Geometry and arithmetic of non-rigid families of Calabi-Yau 3-folds; questions and examples

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Abstract. We speculate about the structure of maximal product subvarieties of moduli stacks of Calabi-Yau manifolds. We discuss an example of a family of quintic hypersurfaces in $\mathbb{P}^4$, parameterized by the product of two ball quotients, one of dimension two, the second one of dimension 12.

Let $\mathcal{M}_h(\mathbb{C})$ denote the set of isomorphism classes of minimal polarized manifolds F with fixed Hilbert polynomial h, and let $\mathcal{M}_h$ be the corresponding moduli functor, i.e.

$$\mathcal{M}_h(U) = \left\{ \begin{array}{c} (f : V \to U, \mathcal{L}); f \text{ smooth and} \\ (f^{-1}(u), \mathcal{L}|_{f^{-1}}(u)) \in \mathcal{M}_h(\mathbb{C}), \text{ for all } u \in U \end{array} \right\}$$

There exists a quasi-projective coarse moduli scheme M_h for $\mathcal{M}_h$. Fixing a projective manifold $\bar{U}$ and the complement U of a normal crossing divisor, we want to consider

$$\mathbf{H} = \left\{ \begin{array}{c} \varphi : (\bar{U}, U) \to (\overline{M}_h, M_h) \quad \text{induced} \\ \text{by polarized families } f : X \to U \end{array} \right\}.$$

Since M_h is just a coarse moduli scheme, it is not clear whether $\mathbf{H}$ has a scheme structure. However, by [**6**], if all $F \in \mathcal{M}(\mathbb{C})$ admit a locally injective Torelli map, there exists a fine moduli scheme M_h^N with a level structure N and étale over M_h. By abuse of notations, we will replace $\mathcal{M}_h$ by the moduli functor of polarized manifolds with a level N structure, and fix some compactification $\overline{M}_h$. Then $\mathbf{H}$ parameterizes all morphisms from

$$\varphi : (\bar{U}, U) \longrightarrow (\overline{M}_h, M_h),$$

hence it is a scheme. Moreover there exists a universal family $f : X \to \mathbf{H} \times U$. As Kovács, Bedulev-Viehweg, Oguiso-Viehweg, and Viehweg-Zuo have shown $\mathbf{H}$ is of finite type.

Definition 1. $\varphi : U \to M_h$ called rigid if the component of $\mathbf{H}$ containing φ is zero-dimensional.

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Problem 2. Study the geometry of $\mathbf{H}$ and the arithmetic properties (for example the Mumford-Tate group) of the universal family $f : X \longrightarrow \mathbf{H} \times U$.

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1. Splitting of variations of Hodge structures

Let us start by recalling some of the properties of complex polarized variations of Hodge structures, and of families of Calabi-Yau manifolds.

Proposition 3. *If $\mathbb{V}$ is an irreducible complex polarized variation of Hodge structures over $U_1 \times \cdots \times U_\ell$ then*

$$\mathbb{V} = p_1^*(\mathbb{V}_1) \otimes \cdots \otimes p_\ell^*(\mathbb{V}_\ell),$$

for complex polarized variations of Hodge structures $\mathbb{V}_i$ over U_i.

Proof. The proof (see [**8**], Section 3, for the details) uses Schur's Lemma and Deligne's semi-simplicity of complex polarized variations of Hodge structures. □

2. Products in moduli stacks of Calabi-Yau manifolds

Since Calabi-Yau manifolds are un-obstructed, the fine moduli scheme M_h is smooth, and we choose a smooth projective compactification $\overline{M}_h$ such that $\overline{M}_h \setminus M_h$ is a normal crossing divisor. Let $g : \mathcal{X} \to M_h$ be the universal family. We will assume moreover, that the local monodromies of $R^m g_* \mathbb{C}_{\mathcal{X}}$ around the components of $\overline{M}_h \setminus M_h$ are uni-potent, where $m = \deg(h)$ is the dimension of the fibres.
Consider a smooth family

$$f : X \longrightarrow U_1 \times \cdots \times U_\ell = U$$

of Calabi-Yau m-folds, such that $\varphi : U \to M_h$ is generically finite. We assume that the factors U_i are non singular, and that $\dim(U_i) > 0$. Let $\mathbb{V} \subset R^m f_*(\mathbb{C}_X)$ be the irreducible sub variation of Hodge structures with system of Hodge bundles

$$\bigoplus_{p+q=m} E^{p,q}$$

such that $E^{m,0} = f_* \Omega^m_{X/U}$.

Fact 4. The Kodaira-Spencer map is injective and factors through

$$d\varphi : T_U \longrightarrow E^{m-1,1} \otimes E^{m,0^{-1}} \subset \varphi^* T_{M_h}.$$

By Proposition 3 one has a decomposition $\mathbb{V} = \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_\ell$. Let us write

$$\bigoplus_{p+q=m} F_i^{p,q}$$

for the system of Hodge bundles of $\mathbb{V}_i$, and $\varphi_i : U \to U_i \to \mathcal{D}_i$ for the corresponding period map. Then

$$d\varphi_i : T_{U_i} \longrightarrow F_i^{m_i-1,1} \otimes F_i^{m_i,0^{-1}} \subset \varphi_i^* T_{\mathcal{D}_i},$$

for $1 \leq i \leq \ell$. As in [**8**], 3.5, a comparison of Hodge bundles on both sides gives rise to

Proposition 5.

i. *The cup-product*

$$\bigoplus_{1 \leq i_1 < \cdots < i_k \leq \ell} T_{U_{i_1}} \otimes \cdots \otimes T_{U_{i_k}} \longrightarrow R^k f_* T^k_{X/U}$$

is injective for $1 \leq k \leq \ell$.

ii. *If* $\varphi : U_1 \times \cdots \times U_\ell \to M_h$ *is an embedding and if* $\ell = m$ *is the dimension of the fibres of* f *then* $U_1 \times \cdots \times U_\ell$ *is a product of curves, and uniformized by* $\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_\ell$ *over an algebraic number field.*

Question 6. When will $U_1 \times \cdots \times U_m$ be a product of Shimura curves?

Remark 7. A similar argument shows that part i) of Proposition 5 also holds true for moduli stacks of hyper-surfaces in $\mathbb{P}^n$ (see [**8**], 3.5 b)).

Question 8. Does Proposition 5, 1) hold true for moduli stacks of minimal polarized manifolds?

If $U_1 \times \cdots \times U_\ell$ maps generically finite to a moduli stack $\mathcal{M}_h$ of minimal polarized manifolds, then it has been shown in [**7**], Corollary 6.4, that

$$\ell \leq m = \deg(h).$$

Question 9. Can one improve this bound for certain moduli stacks and, fixing ℓ, what are optimal bounds for the dimensions of the U_i?

Since we assumed M_h to be a fine moduli space, deformations of the morphism $\varphi : U \to M_h$ correspond to deformations of the family $f : X \to U$. If one assumes that U has a compactification $\bar{U}$ such that φ extends to $\varphi : \bar{U} \to \overline{M}_h$, in such a way that the pre-image of $S = \overline{M}_h \setminus M_h$ remains a reduced normal crossing divisor, the first order deformations of the first type are classified by

$$H^0(\bar{U}, \varphi^* T_{\overline{M}_h}(-\log S)).$$

Proposition 10. *Assume in addition that* f *extends to a proper morphism* $f : \bar{X} \to \bar{U}$, *semi-stable in codimension one, and that* $f^* f_* \omega_{\bar{X}/\bar{U}} \to \omega_{\bar{X}/\bar{U}}$ *is an isomorphism outside of* $f^{-1}(Z)$ *for some* $Z \subset \bar{U}$ *closed and of codimension at least two. Then*

$$\dim H^0(\bar{U}, \varphi^* T_{\overline{M}_h}(-\log S))$$

is invariant under infinitesimal deformations.
In particular, by Ran's T^1*-lifting property, deformations of those families* $f : X \to U$ *of Calabi-Yau manifolds with* U *fixed are un-obstructed.*

Remark 11. We expect that Proposition 10 holds true under weaker and more natural conditions on the boundary.

Proof. Since we are only interested in global sections, taking complete intersection we may assume that $\dim \bar{U} = 1$, that all fibres are semi-stable and that

$$f^* f_* \omega_{\bar{X}/\bar{U}} \longrightarrow \omega_{\bar{X}/\bar{U}}$$

is an isomorphism.
Recall that (choosing a level N structure) we assumed the existence of a universal family $f : \mathcal{X} \to M_h$. The pull back of the logarithmic Higgs field

$$\theta : E \longrightarrow E \otimes \Omega^1_{\overline{M}_h}(\log S)$$

of the variation of Hodge structures $R^m f_* \mathbb{Q}_{\mathcal{X}}$ to $\bar{U}$ corresponds to a sub-sheaf

$$\varphi^* T_{\overline{M}_h}(-\log S) \longrightarrow (\mathcal{E}nd(\varphi^* E), \theta^{\mathcal{E}nd}).$$

By ([**10**], Prop. 2.1) $\theta^{\mathcal{E}nd}(\varphi^* T_{\overline{M}_h}(-\log S)) = 0$. This means that the above sub-sheaf is a Higgs sub-sheaf.
We need the following theorem on intersection cohomology and Higgs cohomology of a complex polarized variation of Hodge structures $\mathbb{W}$ with uni-potent local monodromy around S. Let (F, θ) denote the logarithmic Higgs bundle of $\mathbb{W}$. We consider the complex of sheaves defined by the Higgs field

$$F \xrightarrow{\theta} F \otimes \Omega^1_{\bar{U}}(\log S) \xrightarrow{\theta} F \otimes \Omega^2_{\bar{U}}(\log S) \longrightarrow \cdots .$$

In [**9**] (for $\dim \bar{U} = 1$ in an implicit way) and in [**4**] (in general) one finds the definition of an algebraic L_2- sub complex of sheaves

$$\begin{array}{ccccccc} F & \xrightarrow{\theta} & F \otimes \Omega^1_{\bar{U}}(\log S) & \xrightarrow{\theta} & F \otimes \Omega^2_{\bar{U}}(\log S) & \longrightarrow & \cdots \\ \cup & & \cup & & \cup & & \\ F_{(2)} & \xrightarrow{\theta} & (F \otimes \Omega^1_{\bar{U}}(\log S))_{(2)} & \xrightarrow{\theta} & (F \otimes \Omega^2_{\bar{U}}(\log S))_{(2)} & \longrightarrow & \cdots \end{array}$$

determined by an algebraic condition on $F|_S$ imposed by the weight-filtration of

$$\mathrm{res}(\theta) : F|_S \longrightarrow \varphi^* E|_S.$$

Note that for a sub sheaf $F' \subset \mathrm{Ker}(\theta)$, one has $F' \subset F_{(2)}$.

Theorem 12 ([**9**] for $\dim \bar{U} = 1$, [**4**]).

$$\mathbb{H}^i\big(F_{(2)} \xrightarrow{\theta} (F \otimes \Omega^1_{\bar{U}}(\log S))_{(2)} \xrightarrow{\theta} \cdots\big) \simeq H^i_{\mathrm{intersection}}(\mathbb{W}).$$

Back to our situation, the exact sequence of complexes of sheaves

$$0 \longrightarrow (\varphi^* T_{\overline{M}_h}(-\log S), 0) \longrightarrow (\mathcal{E}nd(\varphi^* E), \theta^{\mathcal{E}nd}) \longrightarrow (Q, \theta) \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{array}{ccccl} \cdots & \to & \mathbb{H}^{i-1}(Q,\theta) & \to & H^i(\varphi^* T_{\overline{M}_h}(-\log S)) \to \mathbb{H}^i(\mathcal{E}nd(\varphi^* E)_{(2)}, \theta^{\mathcal{E}nd}) \\ & \to & \mathbb{H}^i(Q,\theta) & \to & H^{i+1}(\varphi^* T_{\overline{M}_h}(-\log S)) \to \mathbb{H}^{i+1}(\mathcal{E}nd(\varphi^* E)_{(2)}, \theta^{\mathcal{E}nd}) \\ & \to & \cdots & & \end{array}$$

Since we assumed the fibres $f^{-1}(p)$ of f to be semi-stable and minimal, [**5**] implies that $f^{-1}(p)$ has no obstruction to deformations in any direction. This means that the pullback of the Kodaira-Spencer map of the moduli space to $\bar{U}$

$$(\varphi^* T_{\overline{M}_h}(-\log S), 0) \longrightarrow (\mathcal{E}nd(\varphi^* E)_{(2)}, \theta^{\mathcal{E}nd}) \longrightarrow (\varphi^* E^{m-1,1} \otimes \varphi^* E^{0,m}, 0)$$

is an isomorphism. Taking in account that those are maps between complexes of sheaves, we find

$$H^i(\varphi^* T_{\overline{M}_h}(-\log S)) \longrightarrow \mathbb{H}^i(\mathcal{E}nd(\varphi^* E)_{(2)}, \theta^{\mathcal{E}nd})$$

to be injective for all i. Hence there is a splitting

$$\mathbb{H}^i(\mathcal{E}nd(\varphi^* E)_{(2)}, \theta^{\mathcal{E}nd}) = H^i((\varphi^* T_{\overline{M}_h}(-\log S)) \oplus \mathbb{H}^i(Q, \theta).$$

By Theorem 12 $\mathbb{H}^i(\mathcal{E}nd(\varphi^* E)_{(2)}, \theta^{\mathcal{E}nd})$ is isomorphic to the intersection cohomology, hence is invariant under small deformations. Using the semi continuity of the hyper-cohomology of complexes of sheaves one shows that both $H^i((\varphi^* T_{\overline{M}_h}(-\log S))$ and $\mathbb{H}^i(Q, \theta)$ are invariant under small deformations. □

Corollary 13. *Under the assumptions made in 10 the scheme* H *is smooth.*

3. Applications

Again $f : X \to U$ denotes a smooth family of Calabi-Yau 3-folds, such that $\varphi : U \to M_h$ is generically finite. We keep the assumption, that M_h has a universal family. Moreover, we choose a compactification $\overline{M}_h$ with $\overline{M}_h \setminus M_h$ a normal crossing divisor, such that $U \to M_h$ extends to $\bar{U} \to \overline{M}_h$.
Staring with

$$\mathbf{H_1} = \mathrm{Hom}((\overline{U}, U), (\overline{M}_h, M_h)),$$

consider

$$\mathbf{H_2} = \mathrm{Hom}((\overline{\mathbf{H}}_{\mathbf{1}} \times \{0\}, \mathbf{H_1} \times \{0\}), (\overline{M}_h, M_h)), \quad \{0\} \in U,$$

together with the induced family $f : X \to \mathbf{H_1} \times \mathbf{H_2} = \mathbf{H}$.
Let $\mathbb{V} \subset R^3 f_*(\mathbb{C}_X)$ be the irreducible sub variation of Hodge structures with Hodge decomposition

$$\bigoplus_{p+q=3} F^{p,q} \quad \text{with} \quad F^{3,0} = f_* \Omega^3_{X/\mathbf{H}}.$$

Recall that by Proposition 3 one has a decomposition $\mathbb{V} = \mathbb{V}_1 \otimes \mathbb{V}_2$, where $\mathbb{V}_i$ is the pull back of a $\mathbb{C}$ variation of Hodge structures on $\mathbf{H_i}$. Comparing the possible Hodge numbers, one finds:

Proposition 14. $\mathbb{V}_i$ *has one of the following Hodge types:*

a. $F_i^{1,0} \oplus F_i^{0,1}$, $\quad \mathrm{rk} F_i^{1,0} = 1$.
b. $F_i^{2,0} \oplus F_i^{1,1} \oplus F_i^{0,2}$, $\quad \mathrm{rk} F_i^{2,0} = \mathrm{rk} F_i^{0,2} = 1$, *and* $\mathbb{V}_i$ *is real.*
c. *Moreover, if* $\mathbb{V}_1$ *is of type b), then* $\mathrm{rk} \mathbb{V}_2 = 2$.

It is well known that the period domains $\mathcal{D}_i$ of Hodge structures of types a) or b) are the bounded symmetric domain of the algebraic group $U(1, \mathrm{rk}\mathbb{V}_i^{0,1})$, or $SO(2, \mathrm{rk}\mathbb{V}_i^{1,1})$, respectively.
The un-obstructedness for deformations of families implies that the generically finite period map $\widetilde{\mathbf{H}}_{\mathbf{i}} \to \mathcal{D}_i$ has to be dominant. Let us assume that $U \to M_h$ is injective.

QUESTION 15.

1. Is $\mathbf{H}_\mathbf{i}^\mathbf{s} \simeq \mathcal{D}_i/\Gamma_i$ for some Γ_i a partial compactification of $\mathbf{H}_\mathbf{i}$?
2. What is the moduli-interpretation of points in $\mathbf{H}_\mathbf{i}^\mathbf{s} \setminus \mathbf{H}_\mathbf{i}$?

4. An example of a non-rigid family of Calabi-Yau quintic threefolds

Let $f_5(x_2, x_1, x_0) \in \mathbb{C}[x_2, x_1, x_0]$ be the polynomial of a quintic plane curve in $\mathbb{P}^2$. Then

$$x_3^5 + f_5(x_2,\, x_1,\, x_0)$$

defines a quintic hypersurface in $\mathbb{P}^3$, and

$$x_4^5 + x_3^5 + f_5(x_2,\, x_1,\, x_0)$$

a Calabi-Yau quintic 3-fold in $\mathbb{P}^4$.
Obviously this construction can also be done locally over the moduli stack $M_{5,2}$ of quintic plane curves in $\mathbb{P}^2$, starting with the universal family $f : X \to M_{5,2}$ of curves. Replacing $M_{5,2}$ by some covering, one can glue those families as family of subvarieties in some projective bundle (see [**8**]). The resulting family of surfaces will be denoted by $g_1 : Z_1 \to M_{5,2}$, and the one of threefolds by $g_2 : Z_2 \to M_{5,2}$.

REMARK 16. As pointed out by S.T. Yau, this family has been studied by S. Ferrara and J. Louis [**3**]. They have shown that the Yukawa-coupling is zero and that and the monodromy lies in $SU(2,1)$. In [**8**] the exact length of the Yukawa coupling is calculated for such families.

One can play a similar game, starting with 5 points in $\mathbb{P}^1$. say with equation $h_5(x_1,\, x_0) \in \mathbb{C}[x_1,\, x_0]$. Then $x_2^5 + h_5(x_1,\, x_0)$ defines a quintic plane curve. Again, one can do such a construction starting with the universal family $P \to M_{5,1}$ of 5 points in $\mathbb{P}^1$, and one obtains a family $g_0 : Z_0 \to M_{5,1}$ of quintic plane curves.
Finally Σ_5 denotes the Fermat curve $x_2^5 + x_1^5 + x_0^5 = 0$ of degree 5.

PROPOSITION 17. *The fibre product $Z_1 \times \Sigma_5 \to M_{5,2}$ admits an $\mathbb{Z}_5$-action over $M_{5,2}$, given fibrewise by*

$$(x_3, x_2, x_1, x_0), (y_2, y_1, y_0) \mapsto (e^{2\pi i/5} x_3, x_2, x_1, x_0), (e^{2\pi i/5} y_2, y_1, y_0).$$

1 *The family of Calabi-Yau quintics $g_2 : Z_2 \to M_{5,2}$ can be reconstructed as:*

$$\begin{tikzcd}
(Z_1 \times \Sigma_5)/\mathbb{Z}_5 \arrow[dr, "g_1"] & (\widehat{Z_1 \times \Sigma_5})/\mathbb{Z}_5 \arrow[l, "\text{blowup}"'] \arrow[r, "\text{blowdown"] \arrow[d] & Z_2 \arrow[dl, "g_2"] \\
& M_{5,2}. &
\end{tikzcd}$$

2. *The construction in 1) extends to the product family*

$$
\begin{tikzcd}
(Z_1 \times Z_0)/\mathbb{Z}_5 & (\widehat{Z_1 \times Z_0})/\mathbb{Z}_5 \arrow[l, "\text{blow up}"'] \arrow[r, "\text{blow down}"] \arrow[d] & \mathcal{Z}_2 \arrow[dl, "h_2"] \\
& M_{5,2} \times M_{5,1}. \arrow[from=1-1, "(g_1, g_0)"] &
\end{tikzcd}
$$

3. *The family* $h_2 : \mathcal{Z}_2 \to M_{5,2} \times M_{5,1}$ *of Calabi-Yau quintics is a universal family of the form*

$$h_2 : \mathcal{Z}_2 \longrightarrow \mathbf{H_1} \times \mathbf{H_2},$$

i.e. for suitable compactifications $\overline{M}_h$, $\overline{M}_{5,2}$ *and* $\overline{M}_{5,1}$ *and for some base point* $u \in M_{5,2}$ *and* $u' \in M_{5,1}$

$$M_{5,2} = \mathbf{H_1} = \mathrm{Hom}((\{u\} \times \overline{M}_{5,1}, \{u\} \times M_{5,1}), (\overline{M}_h, M_h)),$$

and

$$M_{5,1} = \mathbf{H_2} = \mathrm{Hom}((\overline{M}_{5,2} \times \{u'\}, M_{5,2} \times \{u'\}), (\overline{M}_h, M_h)).$$

Moreover, a partial compactification $\mathbf{H_1^s}$ *of* $\mathbf{H_1}$ *is a 2-dimensional complex arithmetic ball quotient, and a partial compactification* $\mathbf{H_2^s}$ *of* $\mathbf{H_2}$ *is a 12-dimensional complex non-arithmetic ball quotient.*

Sketch of the proof. 1) and 2) have been shown in ([**8**], Proposition 6.4). For 3) consider the eigen-space decompositions

$$R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0}) = \bigoplus_{i=1}^{4} R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_i, \quad \text{and} \quad R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1}) = \bigoplus_{i=1}^{4} R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_i$$

for the $\mathbb{Z}_5$-action. Recall that the restriction of $R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_i$ to a point in $M_{5,1}$ is a Hodge structure with $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(5-i))$ in degree $(1,0)$ and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-i))$ in degree $(0,1)$. Hence $R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_i$ is unitary for $i = 1$ and $i = 4$.
$R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_3$ and $R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_2$ are not unitary, and dual to each other. By Deligne-Mostow [**1**] $M^s_{5,1}$ is uniformized by $R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_3$ as a 2-dimensional arithmetic ball quotient, which is a component of the moduli space, parameterizing Abelian varieties of dimension 6 with complex multiplication $\mathbb{Q}(\zeta)$ for $\zeta = e^{2\pi i/5}$.
For g_1 the situation is more complicated. One easily computes that the Higgs bundle of $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2$ is trivial in degree $(0,2)$, of a rank one in degree $(2,0)$ and of a rank 12 in degree $(1,1)$. A similar argument as the one used by Deligne-Mostow allows to show that $M^s_{5,2}$ is uniformized by $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2$ as a 12-dimensional complex ball quotient.
There is a Galois conjugate $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2^{\sigma}$, which is neither the dual of $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2$, nor unitary. As in Deligne-Mostow this implies that the ball quotient is not arithmetic.

The quotient by $\mathbb{Z}_5$, together with blowing up and blowing down, gives rise to a $\mathbb{Q}$-Hodge isometry (see [**8**], 7.4) $(R^3h_{2*}\mathbb{Q}_{\mathcal{Z}_2}) \simeq \mathbb{V}' \oplus \mathbb{T}$ with

$$\mathbb{V}' \otimes \mathbb{Q}(\zeta) = \bigoplus_{i=1}^{4} \mathbb{V}_{i,5-i} \quad \text{for} \quad \mathbb{V}_{i,5-i} = R^2g_{1*}(\mathbb{Q}(\zeta)_{Z_1})_i \otimes R^1g_{0*}(\mathbb{Q}(\zeta)_{Z_0})_{5-i}$$

$$\text{and with} \quad \mathbb{T} = \bigoplus^{4} R^1f_*\mathbb{Q}_X(1),$$

where (1) denotes the Tate-twist. Of course, we should write

$$\mathbb{V}_{i,5-i} = \mathrm{pr}_1^* R^2g_{1*}(\mathbb{Q}(\zeta)_{Z_1})_i \otimes \mathrm{pr}_2^* R^1g_{0*}(\mathbb{Q}(\zeta)_{Z_0})_{5-i},$$

but we suppress the pullback under the projections in our notation. Remark that $\mathbb{T}$ is the part of the variation of Hodge structures, coming from the blowing ups. $\mathbb{V}_{i,5-i}$ is an irreducible sub-variation of Hodge structures in $(R^3h_{2*}\mathbb{Q}(\zeta)_{\mathcal{Z}_2})$, and for the corresponding $\mathbb{C}$ variation of Hodge structures, the fibre over a point y has

$$H^0(\mathbb{P}^2, \Omega^2_{\mathbb{P}^2}(5-i)) \otimes H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(i))$$

in degree $(3,0)$. This is zero for $i=1$ and $i=4$, and the $\mathbb{C}$ variation of Hodge structures given by

$$\mathbb{V} = \mathbb{V}_{2,3} \oplus \bar{\mathbb{V}}_{2,3} = \mathbb{V}_{2,3} \oplus \mathbb{V}_{3,2}$$

contains the Hodge bundle $h_{2*}\Omega^3_{\mathcal{Z}_2/M_{5,2}\times M_{5,1}}$.
By abuse of notations we will regard $\mathbb{V}$ and the $\mathbb{V}_{i,5-i}$ as $\bar{\mathbb{Q}}$ variations of Hodge structures. Write $(R^3h_{2*}\bar{\mathbb{Q}}_{\mathcal{Z}_2}) = \mathbb{V} \oplus \mathbb{W}$, and let $(F^{2,1} \oplus F^{1,2} \oplus F^{0,3}, \theta)$ denote the system of Hodge bundles corresponding to $\mathbb{W}$. The missing part of 17, 3), follows from the next two Claims. $\square$

CLAIM 18. There is no nontrivial extension

$$\begin{array}{ccc} \mathcal{Z}_2 & \xrightarrow{h_2} & M_{5,2} \times M_{5,1} \\ \subset\big\downarrow & & \big\downarrow\subset \\ \mathcal{Z}_2' & \xrightarrow{h_2'} & N \times M_{5,1}, \end{array}$$

such that the induced morphism $\varphi : N \times M_{5,1} \to M_h$ is generically finite over its image.

PROOF. A deformation $N \times M_{5,1}$ of $M_{5,1} = \{u\} \times M_{5,1}$, which does not lie in $M_{5,2} \times M_{5,1}$, induces a non-zero flat section τ of $\mathrm{End}(\mathbb{V} \oplus \mathbb{W})|_{M_{5,1}}$ of type $(-1,1)$, which does not respect the direct sum decomposition $\mathbb{V} \oplus \mathbb{W}$.
In fact, if it does, one has $\tau(\mathbb{V}|_{M_{5,1}}) \subset \mathbb{V}|_{M_{5,1}}$. The restriction of $\mathbb{V}_{2,3}$ and $\mathbb{V}_{3,2}$ to $M_{5,1}$ are direct sums of local systems isomorphic to $R^1g_{0*}(\bar{\mathbb{Q}}_{Z_0})_3$ or $R^1g_{0*}(\bar{\mathbb{Q}}_{Z_0})_2$, respectively. As uniformizing variations of Hodge structures of a ball quotient, both are irreducible and, since the Hodge numbers are different, $R^1g_{0*}(\bar{\mathbb{Q}}_{Z_0})_3$ is not isomorphic to $R^1g_{0*}(\bar{\mathbb{Q}}_{Z_0})_2$. So τ respect the decomposition $\mathbb{V} = \mathbb{V}_{2,3} \oplus \mathbb{V}_{3,2}$ and it is induced by an endomorphism of $R^2g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2|_{\{u\}}$ of type $(-1,1)$. The calculation of Hodge numbers, indicated above, shows that those are lying in a 12 dimensional vector space. So they correspond all to the deformations along $M_{5,2}$.

On the other hand, if τ does not respect the direct sum decomposition, one obtains a non trivial morphism $\mathbb{V}|_{M_{5,1}} \to \mathbb{W}|_{M_{5,1}}$. Since $\mathbb{W}|_{M_{5,1}}$ is a direct sum of unitary local systems, and since for $i = 2$ or $i = 3$ there exists no non-trivial morphism

$$R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_i \longrightarrow \text{unitary local system},$$

this leads to a contradiction. □

Claim 19. There is no nontrivial extension

$$\begin{array}{ccc} \mathcal{Z}_2 & \xrightarrow{h_2} & M_{5,2} \times M_{5,1} \\ \subset\downarrow & & \downarrow\subset \\ \mathcal{Z}'_2 & \xrightarrow{h'_2} & M_{5,2} \times N, \end{array}$$

such that the induced morphism $\varphi : M_{5,2} \times N \to M_h$ is generically finite over its image.

Proof. Again, the deformations of $M_{5,2} \times \{u\}$ correspond to flat section τ of $\mathrm{End}(\mathbb{V} \oplus \mathbb{W})|_{M_{5,2}}$ of type $(-1, 1)$. If τ respects the direct sum decomposition, the irreducibility of $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2$ and $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_3$ implies that τ is induced by an endomorphism of $R^1 g_{0*}(\bar{\mathbb{Q}}_{Z_0})_3|_{\{u\}}$ of type $(-1, 1)$. Those form a 2 dimensional vector space, corresponding to the deformations along $M_{5,1}$. So a deformation $M_{5,2} \times N$, which does not lie in $M_{5,2} \times M_{5,1}$, induces a non-zero flat section τ of $\mathrm{End}(\mathbb{V} \oplus \mathbb{W})|_{M_{5,2}}$ of type $(-1, 1)$. which does not respect the direct sum decomposition. So one finds a non-trivial morphism

$$\tau : \mathbb{V}|_{M_{5,2}} = \bigoplus R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_3 \oplus \bigoplus R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2 \longrightarrow \mathbb{W}|_{M_{5,2}}.$$

On the other hand, $\mathbb{W}|_{M_{5,2}}$ is a direct sum of several copies of the local systems

$$R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_1,\ R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_4, \quad \text{and} \quad R^1 f_* \bar{\mathbb{Q}}_X(1).$$

Remark that the uniformization local system $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_2$ for $M_{5,2}$ is irreducible, as well as its dual $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_3$.
The local system $R^1 f_* \bar{\mathbb{Q}}_X(1)$ is the variation of Hodge structures attached to the universal family of plane curves of degree 5, hence it is irreducible by Deligne's irreducibility theorem [**2**]. $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_1$ and $R^2 g_{1*}(\bar{\mathbb{Q}}_{Z_1})_4$ are both irreducible, by a generalization of Deligne's irreducibility theorem proved in [**8**], Lemma 4.1.
On the other hand, all the irreducible local systems considered above have different Hodge types. So there exists no non-zero morphism between them, a contradiction. □

Remark 20. In [**8**] we consider the subscheme

$$M_{5,1} \times M_{5,1} \subset M_{5,2} \times M_{5,1},$$

and the restriction of

$$h_2 : \mathcal{Z}_2 \longrightarrow M_{5,2} \times M_{5,1}$$

to this subscheme. It is shown there, that the set of CM-points $y \in M_{5,1} \times M_{5,1}$ is dense in $M_{5,1} \times M_{5,1}$, i.e. the set of points y for which the Hodge structure $H^3(h_2^{-1}(y), \mathbb{Q})$ has complex multiplication.

Since $M^s_{5,2}$ is a non-arithmetic ball quotient one should expect, according to the André-Oort conjecture, that the only positive dimensional component of Zariski closure of the set of CM-points in $M_{5,2} \times M_{5,1}$ is $M_{5,1} \times M_{5,1}$.

References

[1] Deligne, P., Mostow, G.:Monodromy of hypergeometric functions and non-lattice integral monodromy. IHES **63** (1986) 5–90

[2] Deligne, P., Katz, N.: Groupes de Monodromy en Géométrie Algébrique (SGA VII, 2) Lecture Notes in Math. **340** (1973) Springer, Berlin Heidelberg New York

[3] Ferrara, S., Louis, J.: Picard-Fuchs equations and flat holomorphic connections from $N = 2$ supergravity. Essays on Mirror Manifolds, edited by Shing-Tung Yau, International Press (1992) 301–315

[4] Jost, J., Yang,Y-H., Zuo, K.: The cohomology of a variation of polarized Hodge structures over a quasi-compact Kähler manifold. preprint, AG/0312145

[5] Kawamata, Y., Namikawa, Y.: Logarithmic deformations of normal crossing varieties and smoothing degenerated Calabi-Yau varieties. Invent. Math. **118** (1994) 395–409

[6] Popp, H.: Moduli theory and classification theory of algebraic varieties. Lect. Notes in Math. **620** Springer-Verlag, Berlin-New York, 1977

[7] Viehweg, E., Zuo K.: Base spaces of non-isotrivial families of smooth minimal models. In: Complex Geometry (Collection of Papers dedicated to Hans Grauert), Springer Verlag (2002) 279–328

[8] Viehweg, E., Zuo, K.: Complex multiplication, Griffiths-Yukawa couplings, and rigidity for families of hypersurfaces. preprint, AG/0307398. To appear in J. of Alg. Geom.

[9] Zucker, S.: Hodge theory with degenerating coefficients: L^2-cohomology in the Poincaré metric. Ann. of Math. **109** (1979) 415–475

[10] Zuo, K.: On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. The Asian J. of Math. **4** (2000) 279–302

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