ON THE ALGEBRAIC FUNDAMENTAL GROUP OF SMOOTH VARIETIES IN CHARACTERISTIC $p > 0$

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Abstract. We define an analog in characteristic $p > 0$ of the proalgebraic completion of the topological fundamental group of a complex manifold.

1. Introduction

Let $X$ be a smooth algebraic variety defined over a field $k$ endowed with a rational point $x \in X(k)$.

If $k$ is the field of complex numbers $\mathbb{C}$, the proalgebraic completion $\pi_{\text{alg,rs}}^{\text{top}}(X,x)$ of the topological fundamental group $\pi_{\text{top}}^{\text{top}}(X,x)$ is defined as the prosystem $\lim_{\leftarrow} H$, where $H \subseteq GL(n, \mathbb{C})$ runs over the Zariski closures of the monodromy groups $\rho(\pi_{\text{top}}^{\text{top}}(X,x))$ of complex linear representations $\rho : \pi_{\text{top}}^{\text{top}}(X,x) \to GL(n, \mathbb{C})$. The profinite completion $\lim_{\leftarrow} H$, where $H$ runs over the finite quotients of $\pi_{\text{top}}^{\text{top}}(X,x)$, is, via the Riemann existence theorem, identified with Grothendieck’s étale fundamental group $\pi_{\text{ét}}^{\text{1}}(X,x)$. Since any finite group is embeddable in $GL(n, \mathbb{C})$ for some $n$, this defines, thinking of $\pi_{\text{ét}}^{\text{1}}(X,x)$ as a complex (constant) proalgebraic group, a surjective homomorphism $\phi_{\text{rs}}^{\text{top}} : \pi_{\text{alg,rs}}^{\text{top}}(X,x) \to \pi_{\text{ét}}^{\text{1}}(X,x)$, and in fact $\pi_{\text{ét}}^{\text{1}}(X,x)$ is the profinite quotient of $\pi_{\text{alg,rs}}^{\text{top}}(X,x)$. By the Riemann-Hilbert correspondence, $\pi_{\text{alg,rs}}^{\text{top}}(X,x)$ is the Tannaka group-scheme of the category of $\mathcal{O}_X$-coherent regular singular $\mathcal{D}_X$-modules, which is a full subcategory of the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules. We denote by $\pi_{\text{alg}}^{\text{top}}(X,x)$ the corresponding Tannaka group-scheme, and by $\varphi_{\mathbb{C}} : \pi_{\text{alg}}^{\text{top}}(X,x) \to \pi_{\text{alg,rs}}^{\text{top}}(X,x) \xrightarrow{\varphi_{\text{rs}}^{\text{top}}} \pi_{\text{ét}}^{\text{1}}(X,x)$ the composite morphism. It is surjective as well, and since any flat connection with finite monodromy is regular singular, $\pi_{\text{ét}}^{\text{1}}(X,x)$ is the profinite quotient of $\pi_{\text{alg}}^{\text{top}}(X,x)$.

If $k$ is a characteristic 0 field, $\pi_{\text{alg}}^{\text{top}}(X,x)$ is defined as the Tannaka group-scheme of the $k$-linear tensor category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules equipped with the fiber functor defined as the restriction of the module on $x$. The full subcategory of finite objects, that is objects with finite monodromy group-scheme, or said differently, objects which have the property that the full Tannaka subcategory which is spanned by it has a finite Tannaka group-scheme, defines a pro-finite $k$-group-scheme $\pi_{\text{ét}}^{\text{1}}(X,x)$. Since $\pi_{\text{ét}}^{\text{1}}(X,x)(\bar{k}) = \pi_{\text{ét}}^{\text{1}}(X,x)$ ([5, Remark 2.10]), and both $\pi_{\text{alg}}^{\text{top}}(X,x)$ and $\pi_{\text{ét}}^{\text{1}}(X,x)$ satisfy base change for finite extensions $k \subset L$ ([6,

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Property 2.54)), we see that the surjection \( \varphi : \pi^\text{alg}(X, x) \to \pi^\text{et}(X, x) \) is a \( k \)-form of \( \varphi_C \) for any complex embedding \( k \subset \mathbb{C} \). Moreover, by definition, \( \varphi \) induces the pro-finite quotient homomorphism.

If \( k \) is a characteristic \( p > 0 \) field, the category of \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules is again a \( k \)-linear abelian tensor rigid category. It is part of Katz’ theorem asserting that this category is equivalent to the category of stratified \( \mathcal{O}_X \)-coherent sheaves (see [9, Theorem 1.3], [3, Theorem 8], where it is shown over \( k = \bar{k} \)). If \( k = \bar{k} \), its Tannaka group-scheme \( \pi^\text{alg}(X, x) \) is shown to be pro-smooth in [3, Corollary 12] (strictly speaking, it is shown there only for the profinite part, but dos Santos’ proof applies more generally as mentioned in [4, Corollary 7]). The homomorphism \( \varphi \) is then defined by the full embedding of the subcategory of objects with finite monodromy group-scheme. So by definition, \( \varphi \) induces the pro-finite quotient homomorphism.

On the other hand, if \( X \) is a reduced connected scheme over a characteristic \( p > 0 \) field \( k \), endowed with a rational point \( x \in X(k) \), Nori [10, Chapter II] constructed a fundamental group-scheme \( \pi^N(X, x) \) as the projective system of finite \( k \)-group-schemes \( G \) for which there is a \( G \)-torsor \( h : Y \to X \) under \( G \) with trivialization at \( x \). The pro-étale quotient of \( \pi^N(X, x) \) is precisely \( \pi^\text{et}(X, x) \).

Summarizing, one has a diagram

\[
\begin{array}{ccc}
\pi^\text{alg}(X, x) & \xrightarrow{\text{surj}} & \pi^\text{et}(X, x) \\
\downarrow{\text{surj}} & & \downarrow{\text{surj}} \\
\pi^N(X, x) & &
\end{array}
\]

The aim of our article is to define a Tannaka category \( \text{Strat}(X, \infty) \) over a perfect field \( k \), which contains the category of \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules as a full subcategory, in such a way that its Tannaka group-scheme \( \pi^\text{alg,\infty}(X, x) \), which thus surjects onto \( \pi^\text{alg}(X, x) \), also surjects onto \( \pi^N(X, x) \). In other words, we complete (1.1) to

\[
\begin{array}{ccc}
\pi^\text{alg}(X, x) & \xrightarrow{\text{surj}} & \pi^\text{et}(X, x) \\
\downarrow{\text{surj}} & & \downarrow{\text{surj}} \\
\pi^\text{alg,\infty}(X, x) & \xrightarrow{\text{surj}} & \pi^N(X, x)
\end{array}
\]

As a byproduct, we obtain a purely tannakian geometric description of \( \pi^N(X, x) \) (see Corollary 4.9). Recall that we assume that \( X \) is smooth. If in addition \( X \) is proper, Nori himself described his fundamental group-scheme \( \pi^N(X, x) \) as the Tannaka group-scheme of the category of essentially finite bundles [10, Chapter I]. He extends in [10, Chapter III] his construction to non-proper curves by using parabolic bundles. Lacking desingularization in characteristic \( p > 0 \) makes it difficult to generalize his construction to the higher dimensional case. If \( k \) has
characteristic 0, then, as already mentioned, \( \pi^N(X, x) = \pi^{et}(X, x) \) is the Tannaka group-scheme of the category of finite flat connections [6, Section 2], or, equivalently, of the category of \( \mathcal{O}_X \)-coherent \( \mathcal{D}_X \)-modules with finite monodromy group-scheme.

Our construction (see Section 3, most particularly Definition 3.2) generalizes on a smooth variety defined over a perfect characteristic \( p > 0 \) field \( k \) the construction of the category of flat connections (loc. cit) in characteristic 0, and the construction of the stratified bundles (loc. cit.) in characteristic \( p > 0 \). We now explain the main idea.

For \( i \in \mathbb{N} \), let us define inductively the relative Frobenius \( F^{(i)} : X^{(i)} \to X^{(i+1)} \) over \( k \) in the usual manner. As \( k \) is assumed to be perfect, one defines \( X^{(-1)} = X \otimes_{k,F^{-1}} k \) where \( F_k : \text{Spec} \ k \to \text{Spec} \ k \) is the absolute Frobenius of \( k \), together with the relative Frobenius \( F^{(-1)} : X^{(-1)} \to X^{(0)} \). Then one iterates to define inductively \( F^{(i)} : X^{(i)} \to X^{(i+1)} \) for \( i \in \mathbb{Z}, i < 0 \). For \( a, b \in \mathbb{Z}, a < b \) we define \( F^{(a,b)} : X^{(a)} \xrightarrow{F^{(a)} \circ \cdots \circ F^{(b-1)}} X^{(b)} \).

Recall that a stratified bundle is a sequence \( (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \), where \( E^{(i)} \) is a bundle on \( X^{(i)} \), \( \sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)} \) is a \( \mathcal{O}_{X^{(i)}} \)-isomorphism. For \( t \in \mathbb{N}, t \neq 0 \), we define an object of \( \text{Strat}(X, t) \) to be a sequence \( (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \), where \( E^{(i)} \) is a bundle on \( X^{(i)} \), \( \sigma^{(i)} : E^{(i)} \xrightarrow{\cong} F^{(i)*}E^{(i+1)} \) is a \( \mathcal{O}_{X^{(i)}} \)-isomorphism for all \( i \geq 1 \), but for \( i = 0 \), \( \sigma_0 : F^{(-t,0)*}E^{(0)} \xrightarrow{\cong} F^{(-t,1)*}E^{(1)} \) is a \( \mathcal{O}_{X^{(-t)}} \)-isomorphism. The morphisms are the ones between the bundles which respect all the structures. We show (Theorem 3.4) that the obvious functor \( \text{Strat}(X, t) \subset \text{Strat}(X, t+1) \), which assigns \( (E_i, F^{(-t,0)*}E^{(0)}, \sigma_0, \sigma_i, i \geq 1) \) to \( (E_i, \sigma_0, \sigma_i, i \geq 1) \), induces a full embedding of Tannaka categories, where the fiber functor is simply the restriction of \( E^{(0)} \) to the rational point \( x \). Then \( \text{Strat}(X, \infty) \) is defined as the inductive limit over \( t \to \infty \) of the categories \( \text{Strat}(X, t) \) (Corollary 3.5). In order to show that the Tannaka group-scheme \( \pi^{alg, \infty}(X, x) \) of \( \text{Strat}(X, \infty) \) surjects onto \( \pi^N(X, x) \), we use a slight modification of Nori’s reconstruction theorem [10, Chapter I, Proposition 2.9] of a torsor \( h : Y \to X \) under a finite group scheme \( G \) out of the induced functor \( h^* : \text{Rep}_k(G) \to \text{Coh}(X) \) which assigns to a finite dimensional \( k \)-linear representation \( V \) of \( G \) the vector bundle on \( X \) which is defined by flat descent for \( h \) on \( \mathcal{O}_Y \times_k V \) (Theorem 2.4).

This allows us to define the group-scheme homomorphism \( \pi^{alg, \infty}(X, x) \to \pi^N(X, x) \) (Theorem 4.5). In order to show that this map induces the profinite quotient, we in particular use the categorical translation of injectivity and surjectivity of homomorphisms of Tannaka group-schemes ([2, Proposition 2.12]).

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2. Nori’s fundamental group-scheme

Let \( k \) be a field of characteristic \( p > 0 \) and \( X \) be a \( k \)-scheme. Let \( x \in X(k) \) be a rational point and \( i_x : x \to X \) be the closed embedding.

Nori [10, Chapter II] defines the category \( \mathcal{N}(X,x) \) of triples \((Y \xrightarrow{f} X, G, y)\) where

(a) \( G/k \) is a finite group scheme,
(b) \( f : Y \to X \) is a \( G \)-torsor,
(c) \( y \) is a \( k \)-point of \( Y \) lying above \( x \).

A morphism between two such triples \((Y_i \xrightarrow{f_i} X, G_i, y_i)\) is a pair \((\phi : G_1 \to G_2, \psi : Y_1 \to Y_2)\) such that \( \psi \) an \( X \)-morphism which is \( \phi \)-equivariant and \( \psi(y_1) = y_2 \). Nori shows [10, Chapter II, Proposition 2] that if \( X \) is reduced and geometrically connected, then the projective limit \( \varprojlim \mathcal{N}(X,x) \) exits. He defines

Definition 2.1. Let \( X \) be a reduced geometrically connected \( k \)-scheme, then it’s Nori fundamental group-scheme is the profinite \( k \)-group-scheme \( \pi^N(X,x) = \varprojlim \mathcal{N}(X,x) G \).

Since giving a rational point \( y \in f^{-1}(x) \) is the same as giving a trivialization \( f^{-1}(x) \cong_k G \), \( \mathcal{N}(X,x) \) is equivalent to the category of triples \((h : Y \to X, G, f^{-1}(x) \cong_k G)\), where the morphisms between two such objects are defined by torsor morphisms which respect the trivialization. We will not need this slightly different phrasing.

Definition 2.2. Let \( G \) be a finite \( k \)-group-scheme, and let \( h : Y \to X \) be a \( G \)-torsor. Then it induces a functor \( h^\#: \text{Rep}_k(G) \to \text{Coh}(X) \) which assigns to a finite dimensional \( k \)-representation \( V \) the bundle on \( X \) which comes by flat descent from \( O_Y \otimes_k V \).

Properties 2.3. 1) The functor \( h^\# \) defined in Definition 2.2 is exact, \( k \)-linear and compatible with the tensor structure. Thus it is a fiber functor in the sense of Deligne [1, 1.9]. Since \( \text{Rep}_k(G) \) is a Tannaka category, it follows [1, Corollaire 2.10] that \( h^\# \) is faithful.

2) The functor \( i^*_x : \text{Coh}(X) \to \text{Vec}_k \) defined as the restriction to the rational point, with values in the category of finite dimensional \( k \)-vector spaces, is a fiber functor on the subcategory of vector bundles. The composite functor \( i^*_x \circ h^\# : \text{Rep}_k(G) \to \text{Vec}_k \) is a fiber functor.

3) Let \( h_i : Y_i \to X \) be \( G_i \)-torsors where \( i = 1, 2 \). Let \( \phi : G_1 \to G_2 \) be a group homomorphism and \( \psi : Y_1 \to Y_2 \) be an equivariant map with respect to \( \phi \). We denote by \( \phi^* \) the induced functor \( \text{Rep}_k(G_2) \to \text{Rep}_k(G_1) \). Then
one has the equality $h_2^\# = h_1^\# \circ \phi^*$ of functors. Indeed, if $V$ is a $G$-representation, $\psi^* : \mathcal{O}_Y \otimes_k V \to \psi_*(\mathcal{O}_Y \otimes_k \phi^*(V))$ induces a $\mathcal{O}_X$-linear map $h_2^\#(V) \to h_1^\#(V)$ between those two vector bundles, which, after composing with $i_x^*$, is the identity on $V$. So $h_2^\#(V) = h_1^\# \circ \phi^*(V)$.

4) Let $h : Y \to X$ be a $G$-torsor, let $b : X' \to X$ be a morphism, and let $x' \in X'(k)$ be a rational point with $b(x') = x$. Let $Y' = Y \times_X X' \to X'$ and $h' : Y' \to X'$ denote the projection. Then one has the equality $b^* \circ h^\# = h'^\#$ of functors. Indeed, denoting by $b' : Y' \to Y$ the induced morphism, if $V$ is a $G$-representation, $(b')^* : \mathcal{O}_Y \otimes_k V \to (b')_* \mathcal{O}_{Y'} \otimes_k V$ induces $\mathcal{O}_X$-linear map $b^* \circ h^\#(V) \to (h'^\#(V)$ between vector bundles, which is the identity on $V$ after composing with $i_{x'}$. So $b^* \circ h^\# = (h')^\#$.

The following is a direct consequence of [10, Proposition 2.9].

**Theorem 2.4.** Let $G$ be a finite $k$-group-scheme and let $F : \text{Rep}_k(G) \to \text{Coh}(X)$ be a fiber functor such that $i_x^* \circ F$ is the forgetful functor $F_G : \text{Rep}_k(G) \to \text{Vec}_k$. Then there exists a unique object $(Y \xrightarrow{h} X, G, y)$ of $\mathcal{N}(X, x)$ such that $F = h^\#$ and $(h^{-1}(x), y) = (G, 1)$. For any other object $(Y' \xrightarrow{h'} X, G, y') \in \mathcal{N}(X, x)$ such that $F = h^\#$, there exists a unique isomorphism in $\mathcal{N}(X, x)$ between $(Y \xrightarrow{h} X, G, y)$ and $(Y' \xrightarrow{h'} X, G, y')$.

**Proof.** By Nori’s reconstruction theorem [10, Proposition 2.9], $F(k[G])$, where $k[G]$ is the regular representation of $G$, is a finite $\mathcal{O}_X$-algebra. The $G$-torsor $h : Y \to X$ is defined to be $\text{Spec}_X F(k[G])$. By Property 2.3 2), $i_x^* \circ F(k[G]) = F_G(k[G]) = k[G]$. Said differently, $h^{-1}(x) = \text{Spec}_X k[G] = G$. Then $y$ is the rational point of $h^{-1}(x)$ which is $1 \in G$. By the unicity in loc. cit., $h$ is uniquely defined. If $y' = g \in h^{-1}(x)(k)$ is another rational point, then multiplication $g : Y \to Y$ by $g$, together with the conjugation $G \to G, h \mapsto ghg^{-1}$ defines an isomorphism $(h : Y \to X, G, y) \to (h : Y \to X, G, y')$ in $\mathcal{N}(X, x)$.

3. THE CATEGORY OF GENERALIZED STRATIFIED BUNDLES

The aim of this section is to define the category of generalized stratified bundles. We start with some notations.

**Notations 3.1.** Let $k$ be a perfect field of characteristic $p > 0$, $X$ be a smooth scheme over $k$ which is geometrically irreducible.

For $i \in \mathbb{N}$, we define inductively the relative Frobenius $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over $k$ in the usual manner, by defining $X^{(0)} = X$, $X^{(i+1)}$ to be the fiber product of $X^{(i)} \otimes_{k, F_k} k$ over the absolute Frobenius $F_k : \text{Spec } k \to \text{Spec } k$ of $k$, and $F^{(i)}$ to be the factorization of the absolute Frobenius $F_{X^{(i)}} : X^{(i)} \to X^{(i)}$ morphism.

For $i \in \mathbb{Z}, i < 0$, we define inductively $F^{(i)} : X^{(i)} \to X^{(i+1)}$ over $k$ as follows. First we set $X^{(-1)} = X \otimes_{F_k} k$. Then we define $F^{(-1)} : X^{(-1)} \to X$ to be the
relative Frobenius. Similarly, we define \( X^{(-i-1)} = X^{(-i)} \otimes_{O^{(i+1)}} k \) together with the relative Frobenius \( F^{(-i-1)} : X^{(-i-1)} \to X^{(-i)} \) over \( k \).

For \( a, b \in \mathbb{Z}, a < b \) we define \( F^{(a,b)} : X^{(a)} \to X^{(b)} \).

Recall that a stratified bundle (see [9, Section 1]) is a sequence \( (E^{(i)}, \sigma^{(i)}), i \in \mathbb{N}, \) where \( E^{(i)} \) is a \( O_X \)-coherent sheaf on \( X^{(i)} \), \( \sigma^{(i)} : E^{(i)} \xrightarrow{\sim} F^{(i)} \otimes E^{(i+1)} \) is a \( O^{(i)} \)-isomorphism. One defines the category \( \text{Strat}(X) \) of stratified bundles by defining

\[
\text{Hom}(\{(D^{(i)}, \tau^{(i)}), (E^{(i)}, \sigma^{(i)})\})
\]
to be set of sequences \( f_i : D^{(i)} \to E^{(i)} \) of morphisms of \( O^{(i)} \)-coherent sheaves, which commute with all the \( \sigma_i \) and \( \tau_i \). It is a fact (loc. cit.) that if \( (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \) is a stratified sheaf, the \( E^{(i)} \) are all locally free, and if \( f = (f)_i, i \in \mathbb{N} \) is a morphism of stratified sheaves, then \( f_i \) are vector bundle maps (i.e. locally split), so the category is abelian, rigid, and monoidal. Moreover, the Hom-sets are finite dimensional \( k \)-vector spaces. As \( X \) is geometrically irreducible, the unit object \( \mathbb{I} = (O_X, \text{Id}), i \in \mathbb{N} \) fulfills \( \text{End}(\mathbb{I}) = k \). If now \( X \) is endowed with a rational point \( x \in X(k) \), then \( \omega_x : \text{Strat}(X) \to \text{Vec}_k, (E^{(i)}, \sigma^{(i)}) \mapsto E_0|_x \) is a fiber functor in the sense of Deligne [1, 1.9], and thus yields the structure of a Tannaka category on \( \text{Strat}(X) \). A fundamental property due to dos Santos is that the corresponding Tannaka \( k \)-group-scheme \( \text{Aut}^{\otimes}(\omega_x) \) is pro-smooth ([3, Corollary 12], [4, Corollary 7]).

**Definition 3.2.** Let \( t \geq 0 \) be an integer. A \( t \)-stratified bundle is a sequence

\[
(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}),
\]

where \( E^{(i)} \) is a \( O_X \)-coherent sheaf on \( X^{(i)} \),

\[
\sigma^{(i)} : E^{(i)} \xrightarrow{\sim} F^{(i)} \otimes E^{(i+1)}
\]
is a \( O^{(i)} \)-isomorphism for \( i \geq 1 \) and for \( i = 0 \),

\[
\sigma^{(0)} : F^{(-t,0)} \otimes E^{(0)} \xrightarrow{\sim} F^{(-t,1)} \otimes E^{(1)}
\]
is a \( O^{(-t)} \)-isomorphism.

One defines the category \( \text{Strat}(X,t) \) of \( t \)-stratified bundles by defining

\[
\text{Hom}(\{(D^{(i)}, \tau^{(i)}), (E^{(i)}, \sigma^{(i)})\})
\]
to be set of sequences \( f_i : D^{(i)} \to E^{(i)} \) of morphisms of \( O_X \)-coherent sheaves, which commute with all the \( \sigma_i \) and \( \tau_i \).

In particular, \( \text{Strat}(X,0) = \text{Strat}(X) \).

**Example 3.3.** We now give an example of a non-trivial 1-stratified bundle on \( X = \mathbb{A}_k^1 = \text{Spec}(k[[x]]) \). Thus \( X^{(i)} = \text{Spec}(k[x_{i+1}]) \) where the relative Frobenius \( X^{(i)} \to X^{(i+1)} \) is induced by \( x_{i+1} \to x_{i+1}^p \). For simplicity let us assume \( p = \text{char}(k) = 2 \). Let \( V \) be a 2-dimensional vector space over \( k \) with basis \( e_1, e_2 \). Define

\[
E^{(i)} = O_{X^{(i)}} \otimes_k V \quad \forall \ i \geq 0
\]
and
\[ \sigma^{(i)} : E^{(i)} \to F^{(i)} \ast E^{(i+1)}, \quad i \geq 1 \]
to be the isomorphism induced by the identity on \( V \). We define
\[ \sigma^{(0)} : F^{(-1,0)} \ast E^{(0)} \to F^{(-1,1)} \ast E^{(1)} \]
to be the isomorphism defined by sending
\[ e_1 \to e_1, \quad e_2 \to x_1 e_1 + e_2. \]
We claim that the \(-1\)-stratified bundle thus defined is not isomorphic to the trivial stratified bundle of rank 2. If indeed this were the case, then we would have a \( k[x] \)-module automorphism \( \phi : k[x] \otimes_k V \to k[x] \otimes_k V \), such that
\[ \phi \otimes_{k[x]} k[x] = \sigma^{(0)}. \]
This is impossible since \( x \) is not contained in \( k[x] \). It can be shown (see (4.3)) that this \(-1\)-stratified bundle “arises” from the non-trivial \( \alpha_p \)-torsor on \( A^1_k \) defined by the relative Frobenius of \( A^1_k \).

**Theorem 3.4.** The notations are as in 3.1.

1) For every integer \( t \geq 0 \), \( Strat(X,t) \) is a \( k \)-linear, abelian, rigid, tensor category.

2) The functor
\[ (+) : Strat(X,t) \subset Strat(X,t+1) \]
\[ (E_i, \sigma_0, \sigma_i, i \geq 1) \mapsto (E_i, F^{(-t-1)} \ast \sigma_0, \sigma_i, i \geq 1), \]
induces a full faithful embedding of \( k \)-linear, abelian, rigid, tensor categories.

3) If \( x \in X(k) \) is a rational point, the functor
\[ \omega_x : Strat(X,t) \to Vec_k \]
\[ (E^{(i)}, \sigma^{(i)}) \mapsto E_0|_x \]
is a fiber functor, which makes \( (Strat(X,t), \omega_x) \) a Tannaka category.

**Proof.** We show 1). Since \( Strat(X,0) = Strat(X) \), we assume \( t > 0 \). If \( (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \) is an object in \( Strat(X,t) \), then \( (E^{(i)}, F^{(-t-1)} \ast \sigma_0, \sigma_i, i \geq 1) \) is an object \( Ver(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in Strat(X^{(1)}) \). Since \( E^{(i)} \) is locally free, by the isomorphism \( \sigma^{(0)}, F^{(-1,0)} \ast E^{(0)} \) is locally free. Since \( X \) is smooth, the relative Frobenius is flat, thus by flat descent, \( E^{(0)} \) is locally free as well. So \( Strat(X) \) is rigid and monoidal. On the other hand,

\[ Hom((D^{(i)}, \tau^{(i)}, i \in \mathbb{N}),(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})) \]
\[ \subset Hom(Ver(D^{(i)}, \tau^{(i)}, i \in \mathbb{N}),(E^{(i)}, \sigma^{(i)}, i \in \mathbb{N})) \]
and is obviously a \( k \)-vector space. So the Hom-sets are finite dimensional \( k \)-vector spaces. Moreover, any morphism \( f = (f^{(i)}, i \in \mathbb{N}) \) is such that \( f^i, i \geq 1 \)
is a morphism of vector bundles. Thus by the ismorphisms $\tau_{(0)}, \sigma^0,$ Ker, Im and Coker of $f^{(0)}$ are pulled back to vector bundles on $X^{(-t)}$ via $F^{(-t,0)}$, thus by flat descent again, there are vector bundles on $X$. We conclude that $\text{Strat}(X, t)$ is an abelian category. This shows 1).

2) follows immediately from the factorization of (3.1) through (+).

We show 3): the point $x \in X(k)$ maps to $x^{(1)} \in X^{(1)}(k)$, and the map $x \to x^{(1)}$ is the identity on the residue fields $k(x) = k(x^{(1)}) = k$. If $0 \to A \to B \to C \to 0$ is an exact sequence in $\text{Strat}(X, t)$, then $0 \to \text{Ver}(A) \to \text{Ver}(B) \to \text{Ver}(C) \to 0$ is an exact sequence in $\text{Strat}(X^{(1)})$, thus $0 \to \omega_{x^{(1)}}(\text{Ver}(A)) \to \omega_{x^{(1)}}(\text{Ver}(B)) \to \omega_{x^{(1)}}(\text{Ver}(C)) \to 0$ is an exact sequence in $\text{Vec}_k$. But

$$\omega_{x^{(1)}}(\text{Ver}(A)) = \omega_x(A).$$

This shows that $\omega_x$ is exact. Furthermore, $\omega_x$ is obviously $k$-linear and compatible with the tensor structure. This finishes the proof.

Corollary 3.5. Let the notations be as in Theorem 3.4. The category

$$\text{Strat}(X, \infty) = \lim_{\longrightarrow, t \in \mathbb{N}} \text{Strat}(X, t)$$

is a $k$-linear, abelian, rigid tensor category, on which, if $X$ has a rational point $x \in X(k)$, the functor $\omega_x$ is a fiber functor.

Definition 3.6. The notations are as in Theorem 3.4.

1) We define $\pi_{\text{alg}}(X, x)$ to be the Tannaka $k$-group scheme $\text{Aut}^\otimes(\omega_x)$ of $(\text{Strat}(X), \omega_x)$.

2) We define $\pi_{\text{alg}, \infty}(X, x)$ to be the Tannaka $k$-group scheme $\text{Aut}^\otimes(\omega_x)$ of $(\text{Strat}(X, \infty), \omega_x)$.

The functor $ (+ ) : \text{Strat}(X) \to \text{Strat}(X, \infty)$ defines the homomorphism

$$ (+ )^* : \pi_{\text{alg}, \infty}(X, x) \to \pi_{\text{alg}}(X, x).$$

Lemma 3.7. The homomorphism $(+)^*$ in (3.3) is faithfully flat.

Proof. We apply [2, Proposition 2.21]. As $(+)$ is fully faithful, the lemma is equivalent to saying that if $A$ is an object on $\text{Strat}(X)$, and $B \subset (+)A$ is a subobject in $\text{Strat}(X, \infty)$, then there is a subobject $B' \subset A$ in $\text{Strat}(X)$ such that $B = (+)B'$. One has that $\text{Ver}(B) \subset \text{Ver}(A)$ is a subobject in $\text{Strat}(X^{(1)})$, thus $F^{(0)*}B^{(1)} \subset A^{(0)}$ is a subvector bundle with the property that $F^{(-t,0)*}F^{(0)*}B^{(1)} = F^{(-t,1)*}B^{(1)} = F^{(-t,0)*}B^{(0)}$. Thus $B' = (F^{(0)*}B^{(1)}, B^{(i)}, i \geq 1, F^{(0)*}, \sigma^{(i)}, i \geq 1) \subset A$ is a subobject of $A$ such that $(+)^*B' = B$. This finishes the proof.

4. Comparison of $\pi_{\text{alg}, \infty}(X, x)$ with $\pi_1^N(X, x)$

In order to achieve the comparison, we start with a construction.
Construction 4.1. The notations are as in 3.1, and \( x \in X(\mathbb{k}) \) is a rational point. Let \( (h : Y \to X, G, y) \) be an object of \( N(X, x) \). Using this object, we construct a tensor functor
\[
h^* : \text{Rep}_{\mathbb{k}}(G) \to \text{Strat}(X, \infty)
\]
together with a factorization of functors
\[
\begin{array}{ccc}
\text{Rep}_{\mathbb{k}}(G) & \xrightarrow{h^*} & \text{Strat}(X, \infty) \\
\downarrow F_G & & \downarrow \omega_x \\
\text{Vec}_{\mathbb{k}}
\end{array}
\]
Here \( F_G : \text{Rep}_{\mathbb{k}}(G) \to \text{Vec}_k \) is the forgetful functor.

Recall that if \( G \) is a finite \( \mathbb{k} \)-group-scheme, there is an exact sequence of finite \( \mathbb{k} \)-group schemes
\[
1 \to G_0 \to G \to G_{\text{ét}} \to 1,
\]
where \( G_0 \) is the 1-component of \( G \) and \( G_{\text{ét}} \) is étale. Furthermore, as \( k \) is perfect, \( G_{\text{red}} \subset G \) is a closed subgroup-scheme and the composite \( G_{\text{red}} \to G \to G_{\text{ét}} \) is an isomorphism. Thus \( t \) yields on \( G \) the structure of a semi-direct product of \( G_{\text{ét}} \) by \( G_0 \). The construction of \( h^* \) will be such that the image of \( h^* \) is contained in \( \text{Strat}(X, t) \), where \( t \) is a natural number such that the image of the \( \mathbb{k} \)-group-scheme homomorphism \( G^{(-t)} \to G \) is equal to \( G_{\text{ét}} \).

Let \( V \) be a finite dimensional \( \mathbb{k} \)-representation of \( G \). We set
\[
E^{(0)} = h^\#(V).
\]
For \( i \in \mathbb{N} \setminus \{0\} \), the relative Frobenius is an isomorphism of the étale \( \mathbb{k} \)-group-schemes
\[
F^{(0,i)} : G_{\text{ét}} \xrightarrow{\sim} G_{\text{ét}}^{(i)}.
\]
Thus \( F^{(0,i)} \circ F^{(0,i)-1} : G_{\text{ét}}^{(i)} \subset G \) is a closed embedding and composing with it defines a \( G_{\text{ét}}^{(i)} \)-action on \( V \). Since \( h : Y \to X \) is a \( G \)-torsor, for \( i \geq 0 \), \( h^{(i)} : Y^{(i)} \to X^{(i)} \) is also a \( G^{(i)} \)-torsor. Let \( h_{\text{ét}}^{(i)} : Y_{\text{ét}}^{(i)} \to X^{(i)} \) be the induced \( G_{\text{ét}}^{(i)} \)-torsor obtained by moding out by \( G_0^{(i)} \). We define
\[
E^{(i)} = (h_{\text{ét}}^{(i)})^\#(V).
\]
One has
\[
\sigma^{(i)} : E^{(i)} \xrightarrow{\sim} F^{(i)*}E^{(i+1)}, \quad i \in \mathbb{N} \setminus \{0\}.
\]
The object \( h^*(V) \in \text{Strat}(X, t) \) which we wish to construct will have the property
\[
\text{Ver}(h^*(V)) = (E^{(i)}, \sigma^{(i)}, i \geq 1).
\]
It remains to define \( \sigma^{(0)} \). By definition,
\[
F^{(0)*}E^{(1)} = (h_{\text{ét}}^{(0)})^\#(V) = (h_{\text{ét}})^\#(V).
\]
Let $t$ be a natural number such that the image of $G^{(-t)} \to G$ is equal to $G_{\text{ét}}$. One has the following commutative diagram of $k$-varieties.

\[(4.8)\]

\[
\begin{array}{c}
Y^{(-t)} \xrightarrow{F^{(-t,0)}} Y \\
\downarrow h^{(-t)} \downarrow \quad \exists ! \lambda \downarrow \quad \downarrow h \\
Y^{(-t)}_{\text{ét}} \xrightarrow{F^{(-t,0)}} Y_{\text{ét}} \\
\downarrow h \\
X^{(-t)} \xrightarrow{F^{(-t,0)}} X
\end{array}
\]

The morphism $F^{(-t,0)} : Y^{(-t)} \to Y$ is equivariant under $F^{(-t,0)} : G^{(-t)} \to G$. Likewise, the morphism $F^{(-t,0)} : Y^{(-t)}_{\text{ét}} \to Y_{\text{ét}}$ is equivariant under $F^{(-t,0)} : G^{(-t)}_{\text{ét}} \to G^{(-t)}$. The commutativity of the diagram implies that

\[(4.9)\]

\[
\lambda^*(\mathcal{O}_Y \otimes_k V) = F^{(-t,0)*}(\mathcal{O}_{Y_{\text{ét}}} \otimes_k V) = F^{(-t,1)*}(\mathcal{O}_{Y^{(-1)}} \otimes_k V)
\]
equivariantly for the action of $G^{(-t)}_\text{ét}$. Thus

\[(4.10)\]

\[
(h^{(-t)}_{\text{ét}})^\#(V) = F^{(-t,0)*}E^{(0)} = F^{(-t,1)*}E^{(1)}.
\]

We define $\sigma^{(i)} : F^{(-t,0)*}E^{(0)} = F^{(-t,1)*}E^{(1)}$ to be the equality of (4.10).

Thus, starting with $V \in \text{Rep}_k(G)$, we have constructed an object $h^*(V) = (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \in \text{Strat}(X, t)$. Clearly, any $\phi \in \text{Hom}_{\text{Rep}_k(G)}(V, W)$ induces $h^*(\phi) \in \text{Hom}_{\text{Strat}(X, t)}(h^*(V), h^*(W))$. This defines the functor

\[(4.11)\]

\[
h^* : \text{Rep}_k(G) \to \text{Strat}(X, \infty).
\]

by composing with $(+)$. Moreover, one has

\[(4.12)\]

\[
h^*(V)_x = (\mathcal{O}_Y \otimes_k V)_y = V.
\]

This shows the commutativity of (4.1).

**Remark 4.2.** In the above construction we use the fact that for a finite flat group scheme $G$ over a perfect field $k$, the epimorphism $G \to G_{\text{ét}}$ admits a section (necessarily unique). In other words $G_{\text{ét}}$ can be canonically thought of as a subgroup scheme of $G$ via the identification $G_{\text{red}} = G_{\text{ét}}$. When $k$ is not a perfect field, $G_{\text{red}}$ may not be a subgroup scheme, (for example, $G = \text{Spec } k[t]/(t^2 - atp)$, $a \in k \setminus k^p$, see [8, Chapter III, Exercise (3.2)]), and the above construction of $h^*$ does not make sense. This is the reason why we assume throughout $k$ to be perfect. We thank Nguyễn Duy Tấn for this important remark.

**Example 4.3.** Let $p = \text{char}(k) = 2$ for simplicity and let $G = \alpha_2 = \text{Spec } (k[t]/t^2)$. Let $X = \mathbb{A}_k^1 = \text{Spec } (k[x])$. Let $P = \text{Spec } (k[u])$, and $h : P \to X$ be the relative Frobenius defined by $x \to u^2$. Then $h$ is a $G$-torsor. Thus by Construction (4.1), one has a functor

\[
h^* : \text{Rep}_k(G) \to \text{Strat}(X, -1).
\]
We compute now that \( h^*(k[G]) \) is nothing but the \(-1\)-stratified bundle defined in Example 3.3. Here \( k[G] = k[v]/(v^2) \) is the regular representation of \( G \). As in Example (3.3), let \( X(i) = k[x_i] \). Let \( h^*(k[G]) = (E(i), \sigma(i), i \in \mathbb{N}) \). As all schemes are affine, we confuse coherent sheaves with corresponding modules. Since \( G_{\text{et}} \) is trivial, by definition of \( h^* \) we see that

\[
E(i) = k[x_i] \otimes_k k[v]/(v^2) \quad \forall \ i \geq 1
\]

with

\[
\sigma(i) : E(i) \rightarrow F^{(i)} E^{(i+1)} \quad i \geq 1
\]

induced by the identity map on \( k[v]/(v^2) \). Then \( E(0) \) is by definition the \( k[x] \)-module of invariants of \( k[u] \otimes_k k[v]/(v^2) \), where the action of \( G = \text{Spec} k[t]/(t^2) \) is defined by

\[
u \mapsto u + t, \quad v \mapsto v + t.
\]

Since \((u + v)^2 = u^2 = x\), one has \( E(0) = k[x] \cdot 1 \oplus k[x] \cdot (u + v) \). On \( P \) we have an identification

\[
h^* E(0) = k[u] \otimes_k k[v]/(v^2)
\]

defined by \( \tau : 1 \mapsto 1 \otimes 1, u + v \mapsto u \otimes 1 + 1 \otimes v \). The map \( \sigma(0) \) is nothing but the pull back of \( \tau \) via the isomorphism \( X(-1) \rightarrow P \) defined by

\[
k[u] \rightarrow k[x_{-1}], \quad u \mapsto x_{-1}.
\]

We thus see that

\[
\sigma(0) : k[x_{-1}] \cdot 1 \oplus k[x_1] \cdot (u + v) 
\rightarrow k[x_{-1}] \otimes_k k[v]/(v^2)
\]

is defined by \( 1 \mapsto 1 \otimes (u + v) \mapsto u \otimes 1 + 1 \otimes v \). It is then an elementary exercise to see that the stratified bundle \( h^*(k[G]) \) is isomorphic to the \(-1\) stratified bundle defined in Example 3.3.

**Lemma 4.4.** The functor \( h^* \) defined in (4.11) is \( k \)-linear, exact, compatible with the tensor structure and faithful.

**Proof.** As already recalled in the Properties 2.3 1), faithfulness follows from the remaining properties. On the other hand, \( k \)-linearity, and compatibility with the tensor structures are straightforward. Exactness is proven as using Ver as in Theorem 3.4 3). Indeed, \( \text{Ver} \circ h^* \) with values in \( \text{Strat}(X^{(1)}) \) is obviously exact, while a sequence in \( \text{Strat}(X, \infty) \) is exact if and only if it remains exact after applying Ver.

If \((h_i : Y_i \rightarrow X, G_i, y_i)\) are objects in \( \mathbb{N}(X, x) \) for \( i = 1, 2 \) and \((\psi : Y_1 \rightarrow Y_2, \phi : G_1 \rightarrow G_2, y_1 \rightarrow y_2)\) is a morphism in \( \mathbb{N}(X, x) \), then Property 2.3 3) implies that \( h_2^* = h_1^* \circ \phi^* \). On the other hand, the projective system of \( \phi \) in \( \mathbb{N}(X, x) \) induces an inductive system \( \lim_{\rightarrow} \mathbb{N}(X, x), \phi^* \text{Rep}_k(G) \) which is a Tannaka category, with the forgetful functor \( F_G \) as the fiber functor. The Tannaka \( k \)-group-scheme
\[ \text{Aut}^\otimes(F_G) \text{ is simply } \varprojlim_{N(X,x),\phi} G, \text{ which is precisely Nori's fundamental group-scheme } \pi^N(X,x). \text{ As in addition the construction is obviously functorial in } h, \text{ we conclude:} \]

**Theorem 4.5.** Let the notations be as in Construction 4.1. The functor \( h^* \) defined in (4.11) for one object \((h : Y \to X, G, y) \) of \( N(X,x) \) induces a functor of Tannakian categories
\[ h^* : \left( \varprojlim_{N(X,x),\phi} \text{Rep}_k(G), F_G \right) \to \left( \text{Strat}(X,\infty), \omega_x \right), \]
and the Tannaka-dual homomorphism of \( k \)-group-schemes
\[ h^*\vee : \pi_{\text{alg},\infty}(X,x) \to \pi^N(X,x) \]
which is functorial in \( X \).

The aim of the rest of the section is to show that the homomorphism \( h^*\vee \) is faithfully flat and induces the profinite quotient homomorphism.

**Proposition 4.6.** Let \((Y \xrightarrow{h} X, G, y) \) be an object of \( N(X,x) \). The following conditions are equivalent.

1) The induced map \( \pi_{\text{alg},\infty}(X,x) \to G \) (see (4.11)) is an epimorphism.
2) The induced map \( \pi^N(X,x) \to G \) is an epimorphism.
3) The functor \( h^* \) in (4.11) is fully faithful and its image is closed under taking subquotients in \( \text{Strat}(X,\infty) \).

**Proof.** The equivalence \((1) \Leftrightarrow (3) \) follows from [2, Proposition 2.21]. Moreover, since by construction, the map \( \pi_{\text{alg},\infty}(X,x) \to G \) factors through \( \pi^N(X,x) \), \((1) \Rightarrow (2) \) is obvious.

We show \((2) \Rightarrow (3) \). Let \( \mathcal{C} \) denote the full subcategory of \( \text{Strat}(X,\infty) \) generated by subquotients in \( \text{Strat}(X,\infty) \) of objects which are in the image of \( h^* : \text{Rep}_k(G) \to \text{Strat}(X,\infty) \). The property 3) is equivalent to saying that \( h^* : \text{Rep}_k(G) \to \mathcal{C} \) is an equivalence of categories. By standard Tannaka formalism, \( \mathcal{C} \) itself is a \( k \)-linear, abelian, rigid tensor subcategory of \( \text{Strat}(X,\infty) \), thus \((\mathcal{C}, \rho_x) \) is a Tannaka subcategory of \( (\text{Strat}(X,\infty), \omega_x) \), where \( \rho_x = \omega_x|_{\mathcal{C}} \).

We show now that \( h^* : \text{Rep}_k(G) \to \mathcal{C} \) is an equivalence of categories. Let \( H = \text{Aut}(\rho_x) \) be the Tannaka \( k \)-group-scheme of \((\mathcal{C}, \rho_x) \). We claim that the induced homomorphism \( H \to G \) is a closed immersion. This is equivalent ([2, Proposition 2.21]) to saying that every object of \( \mathcal{C} \) is a subquotient in \( \mathcal{C} \) of an object in \( h^*(\text{Rep}_k(G)) \), which is true since by definition of \( \mathcal{C} \), a subquotient in \( \mathcal{C} \) of objects in \( h^*(\text{Rep}_k(G)) \) is the same as a subquotient in \( \text{Strat}(X,\infty) \) of objects in \( h^*(\text{Rep}_k(G)) \). We conclude in particular that \( H \) is a finite group scheme.

The fiber functor (in the sense of Deligne [1, 1.9], see Properties 2.3 1)) \( \omega_X : \text{Strat}(X,\infty) \to \text{Coh}(X) \) defined by \((E_i, \sigma_i, i \in \mathbb{N}) \mapsto E_0 \) restricts to the fiber
functor $\rho_X : C \to \text{Coh}(X)$. One has a commutative diagram of functors

\[
\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{h^*} & C \\
\downarrow_{h^\#} & & \downarrow_{\rho_X} \\
\text{Coh}(X) & & \\
\end{array}
\]

and, upon applying $i_x$, (4.1) implies that $i_x \circ h^\# = F_G$. By applying Theorem 2.4, we obtain a morphism

\[
(h_H : Y_H \to X, H, y_H) \to (h : Y \to X, G, y)
\]

in $\mathcal{N}(X, x)$. This in turn induces a factorization of $\pi^N(X, x) \to G$ as

\[
\begin{array}{ccc}
\pi^N(X, x) & \to & G \\
\downarrow & & \downarrow \\
H
\end{array}
\]

But $\pi^N(X, x) \to G$ is assumed to be an epimorphism. Thus $H \to G$ must be an epimorphism. Since it is also a closed immersion, we conclude

\[
H \xrightarrow{\cong} G.
\]

In other words

\[
h^* : \text{Rep}_k(G) \xrightarrow{\cong} C.
\]

This finishes the proof. \hfill \Box

Recall that $k$ is perfect.

**Lemma 4.7.** Let $G$ be a finite $k$-group-scheme, let $h : Y \to X$ be a $G$-torsor. Then the following conditions are equivalent

(i) $h$ admits a reduction (necessarily unique) of structure group to $G_{\text{red}} = G_{\text{ét}} \subset G$.

(ii) For every natural number $n$, there is a $G$-torsor $h_n : Y_n \to X^{(n)}$ which pulls back via $X \xrightarrow{F^{(0,n)}} X^{(n)}$ to $h$.

**Proof.** We show (i) \implies (ii). Let $h_{\text{ét}} : Y_{\text{ét}} \to X$ be a $G_{\text{ét}}$-torsor which is a reduction of structure of $h$ for the closed embedding $G_{\text{ét}} \subset G$. Thus $Y = Y_{\text{ét}} \times_{G_{\text{ét}}} G$. The isomorphism (4.3) induces a cartesian diagram

\[
\begin{array}{ccc}
Y_{\text{ét}} & \xrightarrow{F^{(0,n)}} & (Y_{\text{ét}})^{(n)} \\
\downarrow_{h_{\text{ét}}} & \xrightarrow{\Box} & \downarrow_{(h_{\text{ét}})^{(n)}} \\
X & \xrightarrow{F^{(0,n)}} & X^{(n)}
\end{array}
\]

We set $Y_n = (Y_{\text{ét}})^{(n)} \times_{G_{\text{ét}}} G$, $h_n = (h_{\text{ét}})^{(n)} \times_{G_{\text{ét}}} G$. 

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We show (ii) ⇒ (i). For a large enough positive integer \(n\), we consider the commutative diagram similar to (4.8):

\[
\begin{array}{ccc}
Y_n^{(-n)} & \xrightarrow{\gamma} & Y_n \\
\downarrow & & \downarrow \\
\left(Y_n^{(-n)}\right)_{\text{ét}} & \xrightarrow{h_n} & X^{(n)}
\end{array}
\]

We explain the terms in the diagram: with Notations 3.1, one has \(Y_n^{(-n)} = Y_n \otimes_F k\), thus \(h_n\) induces \(h_n \otimes_F k : Y_n^{(-n)} \to (X^{(n)})^{(-n)} = X\), which is a principal \(G^{(-n)}\) bundle. The top horizontal map \(\gamma\) is equivariant with respect to \(G^{(-n)} \xrightarrow{F^{(-n,0)}} G\). Since \(n\) is large, the image of \(G^{(-n)} \to G\) is precisely \(G_{\text{ét}} \subset G\). Therefore, \(\gamma\) factors uniquely through \(\left(Y_n^{(-n)}\right)_{\text{ét}}.\) Via the identification \(\left(Y_n^{(-n)}\right)_{\text{ét}} \xrightarrow{F^{(-n,0)}} G_{\text{ét}},\) the morphism \(\left(Y_n^{(-n)}\right)_{\text{ét}} \to X^{(n)}\) is a \(G_{\text{ét}}\)-torsor. The above commutative diagram shows the existence of an equivariant map \(\left(Y_n^{(-n)}\right)_{\text{ét}} \to Y_n \times_{X^{(n)}} X\). We conclude that the \(G\)-torsor \(Y_n \times_{X^{(n)}} X \to X\) has a reduction of structure group to \(G_{\text{ét}}\).

\[\square\]

**Theorem 4.8.** Let the notations are as in 3.1 and let \(x \in X(k)\) be a rational point. Then the homomorphism \(h^*: \pi_{\text{alg}, \infty}(X, x) \to \pi^N(X, x)\) is the profinite quotient homomorphism.

**Proof.** We have already shown in Proposition 4.6 that the homomorphism \(h^*\) is surjective. In order to show that \(h^*\) is the profinite completion homomorphism, we need to show that any epimorphism

\[
\phi : \pi_{\text{alg}, \infty}(X, x) \to G,
\]

where \(G\) is a \(k\)-finite group-scheme, factors through \(\pi^N(X, x)\). This is equivalent to showing that given any finite Tannaka subcategory \(\mathcal{T} \subset \text{Strat}(X, \infty)\), i.e. with \(G = \text{Aut}^{\otimes}(\mathcal{T}, \rho_x)\) finite, where \(\rho_x = \omega_x|_{\mathcal{T}}\), there exists an object \((h : Y \to X, G, y)\) in \(\mathcal{N}(X, x)\) such that \(\mathcal{T}\) is the image of the functor \(h^*\) constructed in (4.11). We do this in two steps.

**Step(1):** For each \(n \geq 0\), we consider the fiber functor

\[
\omega_{X^{(n)}} : \text{Strat}(X, \infty) \to \text{Coh}(X^{(n)}), \quad (E^{(i)}, \sigma^{(i)}, i \in \mathbb{N}) \mapsto E^{(n)}.
\]

It restricts to a fiber functor

\[
P_n : \mathcal{T} \to \text{Coh}(X^{(n)}).
\]
Let $\delta : \text{Rep}_k(G) \to \mathcal{T}$ be the equivalence of Tannaka categories defined by the inverse functor to the equivalence induced by $\rho_x$. Consider $P_n \circ \delta : \text{Rep}_k(G) \to \text{Coh}(X^{(n)})$.

By Theorem 2.4, we obtain $G$-torsors $(h_n : Y_n \to X^{(n)})$ for every $n$, such that

$$h_n^# = P_n \circ \delta. \quad (4.21)$$

Since the $G$-torsors thus obtained are unique upto isomorphism, the equality

$$P_n = F^{(n)*} \circ P_{n+1}, \forall \ n \geq 1 \quad (4.22)$$

implies that the torsor $h_{n+1}$ pulls back to $h_n$. Thus by Lemma 4.7, each $Y_n$ admits a reduction of structure group to $G_\text{ét} \subset G$ for all $n \geq 1$.

Step(2): Composing $\delta$ with the inclusion $\mathcal{T} \hookrightarrow \text{Strat}(X, \infty)$ we obtain a functor from $\text{Rep}_k(G) \to \text{Strat}(X, \infty)$. We also have the functor $h_0^* : \text{Rep}_k(G) \to \text{Strat}(X, \infty)$ (see (4.11)) defined by the $G$-torsor $h_0 : Y_0 \to X$. In order to finish the proof we have to show that these two functors coincide. This is equivalent to saying that the following diagram of functors commutes.

$$
\begin{array}{ccc}
\text{Rep}_k(G) & \xrightarrow{\delta} & \mathcal{T} \\
\downarrow h_0 & & \downarrow \text{incl.} \\
\text{Strat}(X, \infty)
\end{array}
$$

Let $V$ be an object of $\text{Rep}_k(G)$. We will show that there is an isomorphism between $i(V)$ and $h_0^*(V)$, which is functorial in $V$. This will finish the proof. Let $\delta(V) = (\delta(V)^{(n)}, \sigma^{(n)}, n \in \mathbb{N})$ and $h_0^*(V) = (E^{(n)}, \tau^{(n)}, n \in \mathbb{N})$.

We let $h_{n, \text{ét}} : Y_{n, \text{ét}} \to X^{(n)}$ be the $G_\text{ét}$-torsor induced by $h_n$ for $n \geq 1$. Note that by construction 4.1 of the functor $h_0^*$, one has

$$E^{(n)} = h_{n, \text{ét}}^#(V) \forall \ n \geq 1 \quad \text{and} \quad E^{(0)} = h_0^#(V). \quad (4.23)$$

On the other hand, by definition of the functors $P_n$,

$$P_n(i(V)) = i(V)^{(n)}$$

Thus by (4.21), one has

$$i(V)^{(n)} = h_n^#(V) \forall \ n \geq 0. \quad (4.24)$$

But as explained before, for every $n \geq 1$, $h_n : Y_n \to X^{(n)}$ admits a reduction of structure group to $G_\text{ét}$. Thus by Proposition 2.3(3),

$$h_n^#(V) = h_{n, \text{ét}}^#(V) \forall \ n \geq 1. \quad (4.25)$$

Thus we conclude

$$i(V) = h_0^*(V). \quad (4.26)$$

$\square$
If $\mathcal{T}$ is any $k$-linear, abelian, rigid tensor category, together with a neutral fiber functor $\omega : \mathcal{T} \to \text{Vec}_k$, we denote by $\mathcal{T}^{\text{fin}}$ the full subcategory spanned by objects $E$ which have the property that the full tensor subcategory $\langle E \rangle \subset \mathcal{T}$ spanned by $E$ and its dual $E^\vee$ has a finite Tannaka group scheme $\text{Aut}^\circ(\langle E \rangle, \omega|_{\langle E \rangle})$. So by construction, Theorem 4.8 has the following consequence:

**Corollary 4.9.** With the notations as in Theorem 4.8, the full embedding

$$\text{Strat}(X, x)^{\text{fin}} \subset \text{Strat}(X, x)$$

induces via the fiber functor $\omega_x$ the quotient homomorphism

$$\pi^{\text{alg, }\infty}(X_x) \to \pi^N(X, x).$$

**References**


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