only $n-1$ components. It follows that $E_{r+1}^{p,q} \cong E_r^{p,q}$. We conclude from (11.60),

$$H^{2n-5}(Z_n) \cong \mathbb{Q}(0). \tag{11.69}$$

This completes the proof of Theorem 11.9. □

By Lemma 11.8, Theorem 11.2 follows from Theorem 11.9. This completes the proof of Theorem 11.2. □

12. de Rham Class

Let $X_n \subset \mathbb{P}^{2n-1}$ be the graph hypersurface associated to the wheel and spoke graph with n spokes as in Sect. 11. By the results in that section, we know that de Rham cohomology fulfills $H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus X_n) \cong K$. Our objective here is to show this is generated by

$$\eta_n := \frac{\Omega_{2n-1}}{\Psi_n^2} \in \Gamma(\mathbb{P}^{2n-1}, \omega(2X_n)) \tag{12.1}$$

(cf. (6.10)), i.e. we show that $[\eta_n] \neq 0$ in $H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus X_n)$.

To a certain point, the argument is general and applies to the form η_Γ attached to any graph with n loops and $2n$ edges. In this generality it is true that $[\eta_\Gamma]$ lies in the second level of the coniveau filtration. We do not give the proof here.

Lemma 12.1. *Let $U = \operatorname{Spec} R$ be a smooth, affine variety, and let $0 \neq f, g \in R$ be functions. Let $Z : f = g = 0$ in U. We have a map of complexes*

$$\left(\Omega^*_{R[1/f]}/\Omega^*_R\right) \oplus \left(\Omega^*_{R[1/g]}/\Omega^*_R\right) \to \left(\Omega^*_{R[1/fg]}/\Omega^*_R\right). \tag{12.2}$$

Then the de Rham cohomology with supports $H^*_{Z,DR}(U)$ is computed by the cone of (12.2) *shifted by* -2.

Proof. The localization sequence identifies

$$H^*_{\{f=0\},DR}(U) = H^*(\Omega^*_{R[1/f]}/\Omega^*_R[-1]) \tag{12.3}$$

(resp. replace f by g resp. fg.) The assertion of the lemma follows from the exact sequence for $X, Y \subset U$

$$\ldots \to H^*_{X\cap Y} \to H^*_X \oplus H^*_Y \to H^*_{X\cup Y} \to H^{*+1}_{X\cap Y} \to \ldots . \tag{12.4}$$

□

Remark 12.2. Evidently, this cone is quasi-isomorphic to the cone of

$$\Omega^*_{R[1/f]}/\Omega^*_R \to \Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}. \tag{12.5}$$

For the application, $U = \mathbb{P}^{2n-1} \setminus X_n$. To facilitate computations, it is convenient to localize further and invert a homogeneous coordinate as well. We take $a_i = \frac{A_i}{A_{n-1}}$ and $b_i = \frac{B_i}{A_{n-1}}$, (11.4) . (We will check that the forms we work with have no poles along $A_{n-1} = 0$.)

We write $Q_p(i)$ as in (11.33). Let $q_p(i) = \frac{Q_p(i)}{A^p_{n-1}}$ (resp. $\kappa_n = \frac{K_n}{A^n_{n-1}}$ with K_n as in (11.7)). Take $f = q_{n-1}(1),\ g = q_{n-2}(2)$. The local defining equation $X_n : b_0 q_{n-1}(1) + \kappa_n$ has been inverted in U, so κ_n is invertible on $f = 0$ and the element

$$\beta := -db_1 \wedge \ldots \wedge db_{n-1} \wedge da_0 \wedge \ldots \wedge da_{n-2} \frac{1}{\kappa_n}\left(\frac{1}{q_{n-1}(1)} - \frac{b_0}{b_0 q_{n-1}(1) + \kappa_n}\right) \tag{12.6}$$

is defined in $\Omega^{2n-2}_{R[1/f]}/\Omega^{2n-2}_R$ and satisfies

$$d\beta = \eta_n = \frac{db_0 \wedge \ldots \wedge db_{n-1} \wedge da_0 \wedge \ldots \wedge da_{n-2}}{(b_0 q_{n-1}(1) + \kappa_n)^2}. \tag{12.7}$$

Applying the fundamental relation expressed by Lemma 11.6, one obtains

$$\kappa_n q_{n-2}(2) \equiv (a_0 q_{n-2}(2) + (-1)^n a_1 \cdots a_{n-2})^2 \mod q_{n-1}(1). \tag{12.8}$$

Computing now in $\Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}$ we find

$$\begin{aligned}\beta &= -\frac{dq_{n-1}(1)}{q_{n-1}(1)} \wedge \frac{db_2}{\kappa_n q_{n-2}(2)} \wedge db_3 \wedge \ldots \wedge da_{n-2} \left(1 - \frac{b_0 q_{n-1}(1)}{b_0 q_{n-1}(1) + \kappa_n}\right) \\ &= d\Big(\frac{1}{a_0 q_{n-2}(2) + (-1)^n a_1 \cdots a_{n-2}} \cdot \frac{dq_{n-1}(1)}{q_{n-1}(1)} \wedge \frac{dq_{n-2}(2)}{q_{n-2}(2)} \wedge \nu\Big),\end{aligned} \tag{12.9}$$

where

$$\begin{aligned}\nu &= \pm \frac{db_3}{q_{n-3}(3)} \wedge db_4 \wedge \ldots \wedge db_{n-1} \wedge da_1 \wedge \ldots \wedge da_{n-2} \\ &= \pm \frac{dq_{n-3}(3)}{q_{n-3}(3)} \wedge \frac{dq_{n-4}(4)}{q_{n-4}(4)} \wedge \ldots \wedge \frac{dq_1(n-1)}{q_1(n-1)} \wedge da_1 \cdots \wedge da_{n-2}.\end{aligned} \tag{12.10}$$

(Note that a_0 is omitted.)

It follows from (12.8) that in $\Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}$ we have

$$\begin{aligned}\beta &= d\Big(\frac{1}{a_0 q_{n-2}(2) + (-1)^n a_1 \cdots a_{n-2}} \cdot \frac{dq_{n-1}(1)}{q_{n-1}(1)} \wedge \frac{db_2}{q_{n-2}(2)} \wedge db_3 \ldots. \\ &\quad db_{n-1} \wedge da_1 \wedge \ldots \wedge da_{n-2}\Big) = d\theta,\end{aligned} \tag{12.11}$$

$$\begin{aligned}\theta := &\frac{1}{a_0 q_{n-2}(2) + (-1)^n a_1 \cdots a_{n-2}} \cdot \frac{dq_{n-1}(1)}{q_{n-1}(1)} \wedge \frac{db_2}{q_{n-2}(2)} \wedge db_3 \ldots, \\ &db_{n-1} \wedge da_1 \wedge \ldots \wedge da_{n-2}\end{aligned}$$

(defining θ.) One checks easily that neither β nor θ has a pole along $A_{n-1} = 0$, so the pair

$$(\beta, \theta) \in H^{2n-1}_{Z,DR}(U) \tag{12.12}$$

represents a class mapping to $\eta_n \in H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus X_n)$. Here

$$Z : Q_{n-1}(1) = Q_{n-2}(2) = 0.$$

Lemma 12.3. *The map*

$$H^{2n-1}_Z(\mathbb{P}^{2n-1} \setminus X_n) \to H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_n) \tag{12.13}$$

is injective.

Proof. Let $Y : Q_{n-1}(1) = 0$. We have

$$H^{2n-1}_Z(\mathbb{P}^{2n-1} \setminus X_n) \xrightarrow{u} H^{2n-1}_Y(\mathbb{P}^{2n-1} \setminus X_n) \xrightarrow{v} H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_n), \tag{12.14}$$

and it will suffice to show u and v injective. We have projections

$$\mathbb{P}^{2n-1} \setminus (X_n \cup Y) \xrightarrow{B_0} \mathbb{P}^{2n-2} \setminus Y_0 \xrightarrow{A_0, A_{n-1}} \mathbb{P}^{2n-4} \setminus Y_1. \tag{12.15}$$

Here $\mathbb{P}^{2n-1}$ has homogeneous coordinates $A_0, \ldots, A_{n-1}, B_0, \ldots, B_{n-1}$; the arrows are labeled by the variables which are dropped, and Y, Y_0 are cones over Y_1. The arrow on the left is a $\mathbb{G}_m$-bundle and on the right an $\mathbb{A}^2$-bundle. It follows that

$$\begin{aligned} &H^{2n-2}(\mathbb{P}^{2n-1} \setminus (X_n \cup Y)) \\ &\quad \cong H^{2n-2}(\mathbb{P}^{2n-4} \setminus Y_1) \oplus H^{2n-3}(\mathbb{P}^{2n-4} \setminus Y_1)(-1) = (0) \end{aligned} \tag{12.16}$$

by Artin vanishing. As a consequence, the map v in (12.14) is injective.

The locus $Y \setminus Z$ is smooth ($Q_{n-2}(2) = \partial Q_{n-1}(1)/\partial B_1$) so to prove injectivity for u it will suffice to show

$$H^{2n-4}(Y \setminus ((X_n \cap Y) \cup Z)) = (0). \tag{12.17}$$

Consider the projection obtained as in (12.15) by dropping the variables B_0, A_0, A_{n-1} (so Y, Z are cones over Y_1, Z_1)

$$Y \setminus ((X_n \cap Y) \cup Z) \xrightarrow{\pi} Y_1 \setminus Z_1 \subset \mathbb{P}^{2n-4}. \tag{12.18}$$

Note that $X_n \cap Y : Q_{n-1}(1) = K_n = 0$, where K_n is as in (11.7). We can write π as a composition of two projections. First dropping B_0 yields an $\mathbb{A}^1$-fibration. Then dropping A_0, A_{n-1} leads to a fibration with fibre $\mathbb{A}^2$ − quadric. By Lemma 11.6, this quadric is a double line, so the fibres of π are $\mathbb{A}^2 \times \mathbb{G}_m$. It follows that

$$\begin{aligned} H^{2n-4}(Y \setminus ((X_n \cap Y) \cup Z)) &\cong H^{2n-4}(Y_1 \setminus Z_1) \oplus H^{2n-5}(Y_1 \setminus Z_1)(-1) \\ &= H^{2n-5}(Y_1 \setminus Z_1)(-1). \end{aligned} \tag{12.19}$$

(The right hand identity is Artin vanishing since $Y_1 \setminus Z_1$ is affine of dimension $2n-5$.) Dropping the variable B_1 realizes $\{Q_{n-2}(2) = 0\}$ as the cone over a hypersurface $Y_2 \subset \mathbb{P}^{2n-5}$. Using (11.9), we conclude

$$H^{2n-5}(Y_1 \setminus Z_1) \cong H^{2n-5}(\mathbb{P}^{2n-5} \setminus Y_2). \tag{12.20}$$

But the equation defining Y_2 does not involve A_1, so yet another projection is possible, and we deduce vanishing on the right in (12.20) by Lemma 11.4.

Theorem 12.4. *Let X_n be the graph hypersurface for the wheel and spokes graph with n spokes. Let $[\eta_n] \in H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus X_n)$ be the de Rham class* (12.1). *Then*

$$K[\eta_n] = H^{2n-1}_{DR}(\mathbb{P}^{2n-1} \setminus X_n).$$

Proof. We have lifted $[\eta_n]$ to a class $(\beta, \theta) \in H^{2n-1}_{Z,DR}(\mathbb{P}^{2n-1} \setminus X_n)$, (12.12). By Lemma 12.3, it will suffice to show $(\beta, \theta) \neq 0$. We localize at the generic point of Z. It follows from (12.10) and (12.11) that as a class in the de Rham cohomology of the function field of Z, this class is represented by the form

$$\pm d\log(q_{n-3}(3)) \wedge \ldots \wedge d\log(q_1(n-1)) \wedge d\log(a_1) \wedge \ldots \wedge d\log(a_{n-2}). \tag{12.21}$$

It is easy to see that this is a non-zero multiple of

$$d\log(b_3) \wedge \ldots \wedge d\log(b_{n-1}) \wedge d\log(a_1) \ldots d\log(a_{n-2}),$$

and so is nonzero as a form. To see that it is nonzero as a cohomology class, one applies Deligne's mixed Hodge theory which implies that the vector space of logarithmic forms injects into de Rham cohomology of the open on which those forms are smooth. □

13. Wheels and Beyond

13.1. A few words on the wheel with 3 *spokes.* Let $X_3 \subset \mathbb{P}^5$ be the hypersurface associated to the wheel with 3 spokes. $X_3 : \det(A_1 M_1 + \ldots + A_6 M_6) = 0$, where the M_i are symmetric rank one 3×3 matrices. It is easy to see in this case that the M_i span the vector space of all symmetric 3×3-matrices. The mapping $g \mapsto {}^t gg$ identifies $GL_3(\mathbb{C})/O_3(\mathbb{C})$ with the space of invertible symmetric 3×3 complex matrices. It follows that

$$\mathbb{P}^5 - X_3 \cong GL_3(\mathbb{C})/\mathbb{C}^\times O_3(\mathbb{C}). \tag{13.1}$$

From this, standard facts about the cohomology of symmetric spaces yield Theorem 11.2 for X_3. (We thank P. Deligne for this argument.)

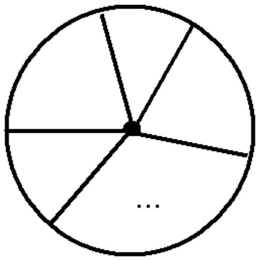
From another point of view, X_3 is the space of singular quadrics in $\mathbb{P}^2$. Such a quadric is a union of two (possibly coincident) lines, so we get

$$X_3 \cong \mathrm{Sym}^2\mathbb{P}^2. \tag{13.2}$$

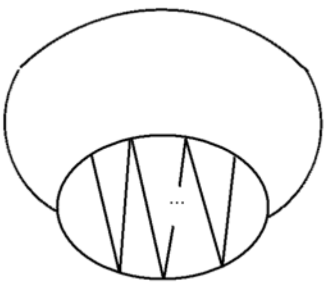
This way we see immediately that $H^4(X) = \mathbb{Q}(-2) \oplus \mathbb{Q}(-2)$, where the 2 generators are the class of the algebraic cycles $p \times \mathbb{P}^2 + \mathbb{P}^2 \times p$ and the diagonal Δ. In particular, Remark 10.5 is clear.

Then $p \times \mathbb{P}^2$ is linearly embedded into $\mathbb{P}^5$ while Δ is embedded by the the complete linear system $\mathcal{O}(-2)$. Thus $\Delta - 2 \cdot (p \times \mathbb{P}^2 + \mathbb{P}^2 \times p)$ spans the interesting class in $H^4(X)_{\mathrm{prim}}$. It is likely that its strict transform in the blow up $\pi : P \to \mathbb{P}^5$ yields a relative class in $H^6_Y(P, B)$, but we haven't computed this last piece.

13.2. Beyond wheels. An immediate observation is that the wheel with n spokes w_n,

$w_n =$  (13.3)

and the zig-zag graphs z_n,

$z_n =$  (13.4) are both obtained by gluing triangles together in a rather obvious way. Both classes of graphs evaluate to rational multiples of $\zeta(2l-3)$ at l-loops [4]. The kinship between these two classes of graphs is not easily seen at the level of their graph polynomials. Suppose we try to look directly at the Feynman period (5.3). Let $\ell = e_1+e_2+e_3 \in H_1(\Gamma)$ be the loop spanned by a triangle. If we choose coordinates on $H_1(\Gamma)$ in such a way that the first coordinate k coincides on $\mathbb{Q}\cdot\ell \subset H_1$ with $e_i^\vee,\ i\leq 3$, and the other coordinates q are pulled back from a system of coordinates on $H_1/\mathbb{Q}\cdot\ell$, then the k coordinate appears only in the quadrics Q_i associated to the edges $e_i,\ i=1,2,3$. Replacing k by $k_1,\dots,k_4$, the period (5.3) can be written

$$\int_{q=-\infty}^{\infty} \frac{dq}{Q_4(q)\cdots Q_n(q)} \int_{k=-\infty}^{\infty} \frac{dk}{Q_1(k,q)Q_2(k,q)Q_3(k,q)}. \tag{13.5}$$

We have the Feynman parametrization

$$\frac{1}{Q_1(k)Q_2(k)Q_3(k)} = \int_0^\infty\int_0^\infty \frac{1+y}{[x(1+y)Q_1(k)+yQ_2(k)+Q_3(k)]^3}dxdy, \tag{13.6}$$

and the elementary integral, valid with appropriate positivity hypotheses on an inhomogeneous quadric $\widetilde{Q}(k_1,\dots,k_4)$,

$$\int_{k_1,\dots,k_4=-\infty}^{\infty} \frac{d^4k}{\widetilde{Q}^3} = \frac{1}{Q}, \tag{13.7}$$

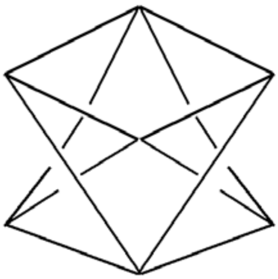
where, up to a scale factor depending on the determinant of the degree 2 homogeneous part of $\widetilde{Q}$, Q is a certain quadratic polynomial in the coefficients of $\widetilde{Q}$. With these substitutions, the period becomes

$$\int_{x,y=0}^{\infty} dxdy \int_{q=-\infty}^{\infty} \frac{dq}{Q(x,y,q)Q_4(q)\cdots Q_n(q)}, \tag{13.8}$$

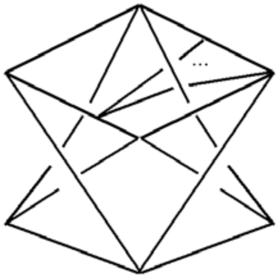
where $Q(x,y,q)$ is quadratic in q with coefficients which are rational functions in the Feynman parameters x, y. It would be of interest to try to make this calculation motivic.

A triangle is the one-loop contribution to the six-point Green function in ϕ^4 theory: its four-valent vertices between any pair of its three edges allow for two external edges, so that these three vertices allow for six external edges altogether.

The message in the above that sequences of triangles increase the transcendental degree (= point at which ζ is evaluated) in steps of two seems to be a universal observation judging by computational evidence. Indeed, let us look at the graph which encapsulates the first appearance of a multiple zeta value, in this case the first irreducible double sum $\zeta(5,3)$ which appears in the graph

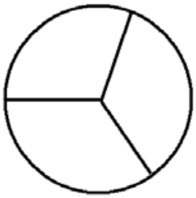
$M =$  .

This graph is the first in a series of graphs

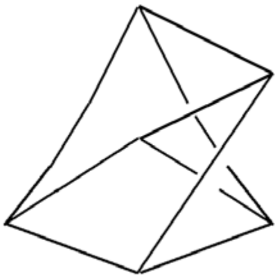
$M_i =$  .

Adding ℓ triangles yields $\zeta(5, 2l+3)$.

Most interestingly, these graphs can be decomposed into zig-zag graphs in a manner consistent with the Hopf algebra structure on the multiple zeta value Hopf algebra MZVs, upon noticing that the replacement of a triangle in

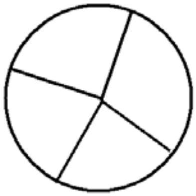


by the six-point function

$g_6 =$ 

delivers the graph M. (Remove the three edges of a triangle from w_3, and attach the remaining graph, which has 3 univalent vertices and one trivalent vertex, to g_6 by identifying the univalent vertices with 3 vertices of g_6 no two of which are connected by a single edge.) Note that indeed g_6 has six vertices of valence three. Each vertex hence will have one external edge attached to it to make it four-valent, and the resulting six external edges make this graph into a contribution to a six-point function. It can hence replace any triangle.

Furthermore, the six-point function g_6 is related to the four-loop graph

$w_4 =$ 

by the operation

$$w_4 = g_6/e, \tag{13.9}$$

where e is any edge connecting two vertices. Indeed, g_6 is the bipartite graph on two times three edges. Shrinking any of those edges to a point combines two valence-three vertices into one four-valent vertex with its four edges connecting to each of the other four remaining vertices.

This suggests constructing a Hopf algebra H on primitive vertex graph in ϕ^4 theory which incorporates the purely graph-theoretic lemma 7.4 such that the following highly symbolic diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\Delta_{2\mathrm{PI}}} & H \otimes H \\ \Big\downarrow \phi & & \Big\downarrow \phi\otimes\phi \\ \mathrm{MZV} & \xrightarrow{\Delta_{\mathrm{MZV}}} & \mathrm{MZV} \otimes \mathrm{MZV} \end{array} \quad . \tag{13.10}$$

First results are in agreement with the expectation that all graphs up to twelve edges are mixed Tate, which they are by explicit calculation [4], and also predict correctly the apperance of a double sum $\zeta(3,5)$ or products $\zeta(3)\zeta(5)$ in six-loop graphs. The seven loop data demand some highly non-trivial checks (currently in process) on the data amassed in [4, 5].

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