Cohomology of graph hypersurfaces associated to certain Feynman graphs

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Abstract

To any Feynman graph (with $2n$ edges) we can associate a hypersurface $X \subset \mathbb{P}^{2n-1}$. We study the cohomology of the middle degree $H^{2n-2}(X)$ of such graph hypersurface. S. Bloch, H. Esnault, and D. Kreimer (Commun. Math. Phys. 267, 2006) have computed this cohomology for the first series of examples, the wheel with spokes WS_n, $n \geq 3$. Using the same technique, we introduce the generalized zigzag graphs and prove that $W_5(H^{2n-2}(X)) = \mathbb{Q}(-2)$ for all of them (with W_* the weight filtration). We also can compute $\#X(\mathbb{F}_q) \equiv 1 + q + 2q^2 \mod q^3$ for the number of rational points of such hypersurface. At the end we study the behavior of graph hypersurfaces under the gluing of graphs.

Introduction

There are interesting zeta and multi zeta values appearing in the calculation of Feynman integrals in physics ([BrKr]). One hopes that there exist Tate mixed Hodge structures with periods given by Feynman integrals at least for some identifiable subset of graphs.

This paper is a natural continuation of the work started in [BEK]. For technical reasons we restrict our attention to primitively log divergent graphs. In [BEK], Sections 11,12 the series WS_n was worked out in all details. Let $X_n \subset \mathbb{P}^{2n-1}$ be the graph hypersurface for the graph "wheel with n spokes" WS_n, it was proved that (as a Hodge structure)

$$H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash X) \cong \mathbb{Q}(-2)$$

and that the de Rham cohomology $H_{DR}^{2n-1}(\mathbb{P}^{2n-1} \backslash X)$ is generated by the integrand of the graph period (24). In this paper we do the same computation for the graph ZZ_5, the zigzag graph with the Betti number equals 5. We define a big series of graphs for which the maximal nontrivial weight piece of Hodge structure is of Tate type: $\mathrm{gr}\,^W_{max} H_c^{2n-1}(\mathbb{P}^{2n-1} \backslash X) \cong \mathbb{Q}(-2)$ (*Theorem 4.6*). Using the same stratification of a graph hypersurface as in this proof and the technique of counting rational points using ℓ-adic cohomology, we get the congruence $\#X(\mathbb{F}_q) \equiv 1+q+q^2 \mod q^3$. We study gluings of primitively log divergent graphs and compute $\mathrm{gr}\,^W_{max} H_c^{2n-1}(\mathbb{P}^{2n-1}\backslash X) = \mathbb{Q}(-3)$ for the case of gluing of two graphs WS_3 and WS_n.

This paper is organized as follows. Section 1 contains some theorems on determinants, this is a key ingredient of our computation. The second section is a remainder of the construction of graph polynomials and periods. The cohomological tools are presented in Section 3.

We define the *generalized zigzag graphs* GZZ and prove that they are primitively log divergent in *Section 4*. Then we present the main result that the minimal nontrivial weight piece of mixed Hodge structure associated to the cohomology of such graph hypersurface is Tate (Theorem 4.6). We get more concrete result for the cohomology of the middle degree of the graph ZZ_5 (Theorem 4.7). Then we count rational points of GZZ (Theorem 4.8).

In *Section 5* it is proved that the integrand (24) is nonzero in the de Rham cohomology $H^{2n-2}_{DR}(\mathbb{P}^{2n-1}\backslash X)$ for the subset $GZZ(n,2)$ of GZZ, $n \geq 2$, and generates it in the case of ZZ_5.

In *Section 6* we define the gluing $\Gamma \times \Gamma'$ of two graphs Γ and Γ and do computation for the cohomology of the middle degree for $WS_3 \times WS_n$.

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1 Determinants

Fix some commutative ring R with 1 and let $\mathcal{M} = (a_{ij})_{0\leq i,j\leq n}$ be an $(n+1)\times(n+1)$-matrix with entries in R. The numeration of rows and columns goes 0 through n. Let $\mathcal{M}(i_0,\dots,i_k;j_0,\dots,j_t)$ be the submatrix which we get from the matrix after removing rows i_0 to i_k and the columns j_0 to j_k. It is very convenient to denote the determinant of $\mathcal{M}$ just by M. We assume that the determinant of zero-dimensional matrix is 1. For example, $M(0,n;0,n) = 1$ for the matix in the definition above with $t = n = 1$.

Theorem 1.1
Let $n \geq 1$. For any $(n+1)\times(n+1)$-matrix $\mathcal{M}$ and any integers $0 \leq i, j, k, t \leq n$, satisfying $i \neq k$ and $j \neq t$, we have

$$M(i;j)M(k;t) - M(k;j)M(i;t) = M \cdot M(i,k;j,t). \tag{1}$$

Proof. The first step of the proof is the reduction to the case $i = j = 0$ and $k = t = n$. After this, the assertion can be proved by expansion of $\mathcal{M}$

along this rows and columns using the induction on n.

$\square$

For a matrix $\mathcal{M} = (a_{ij})_{0\leq i,j\leq n}$ we define the minors

$$\begin{aligned} I_k^i := M(0,1,\ldots,&i-1, i+k, i+k+1,\ldots,n; \\ & 0,1,\ldots,i-1,i+k,i+k+1,\ldots,n). \end{aligned} \tag{2}$$

and

$$S_t := M(t, t+1, \ldots, n; 0, t+1, t+2, \ldots, n), \tag{3}$$

where $1 \leq k \leq n+1$ and $1 \leq t \leq n$. We usually write I_n for I_n^0. For example, $I_{n+1} = M$, $I_n^1 = M(0;0)$ and $I_n = M(n;n)$.

Corollary 1.2
For a symmetric matrix $\mathcal{M} = (a_{ij})_{0\leq i,j\leq n}$ one has the following equality

$$I_n I_n^1 - I_{n-1}^1 I_{n+1} = (S_n)^2. \tag{4}$$

Proof. Since $\mathcal{M}$ is symmetric, $M(0;n) = M(n;0) = S_n$ and the statement follows immediately from Theorem 1.1.

$\square$

Take now an $(n+1)\times(n+1)$-matrix $\mathcal{M} = (a_{ij})$ with entries in R and suppose that the transpose of the last row equals the last column with elements, renumbered by single lower indices.

$$\mathcal{M} = \begin{pmatrix} a_{00} & a_{01} & \vdots & a_{0\,n-2} & a_{0\,n-1} & a_0 \\ a_{10} & a_{11} & \vdots & a_{1\,n-2} & a_{1\,n-1} & a_1 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots \\ a_{n-2\,0} & a_{n-2\,1} & \vdots & a_{n-2\,n-2} & a_{n-2\,n-1} & a_{n-2} \\ a_{n-1\,0} & a_{n-1\,1} & \vdots & a_{n-1\,n-2} & a_{n-1\,n-1} & a_{n-1} \\ a_0 & a_1 & \vdots & a_{n-2} & a_{n-1} & a_n \end{pmatrix}. \tag{5}$$

The determinant of $\mathcal{M}$ is thought of as an element in $R[a_0,\ldots,a_n]$. It can be written as $M = I_{n+1} = a_n I_n - G_n$. Then G_n is computed as

$$G_n := \sum_{0\leq i,j\leq n-1} (-1)^{i+j} a_i a_j I_n(i;j). \tag{6}$$

The entries a_i play the role of variables while the other entries and minors are coefficients. The polynomial $G_n \in R[a_0,\ldots,a_n]$ is of degree 2. We claim

Theorem 1.3
Let $I_{n-1} \not\equiv 0 \mod I_n$*. Then*

$$I_{n-1}G_n \equiv L_n L_n' \mod I_n \tag{7}$$

for some L_n *and* L_n'*, linear as polynomials of the "variables".*

Proof. By Theorem 1.1, we have

$$I_n(i;j)I_n(n-1;n-1) \equiv I_n(i;n-1)I_n(n-1;j) \mod I_n \tag{8}$$

for all $1 \leq i,j \leq n-2$. We multiply G_n by $I_{n-1} = I_n(n-1,n-1)$ and get

$$\begin{aligned} I_{n-1}G_n = \quad & a_{n-1}^2(I_{n-1})^2 + \sum_{0\leq i,j\leq n-2} (-1)^{i+j} a_i a_j I_{n-1} I_n(i;j) \\ & +a_{n-1} \sum_{0\leq i\leq n-2} (-1)^{i+n-1} a_i I_{n-1}\Big(I_n(i;n-1) + I_n(n-1;i)\Big) \\ \equiv \quad & \Big(a_{n-1}I_{n-1} + \sum_{0\leq i\leq n-2} (-1)^{i+n-1} a_i I_n(i;n-1)\Big) \\ & \cdot\Big(a_{n-1}I_{n-1} + \sum_{0\leq j\leq n-2} (-1)^{j+n-1} a_j I_n(n-1;j)\Big) \mod I_n. \end{aligned} \tag{9}$$

We set

$$L_n = a_{n-1}I_{n-1} + \sum_{0\leq i\leq n-2} (-1)^{i+n-1} a_i I_n(i;n-1) \tag{10}$$

and

$$L_n' = a_{n-1}I_{n-1} + \sum_{0\leq j\leq n-2} (-1)^{j+n-1} a_i I_n(n-1;j). \tag{11}$$

□

In the next sections we deal only with symmetric matrices, thus we make a

Corollary 1.4
Let $\mathcal{M} = (a_{ij})$ *be a symmetric* $(n+1)\times(n+1)$*-matrix with entries in a ring* R *(see (5)). If* $I_{n-1} \not\equiv 0 \mod I_n$*, the congruence*

$$I_{n-1}G_n \equiv (L_n)^2 \mod I_n \tag{12}$$

holds, where G_n *and* L_n *are given by (6) and (10) respectively.*

One more fact about G_n will be used frequently in the next sections.

Theorem 1.5
Let $\mathcal{M} = (a_{ij})$ be a symmetric $(n+1) \times (n+1)$-matrix with entries in a ring R and assume that the quotient ring $R/(I_n)$ is a domain. If $I_{n-1} \equiv 0 \mod I_n$, then

$$G_n \equiv \sum_{0 \le i \le n-2} a_i^2 I_n(i,i) + 2 \sum_{0 \le i < j \le n-2} (-1)^{i+j} a_i a_j I_n(i;j) \mod I_n. \tag{13}$$

Proof. We prove that G_n forgets the "variable" a_{n-1}. Since $\mathcal{M}$ is symmetric, we can rewrite G_n (see (6)) as

$$\begin{aligned} G_n = \quad & a_{n-1}^2 I_{n-1} + \sum_{0 \le i \le n-2} (-1)^{i+n-1} a_i a_{n-1} \Big(I_n(i, n-1) + I_n(n-1, i) \Big) \\ & + \sum_{0 \le i \le j \le n-2} (-1)^{i+j} a_i a_j I_n(i;j) \\ = \quad & a_{n-1}^2 I_{n-1} + 2 \sum_{0 \le i \le n-2} (-1)^{i+n-1} a_i a_{n-1} I_n(i, n-1) \\ & + \sum_{0 \le i \le n-2} a_i^2 I_n(i,i) + 2 \sum_{0 \le i < j \le n-2} (-1)^{i+j} a_i a_j I_n(i;j). \end{aligned} \tag{14}$$

Theorem 1.1 implies $(I_n(i,n-1))^2 \equiv I_n(i,i) I_n(n-1,n-1) \mod I_n$ for all $0 \le i \le n-2$. Since $R/(I_n)$ is a domain and $I_{n-1} = I_n(n-1,\, n-1) \equiv 0 \mod I_n$, we get $I_n(i,n-1) \equiv 0 \mod I_n$. Hence, (14) implies the congruence

$$G_n \equiv \sum_{0 \le i \le n-2} a_i^2 I_n(i,i) + 2 \sum_{0 \le i < j \le n-2} (-1)^{i+j} a_i a_j I_n(i;j) \mod I_n. \tag{15}$$

□

Remark 1.6
In our computations we will apply Corollary 1.4 and Theorem 1.3 only for the case where R in a polynomial ring over an algebraically closed field of characteristic zero. Moreover, entries of matrices will be only linear polynomials in R.

2 Graph polynomials

Let Γ be a finite graph with edges E and vertices V. We choose an orientation of edges. For a given vertex v and a given edge e we define $\mathrm{sign}(e,v)$ to be -1 if e enters v and $+1$ if e exits v. Denote by $\mathbb{Z}[E]$ (resp. $\mathbb{Z}[V]$)

the free $\mathbb{Z}$-module generated by the elements of E (resp. V). Consider the homology sequence

$$0 \longrightarrow H_1(\Gamma, \mathbb{Z}) \xrightarrow{\iota} \mathbb{Z}[E] \xrightarrow{\partial} \mathbb{Z}[V] \longrightarrow H_0(\Gamma, \mathbb{Z}) \longrightarrow 0, \tag{16}$$

where the $\mathbb{Z}$-linear map ∂ is defined by $\partial(e) = \sum_{v \in V} sign(v, e)$. The elements $e^\vee$ of a dual basis of $\mathbb{Z}[E]$ define linear forms $e^\vee \circ \iota$ on $H = H_1(\Gamma, \mathbb{Z})$. We view the squares of this functions $(e^\vee \circ \iota)^2 : H \to \mathbb{Z}$ as rank 1 quadratic forms. For a fixed basis of H we can associate a rank 1 symmetric matrix $\mathcal{M}_e$ to each such form.

Definition 2.1
We define the *graph polynomial* of Γ

$$\Psi_\Gamma := \det(\sum_{e \in E} A_e \mathcal{M}_e) \tag{17}$$

in some variables A_e.

The polynomial Ψ is homogeneous of degree rank H. A change of the basis of H only changes Ψ_Γ by $+1$ or -1.

Definition 2.2
The *Betti number* of a graph Γ is defined to be $h_1(\Gamma) := \text{rank } H_1(\Gamma, \mathbb{Z})$.

The definition of Ψ_Γ agrees with the other well-known definition (see *Proposition 2.2, [BEK]*):

Proposition 2.3
One has

$$\Psi_\Gamma(A) = \sum_{T \text{ span tr.}} \prod_{e \notin T} A_e. \tag{18}$$

Corollary 2.4
The coefficients of Ψ_Γ *are all either* 0 *or* $+1$.

For the graph Γ we build the table $Tab(\Gamma)$ with $h(\Gamma)$ rows and $|E(\Gamma)|$ columns. Each row corresponds to a loop of Γ, and these loops form a basis of $H_1(\Gamma, \mathbb{Z})$. For each such loop we choose some direction of loop tracing. The entry $Tab(\Gamma)_{ij}$ equals 1 if the edge e_j in the i's loop is in the tracing direction of the loop and equals -1 if this edge is in the opposite direction; if the edge e_j does not appear in the i's loop, then $Tab(\Gamma)_{ij} = 0$. We take $N = |E(\Gamma)|$ variables $T_1, \ldots, T_N$ and build a matrix

$$\mathcal{M}_\Gamma(T) = \sum_{k=1}^{N} T_k \mathcal{M}^k, \tag{19}$$

where $\mathcal{M}^k$ is a $h_1(\Gamma) \times h_1(\Gamma)$ matrix with entries

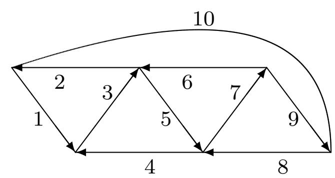
$$\mathcal{M}^d_{ij} = Tab(\Gamma)_{id} \cdot Tab(\Gamma)_{jd}. \tag{20}$$

By definition, the graph polynomial for Γ is

$$\Psi_\Gamma(T) = \det \mathcal{M}_\Gamma(T). \tag{21}$$

Consider the following example which will appear in section 2.1.

Example 2.5
Let Γ be the graph ZZ_5 (see the drawing). This graph has 10 edges and the Betti number equals 5.



	1	2	3	4	5	6	7	8	9	10
1	1	1	1	0	0	0	0	0	0	0
2	0	0	1	1	1	0	0	0	0	0
3	0	0	0	0	1	1	1	0	0	0
4	0	0	0	0	0	0	1	1	1	0
5	0	-1	0	0	0	-1	0	0	1	1

We choose the orientation and the numbering of edges as on the drawing to the left. Following the construction above, we build the table $Tab(ZZ_5)$ to the right and get the following matrix

$$\mathcal{M}_{ZZ_5}(T) = \begin{pmatrix} T_1+T_2+T_3 & T_3 & 0 & 0 & -T_2 \\ T_3 & T_3+T_4+T_5 & T_5 & 0 & 0 \\ 0 & T_5 & T_5+T_6+T_7 & T_7 & -T_6 \\ 0 & 0 & T_7 & T_7+T_8+T_9 & T_9 \\ -T_2 & 0 & -T_6 & T_9 & T_2+T_6+T_9+T_{10} \end{pmatrix} \tag{22}$$

Definition 2.6
The graph hypersurface $X_\Gamma \subset \mathbb{P}^{N-1}$ is the hypersurface cut out by $\Psi_\Gamma = 0$.

Throughout the whole paper we deal with such graph hypersurfaces. Sometimes it is convenient to make a linear change of coordinate in $\mathbb{P}^{N-1}$ to simplify the matrix. Clearly, this new matrix $\widetilde{\mathcal{M}}_\Gamma$ will define a hypersurface isomorphic to X, which we denote again by X. For the graph in Example 2.5, we note that T_0, T_4, T_8 and T_{10} appear only in the diagonal of $\mathcal{M}_{ZZ_5}(T)$. Changing the coordinates and defining new variables, we get the matrix

$$\mathcal{M} = \mathcal{M}_{ZZ_5}(A, B) = \begin{pmatrix} B_0 & A_0 & 0 & 0 & A_5 \\ A_0 & B_1 & A_1 & 0 & 0 \\ 0 & A_1 & C_2 & A_2 & A_4 \\ 0 & 0 & A_2 & B_3 & A_3 \\ A_5 & 0 & A_4 & A_3 & B_4 \end{pmatrix}, \tag{23}$$

where $C_2 = A_1 + A_2 - A_4$.

Definition 2.7
The graph Γ is said to be *convergent* (resp. *logarithmically divergent*) if $N > 2h_1(\Gamma)$ (resp. $N = 2h_1(\Gamma)$). The logarithmically divergent graph Γ is *primitively log divergent* if any connected proper subgraph $\Gamma' \subset \Gamma$ is convergent.

For a primitively log divergent graph the quantity of interest is the period defined by

$$P(\Gamma) := \int_{\sigma^{2n-1}(\mathbb{R})} \eta_\Gamma \qquad \text{with } \eta_\Gamma =: \frac{\Omega_{2n-1}(A)}{\Psi_\Gamma^2}, \tag{24}$$

where $\sigma^{2n-1}(\mathbb{R}) \subset \mathbb{P}^{2n-1}(\mathbb{R})$ is a locus of points with nonnegative coordinates and

$$\Omega_{2n-1}(A) = \sum_{i=1}^{2n} (-1)^i A_i dA_1 \wedge \ldots \widehat{dA_i} \cdots \wedge dA_{2n}. \tag{25}$$

The integral converges (see *Proposition 5.2*, [BEK]).

In Section 7 of [BEK] it was constructed a relative cohomology

$$H := H^{2n-1}(P \backslash Y, B \backslash B \cap Y) \tag{26}$$

with period (i.e. the integration along a homology of H with a de Rham cohomology of H) exactly (24). Here P is some blowing up of $\mathbb{P}^{2n-1}$, Y is a the strict transform of X_Γ and B is the total transform of $A_1 A_2 \ldots A_{2n} = 0$.

There is a hope (see [BEK], 7.25) that for all primitively log divergent graph, or for an identifiable subset of them, the maximal weight piece of the Betti realization H_B is Tate,

$$\mathrm{gr}^W_{max} H_B = \mathbb{Q}(-p)^{\oplus r}. \tag{27}$$

One would like to find a rank 1 sub-Hodge structure $\iota : \mathbb{Q}(-p) \hookrightarrow \mathrm{gr}^W_{max} H_B$ such that the image of η_Γ in $\mathrm{gr}^W_{max} H_{DR}$ spans $\iota(\mathbb{Q}(-p))_{DR}$.

Unfortunately, we cannot compute this even in very simple cases, but something can be done here. Note that by the construction of the blowing up above, we have a natural inclusion $\mathbb{P}^{2n-1} \backslash X \hookrightarrow P \backslash Y$. This implies a morphism

$$H^{2n-1}(P \backslash Y) \xrightarrow{j} H^{2n-1}(\mathbb{P}^{2n-1} \backslash X). \tag{28}$$

Furthermore, the relative cohomology in (26) fits into an exact sequence

$$\longrightarrow H^{2n-2}(B \backslash B \cap Y) \longrightarrow H \longrightarrow H^{2n-1}(P \backslash Y) \longrightarrow . \tag{29}$$

The idea (and the only thing we can do) is to compute $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X)$. We hope that the map j in (28) is nonzero, otherwise our computations give no information about H. In the paper [BEK], Section 11, there was computed $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X_n) \cong \mathbb{Q}(-2n+3)$ for X_n the graph hypersurface of WS_n, $n \geq 3$ (for Betti or l-adic cohomology). Moreover, motivated by discussion above about the weights of realizations of H, for the de Rham cohomology there was proved (see Section 12) that the class of η_Γ lies in the second level of the Hodge filtration (and generates the whole cohomology because $H^{2n-1}_{DR}(\mathbb{P}^{2n-1}\backslash X_n)$) is one dimensional).

In the next sections we compute $H^{2n-1}(\mathbb{P}^{2n-1}\backslash X)$ (or the maximal graded piece of weight filtration) for new examples of primitively divergent graphs. For ZZ_5 we also managed to do the computation for $H^{2n-1}_{DR}(\mathbb{P}^{2n-1}\backslash X)$.

3 Cohomology

In this section we explain the cohomological tools we will use. By *Corollary 2.4*, a graph hypersurface X is always defined over $\mathbb{Z}$. We consider two types of cohomology theories: the ℓ-adic cohomology $H^i(X \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ and the Betti cohomology $H^i(X \otimes_{\mathbb{Z}} \mathbb{C}, \mathbb{Q})$. To unify the notation we write simply $H^i(X)$ and everything below works for both theories.

Let X be a proper scheme over some field K of char. 0 and $Z \subset X$ a closed subscheme, one has the following exact *localization* sequence

$$\longrightarrow H^r_c(X\backslash Z) \longrightarrow H^r(X) \longrightarrow H^r(Z) \longrightarrow \tag{30}$$

where H^r_c is the cohomology with compact support. We assume K to be algebraically closed. We will use the following theorem.

Theorem 3.1
Denote by $\mathrm{cd}(X)$ *the cohomological dimension of a variety* X. *Then*

- $\mathrm{cd}(X) \leq 2\dim(X)$
- $\mathrm{cd}(X) \leq \dim(X)$ *if* X *is affine.*

Proof. For the proof of the first statement in less classical ℓ-adic case see [Mi1], Theorem 15.1. The proof of the second statement is given in [SGA7], XIV, Theorem 3.1; this statement is usually called the *Artin vanishing*. The analytic proof can be found in [Es].

□

Applying the Poincaré duality ([Mi1], Theorem 24.1) in the case of a smooth affine X, we get dually $H^r_c(X) = 0$ for $r < \dim(X)$.
Consider the following situation: $X \subset \mathbb{P}^m$ is defined by the vanishing of one homogeneous polynomial $f \in K[x_0, ..., x_m]$, $m \geq 2$, we write $X = \mathcal{V}(f)$ in such situation. Applying (30) to the inclusion $X \hookrightarrow \mathbb{P}^m$, we get

$$\longrightarrow H^r_c(\mathbb{P}^m \backslash X) \longrightarrow H^r(\mathbb{P}^m) \longrightarrow H^r(X) \longrightarrow . \tag{31}$$

Note that $\mathbb{P}^m \backslash X$ is affine (and smooth) of dimension m, thus, by the Artin vanishing $H^r_c(\mathbb{P}^m \backslash X) = 0$ for $0 \leq r \leq m-1$. This implies

$$H^r(X) \cong H^r(\mathbb{P}^m) \tag{32}$$

for $0 \leq r \leq m-2$ and $H^{m-1}(\mathbb{P}^m) \hookrightarrow H^{m-1}(X)$. So, the first interesting cohomology of a hypersurface in $\mathbb{P}^m$ is in degree $m-1$, we call it sometimes the cohomology of the *middle degree* $H^{mid}(X)$.

Definition 3.2
Define $H^r_{prim}(X) := coker(H^r(\mathbb{P}^m) \longrightarrow H^r(X))$ for all r.

We have no good reference for the following statement and we prove it here.

Theorem 3.3
Let $X \subset \mathbb{P}^n$ be a variety over algebraically closed field of characteristic 0. Then the morphism

$$\phi_r : H^r(\mathbb{P}^n) \longrightarrow H^r(X) \tag{33}$$

is injective for $0 \leq r \leq 2\dim X$.

Proof. First, consider the case of X being a hypersurface, so $\dim X = n-1$. Since $H(X) = H(X_{\text{red}})$, we can assume that X is reduced. For odd r the cohomology $H^r(\mathbb{P}^n)$ vanishes and there is nothing to prove. We start with top cohomology $H^{2n-2}(X)$, $r = 2n-2$. The singular locus Σ of the reduced hypersurface X is of dimension at most $n-2$. Define the complement $U := X - \Sigma$. Consider the localization sequence

$$\longrightarrow H^{2n-3}(\Sigma) \longrightarrow H^{2n-2}_c(U) \longrightarrow H^{2n-2}(X) \longrightarrow H^{2n-2}(\Sigma) \longrightarrow . \tag{34}$$

Both the leftmost and the rightmost terms vanish for dimensional reasons, and we get the isomorphism $H^{2n-2}(X) \cong H^{2n-2}_c(U)$. Let X be a union of irreducible components $X = \bigcup_{i=1}^j X_i$. We resolve singularities and get some $\hat{X} = \coprod_{i=1}^j \hat{X}_i$ with the inclusion $U \hookrightarrow \hat{X}$. This gives us the localization sequence

$$\begin{aligned} \longrightarrow H^{2n-3}(\hat{X} \backslash U) \longrightarrow H^{2n-2}_c(U) \longrightarrow \\ H^{2n-2}(\hat{X}) \longrightarrow H^{2n-2}(\hat{X} \backslash U) \longrightarrow \end{aligned} \tag{35}$$

Again, the term on the left and the term on the right are zero for reason of dimension, and we obtain $H_c^{2n-2}(U) \cong H^{2n-2}(\hat{X})$. Each $\hat{X}_i$ is a smooth projective scheme of dimension $n-1$, and we can compute

$$H^{2n-2}(X) \cong H^{2n-2}(\hat{X}) = \bigoplus_{i=1}^{j} \mathbb{Q}(-n+1). \tag{36}$$

For a general line $\ell \in \mathbb{P}^n$ we have $\ell \cap X \subset U$. Now, $H^{2n-2}(\mathbb{P}^n) \cong H_\ell^{2n-2}(\mathbb{P}^n) \cong \mathbb{Q}(-n+1)$. The intersection with X induces a map

$$H_\ell^{2n-2}(\mathbb{P}^n) \xrightarrow{\alpha} H_{\ell\cap X}^{2n-2}(X). \tag{37}$$

By excision, we have an isomorphism

$$\beta : H_{\ell\cap X}^{2n-2}(X) \cong H_{\ell\cap X}^{2n-2}(U) \cong H_{\ell\cap X}^{2n-2}(\hat{X}). \tag{38}$$

Note that $\ell \cap X$ is a union of $\deg X$ points (lying on U). We have a natural morphism

$$H_{\ell\cap X}^{2n-2}(\hat{X}) \xrightarrow{\gamma} H^{2n-2}(\hat{X}) \cong H^{2n-2}(\coprod_{i=1}^{j} \hat{X}_i). \tag{39}$$

Here γ maps the class of a point $p \in \ell \cap X$ to the class of this point in $H^{2n-2}(\hat{X}_i)$ when $p \in X_i$. All the $\hat{X}_i$ are smooth projective of dimension $n-1$. Thus $H^{2n-2}(\hat{X}_i)$ is one-dimensional and generated by the class of a point. Then the composition $\gamma\beta\alpha$ is a nonzero map. Since $H^{2n-2}(\mathbb{P}^n)$ is one-dimensional, this proves that ϕ_{2n-2} is injective.

Now we consider maps $H^{2i}(\mathbb{P}^n) \xrightarrow{\phi_{2i}} H^{2i}(X)$, $i \leq n-1$, and take $n-1$ general hyperplanes $H_1, \ldots, H_{n-1} \subset \mathbb{P}^n$. The cohomology to the left is generated by the class $[D_i]$ with $D_i := H_1 \cap \ldots \cap H_i$. For injectivity it is enough to show that $\phi_{2i}([D_i]) \neq 0$. Using the cup-product on $H^*(X)$, we obtain

$$\phi_{2i}([D_i]) = \phi_2([D_1])^i \in H^{2i}(X). \tag{40}$$

We see that $D_{n-1} = H_1 \cap \ldots \cap H_{n-1}$ is a general line $\ell \in \mathbb{P}^n$ and it was proved above that $\phi_{2n-2}([\ell]) = \phi_2([D_1])^{n-1} \neq 0$ in $H^{2n-2}(X)$. Thus $\phi_{2i}([D_i]) \neq 0$ and ϕ_{2i} in injective for all $i \leq n-1$.

Suppose now that $X \subset \mathbb{P}^n$ is defined by m homogeneous polynomials, $X = \mathcal{V}(f_1, \ldots, f_m)$, and is of dimension d. We can play the same game for X_{red} to show that ϕ_i are injective for $i \leq 2d$. Indeed, take general hyperplanes H_i, $1 \leq i \leq d$ and define $D_d := H_1 \cap \ldots \cap H_d$. Then $D_d \cap X \subset$

X_{smooth}. Denote by $\hat{X}$ the Hironaka resolution of singularities of X_{res} (exists since $K = \bar{K}$). By the same argument as above,

$$H^{2d}(X) \cong H^{2d}(\hat{X}) \cong \bigoplus_{i=1}^{j} \mathbb{Q}(-d). \tag{41}$$

Note that in $H^{2d}(\hat{X})$ only the summands which correspond to the resolutions of the irreducible components of maximal dimension ($= \dim(X)$) may survive, all other die for reason of dimension.

The intersection $D_d \cap X$ is a union of points. We explain the map ϕ_{2d} as above and conclude that $\phi_{2d}([D_d]) \neq 0$ in $H^{2d}(X)$. Now it follows that $\phi_{2i}([D_i]) \neq 0$ and ϕ_{2i} is injective for all $i \leq d$.

□

Let X be a proper scheme and $Y \subset X$ be a closed subscheme. By the theorem above, the localization sequence for $Y \subset X$ implies that the sequence

$$\longrightarrow H^i_c(X \backslash Y) \longrightarrow H^i_{prim}(X) \longrightarrow H^i_{prim}(Y) \longrightarrow \tag{42}$$

is exact in all terms up to $H^i_{prim}(Y)$ for $i = 2\dim Y$. The Mayer-Vietoris sequence for the closed covering $X = X_1 \cup X_2$ yields the sequence

$$\longrightarrow H^i_{prim}(X) \longrightarrow H^i_{prim}(X_1) \oplus H^i_{prim}(X_2) \longrightarrow H^i_{prim}(X_1 \cap X_2) \longrightarrow \tag{43}$$

which is exact in terms up to $H^i_{prim}(X_1 \cap X_2)$ for $i = 2\dim X_1 \cap X_2$.

The next two theorems are often referred to in the next sections.

Theorem 3.4 (Vanishing Theorem A)
Let Y be a variety $\mathcal{V}(f_1, f_2, \ldots, f_k) \subset \mathbb{P}^N(a_0 : a_1 : \ldots : a_N)$ for some homogeneous polynomials $f_1, \ldots, f_k \in K[a_0, \ldots, a_N]$, and suppose that f_i is independent of the first t variables $a_0, \ldots, a_{t-1}$ for each i, $1 \leq i \leq k$. Then

1) $H^r_{prim}(Y) = 0$ *for* $r < N - k + t$.

2) $H^r(Y) \cong H^{r-2t}(Y')(-t)$ *for* $r \geq 2t$, *where* $Y' \subset \mathbb{P}^{N-t}$ *is defined by the same polynomials.*

Proof. Suppose first that $t = 0$. We prove that $H^r(\mathbb{P}^N \backslash Y) = 0$ for $r \geq N + k$ using induction on k. For $k = 1$ we have an affine $\mathbb{P}^N \backslash Y$ and the statement is exactly Atrin's vanishing. Assume that $k > 1$ and the statement holds for all Y defined by at most $s < k$ polynomials.

Let $Y := \mathcal{V}(f_1, f_2, \ldots, f_k)$ and $U = \mathbb{P}^N \backslash Y$. Define the covering $U_1, U_2 \subset U$ by $U_1 := \mathbb{P}^N \backslash \mathcal{V}(f_1)$ and $U_2 := \mathbb{P}^N \backslash \mathcal{V}(f_2, \ldots, f_k)$. The intersection

$$U_3 := U_1 \cap U_2 = \mathbb{P}^N \backslash (\mathcal{V}(f_1) \cup \mathcal{V}(f_2, \ldots, f_k)) = \mathbb{P}^N \backslash \mathcal{V}(f_1 f_2, \ldots, f_1 f_k) \quad (44)$$

is again the complement of a complete intersection defined by at most $k-1$ polynomials. We write the Mayer-Vietoris sequence

$$\longrightarrow H^{r-1}(U_3) \longrightarrow H^r(U) \longrightarrow H^r(U_1) \oplus H^r(U_2) \longrightarrow \quad (45)$$

By the assumption both the cohomology to the left and the summands to the right vanish for $r-1 \geq N+k-1$. Thus, the sequence implies $H^r(U) = 0$ for $r \geq N+k$. The induction hypothesis follows. By duality, one has $H^r_c(\mathbb{P}^N \backslash Y) = 0$ for $r \leq N-k$. We have an exact sequence

$$\longrightarrow H^{r-1}_{prim}(\mathbb{P}^N) \longrightarrow H^{r-1}_{prim}(Y) \longrightarrow H^r_c(\mathbb{P}^N \backslash Y) \longrightarrow H^r_{prim}(\mathbb{P}^N) \longrightarrow \quad (46)$$

Since $H^i_{prim}(\mathbb{P}^N) = 0$ for all i, the sequence gives us the isomorphism $H^{r-1}_{prim}(Y) \cong H^r_c(\mathbb{P}^N \backslash Y)$. Thus $H^r_{prim}(Y) = 0$ for $r < N-k$.

Suppose now that $t \geq 1$. Define $\Delta := \mathcal{V}(a_t, \ldots, a_N) \cong \mathbb{P}^{t-1}$. Consider the natural projection $\pi : \mathbb{P}^N \backslash \Delta \longrightarrow \mathbb{P}^{N-t}$. Note that $\Delta \subset Y$ is a closed subscheme, thus one has an exact sequence

$$\longrightarrow H^r_c(Y \backslash \Delta) \longrightarrow H^r_{prim}(Y) \longrightarrow H^r_{prim}(\Delta) \longrightarrow . \quad (47)$$

The map π gives us an $\mathbb{A}^t$-fibration over $\pi(Y \backslash \Delta) = Y'$, by homotopy invariance $H^r_c(Y \backslash \Delta) \cong H^{r-2t}(Y')(-t)$. Now, $H^r(Y \backslash \Delta) = 0$ for $r \leq 2n-1$ and $H^r(\Delta) = 0$ for $r \geq 2n-1$. The sequence above implies $H^r_{prim}(Y) \cong H^r_{prim}(\Delta) = 0$ for $r \leq 2n-2$, $H^{2n-1}(Y) = 0$, and $H^r(Y) \cong H^r(Y \backslash \Delta) \cong H^{r-2t}(Y')(-t) = 0$ for $2t \leq r \leq N-k+t$. We applied here the case $t=0$ for Y'. The statement follows.

□

Theorem 3.5 (Vanishing Theorem B)
For homogeneous polynomials $f_1, \ldots, f_k, h \in K[a_0, \ldots, a_N]$, $k \geq 0$, define a subscheme $U \subset \mathbb{P}^N$ by equations $f_1 = \ldots = f_k = 0$ and inequality $h \neq 0$, i.e.

$$U := \mathcal{V}(f_1, \ldots, f_k) \backslash \mathcal{V}(f_1, \ldots, f_k, h).$$

Suppose that all the polynomials are independent of the first t variables $a_0, \ldots, a_{t-1}$, and let $U' \subset \mathbb{P}^{N-t}$ be defined by the same polynomials but in $\mathbb{P}^{N-t}(a_t : \ldots : a_N)$. Then the following equalities hold:

1) $H^i_c(U) = 0$ *for* $i < N-k+t$.

2) $H_c^i(U) \cong H_c^{i-2t}(U')(-t)$.

Proof. Suppose first that $t = 0$. Let $Y := \mathcal{V}(f_1 \dots, f_n) \subset \mathbb{P}^N$, then $U \cong Y \backslash Y \cap \mathcal{V}(h)$. We have an exact sequence

$$\longrightarrow H^{r-1}_{prim}(Y \cap \mathcal{V}(h)) \longrightarrow H^r_c(U) \longrightarrow H^r_{prim}(Y) \longrightarrow \qquad (48)$$

By Theorem A, both the cohomology to the right and the cohomology to the left vanish for $r - 1 < N - k - 1$. Thus $H^r_c(U) = 0$, $r < N - k$.

Let $t \geq 1$, consider the natural projection $\mathbb{P}^N \backslash \Delta \longrightarrow \mathbb{P}^{N-t}$, where $\Delta := \mathcal{V}(a_t, \dots, a_N)$. It maps U onto U' with fibres $\mathbb{A}^t$. Thus $H^r_c(U) \cong H^{r-2t}(U')(-t)$ for all r. The case $t = 0$ applied to $U' \subset \mathbb{P}^{N-t}$ gives us $H_c^{r-2t}(U') = 0$ for $r - 2t < N - t - k$, thus $H^r_c(U) = 0$ for $r < N - k + t$.

□

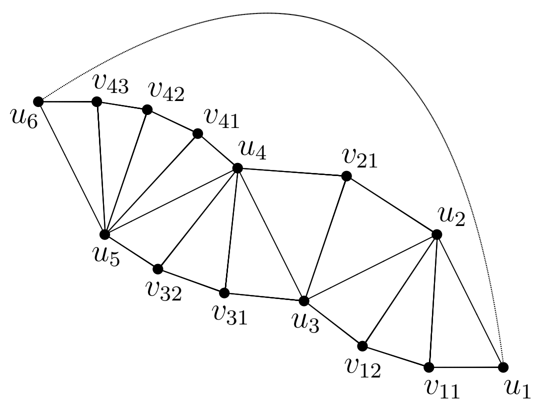
Remark 3.6
In the computation for GZZ (*Theorem 4.6*) we will use the (a priory more weaker result) homotopy invariance *h.i.* instead of the vanishing theorems above because we do not want to apply the Artin vanishing. This makes our proof more motivic because one does not have the Artin vanishing for motivic cohomology.

4 Generalized zigzag graphs

Definition 4.1
Fix some $t \geq 1$ and consider a set $V(\Gamma)$ of $t+2$ vertexes u_i, $1 \leq i \leq t+2$. Define $p(u_1, \dots, u_{t+2})$ to be the set of $t+1$ edges (u_i, u_{i+1}), $1 \leq i \leq t+1$. Let $E(\Gamma) := p(u_1, \dots, u_{t+2})$. Now choose some positive integers l_i for $1 \leq i \leq t$ with $l_1 \geq 2$ and $l_t \geq 2$. For each i, $1 \leq i \leq t$, we add $l_i - 1$ new vertexes v_{ij}, $1 \leq j \leq l_i - 1$, and l_i new edges $p(u_i, v_{i1}, \dots, v_{i\,l_i-1}, u_{i+2})$, and $l_i - 1$ edges (v_{ij}, u_{i+1}), $1 \leq j \leq l_i - 1$. Finally, we add an edge (u_1, u_{t+2}). We call the constructed graph $\Gamma = (V(\Gamma), E(\Gamma))$ the *generalized zigzag* graph $GZZ(l_1, \dots, l_t)$.

For $GZZ(l_1, \dots, l_t)$ we define $n = 1 + \sum_{i=1}^t l_i$. The graph $GZZ(l_1, \dots, l_t)$ has $n+1$ vertexes, $2n$ edges and the Betti number equals n. Thus $GZZ(l_1, \dots, l_t)$ is a logarithmically divergent graph.

Example 4.2
The graph $GZZ(3, 2, 3, 4)$ looks like



Example 4.3
The wheel with spokes graph WS_n is isomorphic to the generalized zigzag graph $GZZ(n-1)$, $n \geq 3$.

Example 4.4
The zigzag graph ZZ_n is isomorphic to the $GZZ(2, 1, \ldots, 1, 2)$ (with $n-5$ 1's in the middle) for $n \geq 5$.

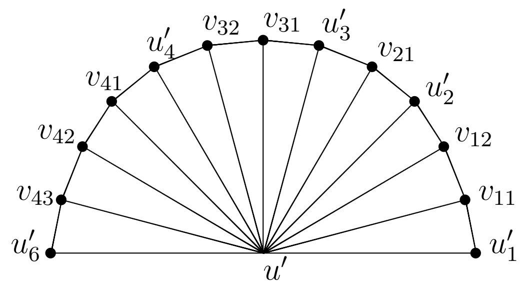
Theorem 4.5
A generalized zigzag graph $\Gamma = GZZ(l_1, \ldots, l_t)$ is primitively log divergent.

Proof. We need to prove that for any proper subgraph $\Gamma' \subset \Gamma$ the inequality $|E(\Gamma')| > 2h_1(\Gamma')$ holds, which means that Γ' is convergent. We do not distinguish between a graph and it's set of edges. Because our graph Γ is planar, it partitions the plain into exactly $h_1 + 1$ pieces. This is a good way to compute h_1. We will call the loops of length 3 simple loops. We can order the simple loops from the right bottom corner to the left top keeping in mind the drawing like in Example 4.2. Formally, let $\Delta_1 = p(u_1, u_2, v_{11}, u_1)$ and for each i we define the next simple loop Δ_{i+1} to be a simple loop which has a common edge with Δ_i but not already labeled. Define $\Gamma_0 := \Gamma \backslash (u_1, u_{t+2})$. The main point of the proof is the following. The graph Γ_0 is a strip of Δ's, for each i, $1 \leq i \leq k$, we can cut this strip along (u_i, u_{i+1}), turn over one piece and glue again along the same edges. Denote this operation by ϕ_i. This gives a map

$$\phi := \phi_t \circ \ldots \circ \phi_2 : \Gamma_0 \longrightarrow \hat{\Gamma}_0, \qquad (49)$$

where $\hat{\Gamma}_0$ is isomorphic to WS_n without one boundary edge; this graph is topologically the same as a half of WS_n, we denote it by hWS_n. Note that the maps ϕ_i and ϕ are the isomorphisms between sets of edges of the graphs in the described way. On some vertexes this map is not single-valued. For

Example 4.2, we have the following $\hat{\Gamma}_0 = hWS_{13}$



The vertex u_i under described operations goes to u'_{i-1}, or u'_i, or u' depending on the edge that we take. We can label the simple loops of hWS_n from the right to the left by $\hat{\Delta}_1, \ldots, \hat{\Delta}_{n-1}$, these are the images of Δ's

$$\phi(\Delta_i) = \hat{\Delta}_i. \tag{50}$$

Each ϕ_i preserves loops; this means that a subgraph $\gamma \subset \phi_{i-1} \ldots \phi_2(\Gamma)$ is a loop if and only if $\phi_i(\gamma)$ is a loop of the same length. Thus, this condition holds for ϕ. It follows that Γ_0 and hWS_n have the same Betti numbers. Moreover, for each subgraph $\Gamma''_0 \subset \Gamma_0$ we have

$$h_1(\Gamma''_0) = h_1(\phi(\Gamma''_0)). \tag{51}$$

To involve the "special" edge (u_1, u_{t+2}) into consideration, note that if the graph Γ''_0 is disconnected and we have no path $p'(u_1, \ldots, u_{t+1})$ with endpoints u_1 and u_{t+1}, then the adding of (u_1, u_{t+2}) doesn't change the Betti number; otherwise this increases the number by one.

$$h(\Gamma''_0 \cup (u_1, u_{t+1})) = \begin{cases} h(\Gamma''_0), & p'(u_1, \ldots, u_{t+2}) \not\subset \Gamma''_0, \\ h(\Gamma''_0) + 1, & \text{otherwise.} \end{cases} \tag{52}$$

This proves that we can extend the map ϕ to

$$\bar{\phi} : \Gamma \longrightarrow \hat{\Gamma} \tag{53}$$

which maps our graph to $\hat{\Gamma}$, that is nothing but $\hat{\Gamma}_0 \cong hWS_n$ compactified by adding the missing boundary edge and is isomorphic to WS_n. The map $\bar{\phi}$ satisfies the same condition as ϕ in (51). For the example of hWS_{13} above, we add the edge (u_1, u_6) on the drawing and get WS_{13}.

So, we reduced the statement to the case WS_n. It is known, that these graphs are primitively log divergent. This concludes the proof.

□

Let $X \subset \mathbb{P}^{2n-1}$ be a graph hypersurface. We consider the Betti cohomology of the middle degree $H^{mid}(X) = H^{2n-2}(X)$. By Deligne's theory of MHS ([De2], [De3]), there is a $\mathbb{Q}$-mixed Hodge structure associated to $H^{mid}(X)$. We can try to study the graded pieces of weight filtration W: $\mathrm{gr}_i^W(H^r(X))$, $0 \leq i \leq 2n-2$, $r \geq mid$.

Theorem 4.6
For the hypersurface X associated to a generalized zigzag graph GZZ, one has the isomorphisms

$$\begin{aligned} \mathrm{gr}_4^W(H^{mid}_{prim}(X)) \cong W_4(H^{mid}_{prim}(X)) \cong W_5(H^{mid}_{prim}(X)) \cong \mathbb{Q}(-2), \\ W_5(H^r_{prim}(X)) = 0, \quad r > mid. \end{aligned} \tag{54}$$

Proof. We consider the case when t is even and start with labeling of edges and choosing orientations. For simplicity, let $n_0 := 0$ and

$$n_i := \sum_{j=1}^{i} l_j, \quad \text{for } 1 \leq i \leq t. \tag{55}$$

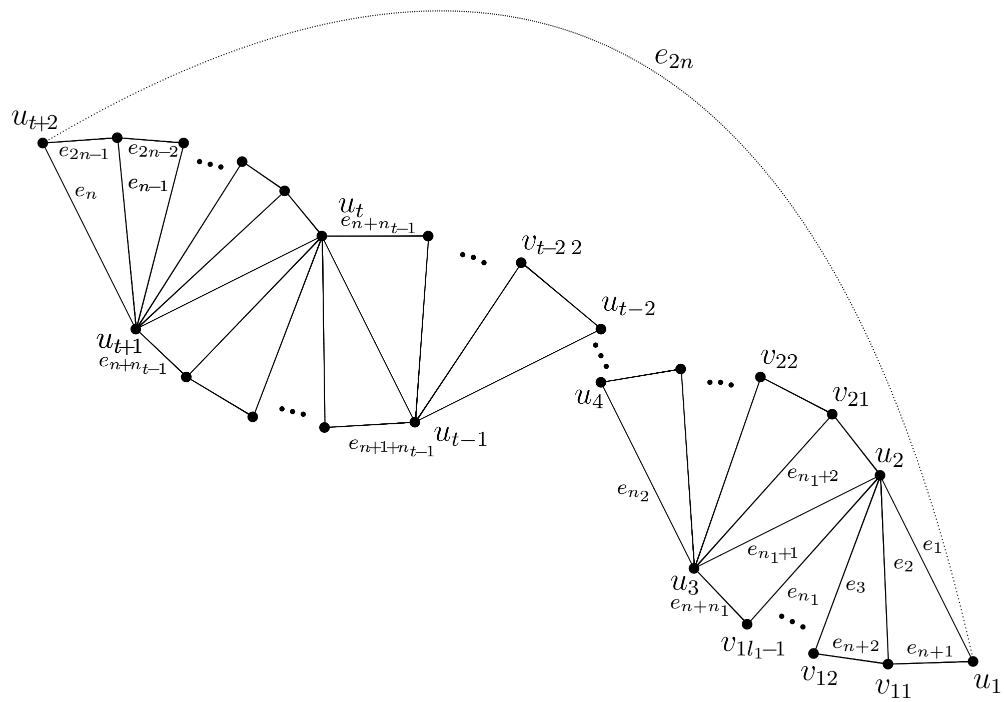
For each i, $1 \leq i \leq t$, define $e_{n_{i-1}+1} := (u_{i+1}, u_i)$ for odd i and $e_{n_{i-1}+1} := (u_i, u_{i+1})$ for even i,

$$e_{n_{i-1}+j} := \begin{cases} (u_{i+1}, v_{i\,j-1}) & \text{for } 2 \leq j \leq l_i, \ i \text{ odd}, \\ (v_{i\,j-1}, u_{i+1}) & \text{for } 2 \leq j \leq l_i, \ i \text{ even}. \end{cases} \tag{56}$$

Together with $e_{n_t+1} := (u_{t+2}, u_{t+1})$ for even t and $e_{n_t+1} := (u_{t+1}, u_{t+2})$ for odd t, these are the first $n_t + 1 =: n$ edges. Now, for each i, $1 \leq i \leq t$, define $e_{n+n_{i-1}+1} := (v_{i1}, u_i)$,

$$e_{n+n_{i-1}+j} := (v_{ij}, v_{ij-1}), \quad \text{for } 2 \leq j \leq l_i - 1, \tag{57}$$

and $e_{n+n_{i-1}+l_i} := (u_{i+2}, v_{i\,l_i-1})$. Roughly speaking, all edges are oriented from the left top corner to the right bottom and from the right top corner to the left bottom corner. Define $e_{2n} := (u_1, u_{t+2})$.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
1	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
4	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
5	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
6	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
7	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
8	0	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
9	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	0	1	0
13	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	1	1

For building the table, we take the small loops from right bottom corner of the drawing to the left top corner, and the last loop to be chosen is the loop with the edge (u_1, u_{t+2}). Because of lack of space, we draw the table for the graph in Example 4.2.

Now we take $2n$ variables $T_1, \ldots, T_{2n}$ and build a matrix $\mathcal{M}(T)$ as the sum of elementary matrices. After a change of the variables, we get the matrix

$$\mathcal{M}_{GZZ} = \begin{pmatrix} B_0 & A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{17} \\ A_0 & B_1 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_2 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & C_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{16} \\ 0 & 0 & 0 & A_3 & C_4 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & A_{15} \\ 0 & 0 & 0 & 0 & A_4 & B_5 & A_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_5 & B_6 & A_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_6 & B_7 & A_7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_7 & C_8 & A_8 & 0 & 0 & A_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_8 & C_9 & A_9 & 0 & A_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_9 & C_{10} & A_{10} & A_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{10} & B_{12} & A_{11} \\ A_{17} & 0 & 0 & A_{16} & A_{15} & 0 & 0 & 0 & A_{14} & A_{13} & A_{12} & A_{11} & B_{13} \end{pmatrix}. \tag{58}$$

In the last row the A's appear in the zero column and in the columns n_i+j-1 for all $i \not\equiv t \bmod 2$, $1 \leq i \leq t$, and all $1 \leq j \leq l_i$. In the same columns (but 0 and $n-2$) we have C's in the main diagonal. This C's are defined by

$$C_k := \begin{cases} A_v + A_{k-1} - A_k, & k = n_i, l_{i+1} > 1, i \neq 0, \\ A_v - A_{k-1} - A_k, & k = n_i + j, 1 \leq j \leq l_{i+1} - 2, \\ A_v - A_{k-1} + A_k, & k = n_{i+1} - 1, l_{i+1} > 1, i \neq t-1, \\ A_v + A_{k-1} + A_k, & k = n_i, l_{i+1} = 1, \end{cases} \tag{59}$$

where $i \not\equiv t \bmod 2$, and A_v is always in the last row in the same column as C_k. Formally, if $k = n_i + j - 1$, then

$$v = v(k) = n - 2 + \sum_{\substack{r=i+2 \\ r \not\equiv t \bmod 2}}^{t-1} l_r + l_{i+1} - j. \tag{60}$$

Sometimes we denote by A_m the entry in the left bottom corner of $\mathcal{M}_{GZZ}$.

For the case of odd t we can derive the tables and the matrices from the even case. Indeed, consider some $\Gamma' = GZZ(l_1, \ldots, l_t)$ with even t and let Γ be the graph which we get from Γ' after forgetting edges of simple loops $\Delta_1, \ldots, \Delta_{l_1}$ (see Theorem 4.5 for definition), we assume that (u_2, u_3) remains, and we take (u_{t+2}, u_2) instead of (u_{t+2}, u_1). So, $\Gamma = GZZ(l_2, \ldots, l_t)$. Constructing everything similar, the table for Γ is that for Γ' without first l_1 rows. The matrix of Γ looks similar to that of Γ' with the same assumptions on A's in the last row and on C's.

Consider the projective space $\mathbb{P}^{2n-1}$ with coordinates all the A_i's and B_j's appearing in the matrix and define $X := \mathcal{V}(\det(\mathcal{M}_{GZZ})) = \mathcal{V}(I_n) \subset \mathbb{P}^{2n-1}$, where

$$\mathcal{M}_{GZZ} = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots \\ \dots & C_{n-3} & A_{n-3} & A_{n-1} \\ \dots & A_{n-3} & B_{n-2} & A_{n-2} \\ \dots & A_{n-1} & A_{n-2} & B_{n-1} \end{pmatrix}. \tag{61}$$

Since $l_t > 1$, the entry $a_{n-3\,n-3}$ is really not independent, thus C_{n-3}.

<u>*Step 1.*</u> Fix some r, $2n-2 \leq r \leq 4n-4$. We are going to compute $H^r(X)$. For the closed subscheme $\mathcal{V}(I_n, I_{n-1}) \subset X$ we have the localization sequence

$$\begin{aligned} \to H^r_c(X\backslash\mathcal{V}(I_n, I_{n-1})) \to H^r(X) \to \\ H^r(\mathcal{V}(I_n, I_{n-1})) \to H^{r+1}_c(X\backslash\mathcal{V}(I_n, I_{n-1})) \to . \end{aligned} \tag{62}$$

We can write

$$I_n = B_{n-1}I_{n-1} - G_{n-1}, \tag{63}$$

where G_{n-1} is independent of B_{n-1}. Projecting from the point where all the variables but B_{n-1} are zero, we get

$$X\backslash\mathcal{V}(I_n, I_{n-1}) \cong \mathbb{P}^{2n-2}\backslash\mathcal{V}(I_{n-1}). \tag{64}$$

Since I_{n-1} is independent of A_{n-2} and A_m, the *h.i.* for scheme on the right hand side of (64) implies

$$H^q_c(X\backslash\mathcal{V}(I_n, I_{n-1})) \cong H^{q-4}_c(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I_{n-1}))(-2) \tag{65}$$

for $q = r$ and $r+1$. We make the change of the coordinates of $\mathbb{P}^{2n-4}$(no B_{n-1}, A_{n-2}, A_m) $C_i := A_{v(i)}$ for all C_i (see 60) and denote the image of I_{n-1} by I'_{n-1}. Using the localization sequence for the closed embedding $I'_{n-1} \subset \mathbb{P}^{2n-4}$, we get

$$H^{q-4}_c(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I'_{n-1})) \cong H^{q-5}_{prim}(\mathcal{V}(I'_{n-1})). \tag{66}$$

Consider $T := \mathcal{V}(I'_{n-1}, I'_{n-2}) \subset \mathcal{V}(I'_{n-1})$. One has the localization sequence

$$\to H^{q-5}_c(\mathcal{V}(I'_{n-1})\backslash T) \to H^{q-5}_{prim}(\mathcal{V}(I'_{n-1})) \to H^{q-5}_{prim}(T) \to . \tag{67}$$

We can rewrite $T = \mathcal{V}(I'_{n-2}, A_{n-3}I'_{n-3})$. So, the defining polynomials of T do not depend on B_{n-2}. Now, on $\mathcal{V}(I'_{n-1})\backslash T$ we can express B_{n-2} from the equation and get $\mathcal{V}(I'_{n-1})\backslash T \cong \mathbb{P}^{2n-5}\backslash\mathcal{V}(I'_{n-2})$. The polynomial I'_{n-2} is independent of A_{n-3}. Thus we can apply *h.i.* to T and $\mathcal{V}(I'_{n-1})\backslash T$, and then applying gr_i^W to (67), we obtain

$$\mathrm{gr}_i^W H^{q-5}_{prim}(\mathcal{V}(I'_{n-1})) = 0, \quad i = 0, 1, \text{ any } q. \tag{68}$$

By (65), (66) and (68), the sequence (62) yields

$$\mathrm{gr}_i^W H^r(X) \cong \mathrm{gr}_i^W H^r(\mathcal{V}(I_n, I_{n-1})), \qquad i = 0, \ldots, 5. \tag{69}$$

By (63), one has $\mathcal{V}(I_n, I_{n-1}) \cong \mathcal{V}(I_{n-1}, G_{n-1})^{(2n-1)}$. Both polynomials to the right are independent of B_{n-1}. *H.i.* for $\mathcal{V}(I_{n-1}, G_{n-1})^{(2n-1)}$ and (69) imply

$$\mathrm{gr}_i^W H^r(X) \cong \mathrm{gr}_{i-2}^W H^{r-2}(\mathcal{V}(I_{n-1}, G_{n-1})), \quad i = 0, \ldots, 5. \tag{70}$$

The variety to the right lives in $\mathbb{P}^{2n-2}$(no B_{n-1}). Define the closed subscheme $\hat{V} \subset \mathcal{V}(I_{n-1}, G_{n-1})$ by $\hat{V} := \mathcal{V}(I_{n-1}, I_{n-2}, G_{n-1}) \subset \mathbb{P}^{2n-2}$ (no B_{n-1}). One has an exact sequence

$$\begin{aligned} \to H_c^{r-2}(\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V}) \to H^{r-2}(\mathcal{V}(I_{n-1}, G_{n-1})) \to \\ H^{r-2}(\hat{V}) \to H_c^{r-1}(\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V}) \to . \end{aligned} \tag{71}$$

The polynomial I_{n-1} is independent of A_{n-2} and the coefficient of A_{n-2}^2 in G_{n-1} is I_{n-2}. By *Corollory 1.4*, we have $I_{n-2}G_{n-1} = (L_{n-1})^2$ on $\mathcal{V}(I_{n-1})$, so

$$\begin{cases} I_{n-1} = G_{n-1} = 0 \\ \qquad I_{n-2} \neq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} I_{n-1} = L_{n-1} = 0 \\ \qquad I_{n-2} \neq 0. \end{cases} \tag{72}$$

Now,

$$L_{n-1} = A_{n-2}I_{n-2} + \sum_s (-1)^{s+n-2} A_v(s) I_{n-1}(s; n-2), \tag{73}$$

where the sum goes over all s such that $a_{ss} = C_s$, and also $s = 0$ (assuming $v(0) = m$), see (60), (10). Solving on A_{n-2} and projecting from the point where all the coordinates but A_{n-2} are zero, we get

$$\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V} \cong \mathcal{V}(I_{n-1})\backslash\mathcal{V}(I_{n-1}, I_{n-2}). \tag{74}$$

Expressing B_{n-2} from $I_{n-1} = 0$ and projecting further, we obtain an isomorphism $\mathcal{V}(I_{n-1})\backslash\mathcal{V}(I_{n-1}, I_{n-2}) \cong \mathbb{P}^{2n-4}\backslash\mathcal{V}(I_{n-2})$. The polynomial I_{n-2} is independent of A_m and A_{n-1} (after $C_{n-3} := A_{n-3}$). Applying *h.i.*, one gets $\mathrm{gr}_i^W H_c^q(\mathcal{V}(I_{n-1}, G_{n-1})\backslash\hat{V}) \cong \mathrm{gr}_{i-4}^W H_c^{q-4}(\mathbb{P}^{2n-6}\backslash\mathcal{V}(I_{n-2})) = 0$ for $i = 0, \ldots, 3$ and any q. The sequence (71) yields

$$\mathrm{gr}_i^W H^{r-2}(\mathcal{V}(I_{n-1}, G_{n-1})) \cong \mathrm{gr}_i^W H^{r-2}(\hat{V}), \quad i = 0, \ldots, 3. \tag{75}$$

By Theorem 1.5, the polynomial G_{n-1} is independent of A_{n-2} on $\hat{V}$. Thus, $\hat{V}$ is defined by the vanishing of three polynomials which are independent of A_{n-2}. Applying *h.i.*, one gets

$$H^{r-2}(\hat{V}) \cong H^{r-4}(V)(-1), \tag{76}$$

where $V := \mathcal{V}(I_{n-1}, I_{n-2}, G'_{n-1}) \subset \mathbb{P}^{2n-3}$ (no B_{n-1}, A_{n-1}) and

$$G'_{n-1} := G_{n-1}|_{A_{n-2}=0}. \tag{77}$$

Combining (70), (75) and (76), we get

$$\mathrm{gr}_i^W H^r(X) \cong \mathrm{gr}_{i-4}^W H^{r-4}(V), \quad i = 0, \dots, 5. \tag{78}$$

<u>*Step 2.*</u> Now we get rid of B_{n-2}. We can write

$$G'_{n-1} = B_{n-2}G_{n-2} - A_{n-3}^2 G_{n-3}, \tag{79}$$

where G_{n-2} and G_{n-3} are considered to be polynomials of variables $A_{n-1}, \dots, A_m$ and $A_n, \dots, A_m$ with "coefficients" from the matrices of I_{n-2} and of I_{n-3} respectively. The decomposition follows from the fact that each coefficient of G'_{n-1} is a factor of some I^j_{n-j-1} for $0 \le j \le n-2$, and the 3-diagonal matrix of I^j_{n-j-1} has the right bottom entry B_{n-2}. Define the variety $\hat{T}_{n-2} := V \cap \mathcal{V}(G_{n-2}) = \mathcal{V}(A_{n-3}I_{n-3}, I_{n-2}, G_{n-2}, A_{n-3}G_{n-3})) \subset V$. One has an exact sequence

$$H^{r-5}_{prim}(\hat{T}) \to H^{r-4}_c(V \backslash \hat{T}) \to H^{r-4}_{prim}(V) \to H^{r-4}_{prim}(\hat{T}) \to . \tag{80}$$

Since the defining polynomials of $\hat{T}$ are independent of B_{n-2}, *h.i.* for $\hat{T}$ implies

$$\to H^{r-7}_{prim}(T)(-1) \to H^{r-4}_c(V \backslash \hat{T}) \to H^{r-4}_{prim}(V) \to H^{r-6}_{prim}(T)(-1) \to \tag{81}$$

for $T \subset \mathbb{P}^{2n-4}$ (no B_{n-1}, A_{n-2}, B_{n-2}) defined by the same equations as $\hat{T}$. Applying the exact functor gr_*^W to the sequence above, we obtain

$$\mathrm{gr}_i^W H^{r-4}_{prim}(V) \cong \mathrm{gr}_i^W H^{r-4}_c(V \backslash \hat{T}), \quad i = 0, 1. \tag{82}$$

The subscheme $V \backslash \hat{T} \subset V$ is defined by the system

$$\begin{cases} I_{n-2} = A_{n-3}I_{n-3} = 0 \\ B_{n-2}G_{n-2} - A_{n-3}^2 G_{n-3} = 0 \\ G_{n-2} \neq 0. \end{cases} \tag{83}$$

Projecting from the point where all the variables but B_{n-2} are zero and solving the middle equation on B_{n-2}, we get an isomorphism

$$\begin{aligned} V\backslash\hat{T} \cong \mathcal{V}(I_{n-2}, A_{n-3}I_{n-3})\backslash\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2}) \\ =: U_1 \subset \mathbb{P}^{2n-4}(\text{ no } B_{n-1}, A_{n-2}, B_{n-2}). \end{aligned} \tag{84}$$

One has an exact sequence

$$\begin{aligned} H^{r-5}_{prim}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3})) \to H^{r-5}_{prim}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})) \to \\ H^{r-4}_c(U_1) \to H^{r-4}_{prim}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3})) \to . \end{aligned} \tag{85}$$

The variety $\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}) \subset \mathbb{P}^{2n-4}$ is defined by the polynomials independent of A_m. After applying of *h.i.* and then gr^W_*, the sequence yields

$$\operatorname{gr}^W_i H^{r-4}_c(U_1) \cong \operatorname{gr}^W_i H^{r-5}_{prim}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})), \quad i = 0, 1. \tag{86}$$

Define $\hat{S} := \mathcal{V}(I_{n-2}, I_{n-3}, G_{n-2})$ and $U_2 := \mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})\backslash\hat{S}$ in $\mathbb{P}^{2n-4}$. One has an exact sequence

$$\begin{aligned} \longrightarrow H^{r-6}_{prim}(\hat{S}) \longrightarrow H^{r-5}_c(U_2) \longrightarrow \\ H^{r-5}_{prim}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})) \longrightarrow H^{r-5}_{prim}(\hat{S}) \longrightarrow . \end{aligned} \tag{87}$$

The only appearance of A_{n-3} in the defining polynomials of S is in G_{n-2}, namely in C_{n-3}. After a linear change of the variables we may assume that $C_{n-3} := A_{n-3}$ is independent. Furthermore, the same argument as for $\hat{V}$ at *Step 1* (see (76)) gives us $H^{r-5}(\hat{S}) \cong H^{r-7}(S)(-1)$ with $S := \mathcal{V}(I_{n-2}, I_{n-3}, G''_{n-2}) \subset \mathbb{P}^{2n-5}(\text{no } B_{n-1}, A_{n-2}, B_{n-2}, A_{n-1})$. The sequence (87) simplifies to

$$\begin{aligned} \longrightarrow H^{r-7}_{prim}(S)(-1) \longrightarrow H^{r-5}_c(U_2) \longrightarrow \\ H^{r-5}_{prim}(\mathcal{V}(I_{n-2}, A_{n-3}I_{n-3}, G_{n-2})) \longrightarrow H^{r-7}_{prim}(S)(-1) \longrightarrow . \end{aligned} \tag{88}$$

Applying the functors gr^W_i to the sequence, by (82), (84), (86) and (88), we get

$$\operatorname{gr}^W_i H^{r-4}_{prim}(V) \cong \operatorname{gr}^W_i H^{r-5}_c(U_2) \quad \text{for } i = 0, 1. \tag{89}$$

Now, the scheme U_2 is defined by the system

$$\left\{ \begin{aligned} I_{n-2} = G_{n-2} = 0 \\ A_{n-3}I_{n-3} = 0 \\ I_{n-3} \neq 0 \end{aligned} \right. \quad \Leftrightarrow \quad \left\{ \begin{aligned} G_{n-2} = I_{n-2} = 0 \\ A_{n-3} = 0. \\ I_{n-3} \neq 0 \end{aligned} \right. \tag{90}$$

Eliminating A_{n-3}, which is zero on U_2, we get an isomorphism

$$U_2 \cong U_2' \tag{91}$$

with $U_2' := \mathcal{V}(I_{n-2}', G_{n-2}') \backslash \mathcal{V}(I_{n-2}', G_{n-2}', I_{n-3}) \subset \mathbb{P}^{2n-5}$ (no B_{n-1}, A_{n-2}, B_{n-2},A_{n-3}), where primes mean that we set $A_{n-3} = 0$ in the polynomials, namely in C_{n-3}. Now we write

$$C_{n-3}' = A_{n-1} \pm A_{n-4}, \tag{92}$$

with "+" only when $a_{n-4n-4} = B_{n-4}$ in the matrix.

By *Corollary 1.4*, it follows that U_2' is defined by the system

$$\begin{cases} C_{n-3}' I_{n-3} - A_{n-4}^2 I_{n-4} = 0 \\ L_{n-2} = 0 \\ I_{n-3} \neq 0 \end{cases} \tag{93}$$

with

$$L_{n-2} := A_{n-1} I_{n-3} + \sum_s (-1)^{s+n-1} A_{v(s)} I_{n-2}(s, n-3) = \\ A_{n-1} I_{n-3} + \sum_s (-1)^{s+n-1} A_{v(s)} I_s \prod_{k=s}^{n-4} A_k. \tag{94}$$

The sum goes over all $s = n_i + j - 1 < n - 3$, $i \not\equiv t \bmod 2$, $1 \le i \le t$, $1 \le j \le l_i$, so over all $s < n - 3$ such that $a_{ss} = C_s$. It is convenient to use the recurrence formula

$$L_{s+1} = \begin{cases} A_{v(s)} I_s - A_{s-1} L_s, & a_{s+1s+1} = C_{s+1}, \\ -A_{s-1} L_s, & a_{s+1s+1} = B_{s+1}. \end{cases} \tag{95}$$

We can express A_{n-1} from the second equation of the system (93) and C_2 from the first one.

$$\begin{cases} A_{n-1} \pm A_{n-4} = C_{n-3}' = A_{n-4}^2 I_{n-4} / I_{n-3} \\ A_{n-1} = A_{n-4} L_{n-3} / I_{n-3} \\ I_{n-3} \neq 0. \end{cases} \tag{96}$$

These two expressions for A_{n-1} must be equal on U_2'. We introduce the polynomials N_s defined by $A_{s-1} N_s = \pm A_{s-1} I_s + A_{s-1}^2 I_{s-1} - A_{s-1} L_s$. Sometimes we write N_s^- and N_s^+ to indicate the sign taken in the expression on the right. The natural projection from the point where all the variables but A_{n-1} are zero induces an isomorphism

$$U_2' \cong U_3 := \mathcal{V}(A_{n-4} N_{n-3}) \backslash \mathcal{V}(A_{n-4} N_{n-3}, I_{n-3}) \tag{97}$$

with $U_3 \subset \mathbb{P}^{2n-6}$(no B_{n-1}, A_{n-2}, B_{n-2}, A_{n-3}, A_{n-1}). By (89), and (91),

$$\operatorname{gr}_i^W H^{r-4}_{prim}(V) \cong \operatorname{gr}_i^W H^{r-5}_c(U_3) \quad \text{for } \ i = 0, 1. \tag{98}$$

We have two possibilities : $a_{n-4\,n-4} = C_{n-4}$ or $a_{n-4\,n-4} = B_{n-4}$. When the latter holds, go to *Step 4* with $N_{n-3} = N^-_{n-3}$; do the next step with $N_{n-3} = N^+_{n-3}$ otherwise.

Step 3. Suppose that the entry a_{ss} of M_{GZZ} is C_s and $a_{s+1s+1} = C_{s+1}$. With other words, $n_i \leq s \leq n_i + l_i - 2$ for some $i \not\equiv t \bmod 2$. This corresponds to the case $s = n-4$ if we had come from *Step 2*. One has

$$C_s = A_v - A_s \pm A_{s-1} \tag{99}$$

with "+" only when $a_{s-1\,s-1} = B_s$. We work in $\mathbb{P}^N$(no DV_s) for $N = 2n-1-2(n-1-s-1)-1 = 2s+2$, and the Dropped Variables (DV_s) are all the variables in I^{s+1}_{n-1-s} but A_s. We are going to compute $H^q_c(U)$ for $q \geq 2s+1$ and U defined by

$$U := \mathcal{V}(A_s N_{s+1}) \backslash \mathcal{V}(A_s N_{s+1}, I_{s+1}), \tag{100}$$

where $N_{s+1} = I_{s+1} + A_s I_s - L_{s+1}$. Let $T := \mathcal{V}(A_s N_{s+1})$, $Y := \mathcal{V}(A_s N_{s+1}, I_{s+1})$ in $\mathbb{P}^{2s+2}$(no DV_s). One has an exact sequence

$$\to H^{q-1}_{prim}(T) \to H^{q-1}_{prim}(Y) \to H^q_c(U) \to H^q_{prim}(T) \to . \tag{101}$$

Using (99), we rewrite

$$\begin{aligned} N_{s+1} = (A_v - A_s \pm A_{s-1})I_s - A^2_{s-1} I_{s-1} + A_s I_s - A_v I_s + \\ A_{s-1} L_s = -A_{s-1}(\pm I_s + A_{s-1} I_{s-1} - L_s) = -A_{s-1} N_s. \end{aligned} \tag{102}$$

and see that N_{s+1} is actually independent of A_v and A_s. After applying of *h.i.* and gr_*^W, the sequence (101) yields the isomorphism

$$\operatorname{gr}_i^W H^q_c(U) \cong \operatorname{gr}_i^W H^{q-1}_{prim}(Y), \quad i = 0, 1. \tag{103}$$

Define $\hat{Y}_1 := Y \cap \mathcal{V}(I_s) = \mathcal{V}(A_s N_{s+1}, I_s, A_{s-1} I_{s-1})$. The polynomial N_{s+1} is independent of A_v by (102). Using *h.i.* for $\hat{Y}_1$, we come to an exact sequence

$$H^{q-4}_{prim}(Y_1)(-1) \to H^{q-1}_c(Y \backslash \hat{Y}_1) \to H^{q-1}_{prim}(Y) \to H^{q-3}_{prim}(Y_1)(-1) \to \tag{104}$$

with $Y_1 \subset \mathbb{P}^{2s+1}$(no DV_s, A_v) defined by the same polynomials. The scheme $Y \backslash \hat{Y}_1$ is given by the system

$$\begin{cases} A_s A_{s-1} N_s = 0 \\ C_s I_s - A^2_{s-1} I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{105}$$

By (99), we express A_v from the second equation. Projecting from the point where all the variables but A_v are zero, we get isomorphisms

$$Y\backslash\hat{Y}_1 \cong R \qquad \text{and} \qquad H_c^{q-1}(Y\backslash\hat{Y}_1) \cong H_c^{q-1}(R), \tag{106}$$

where $R := \mathcal{V}(A_sA_{s-1}N_s)\backslash\mathcal{V}(A_sA_{s-1}N_s, I_s) \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_v)$. Define $R_1 := \mathcal{V}(A_{s-1}N_s)\backslash\mathcal{V}(A_{s-1}N_s, I_s)$ and $R_2 := \mathcal{V}(A_s)\backslash\mathcal{V}(A_s, I_s)$. One has the Mayer-Vietoris sequence

$$\begin{aligned}\longrightarrow H_c^{q-2}(R_1) \oplus H_c^{q-2}(R_2) \longrightarrow H_c^{2s-1}(R_3) \longrightarrow \\ H_c^{q-1}(R) \longrightarrow H_c^{q-1}(R_1) \oplus H_c^{q-1}(R_2) \longrightarrow\end{aligned} \tag{107}$$

with $R_3 := R_1 \cap R_2$. The defining polynomials of R_1 and R_2 are independent of A_s and A_m respectively. Applying *h.i.* to them, and applying functors gr_i^W to the sequence above, we get isomorphisms

$$\operatorname{gr}_i^W H_c^{q-1}(R) \cong \operatorname{gr}_i^W H_c^{q-2}(R_3), \quad i = 0, 1. \tag{108}$$

Now, $R_3 := \mathcal{V}(A_s, A_{s-1}N_s)\backslash\mathcal{V}(A_s, A_{s-1}N_s, I_s) \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_v)$. Projecting from the point where all the variables but A_s are zero, we get isomorphisms

$$R_3 \cong U' \qquad \text{and} \qquad H_c^{q-2}(R_3) \cong H_c^{q-2}(U') \tag{109}$$

for $U' = \mathcal{V}(A_{s-1}N_s)\backslash\mathcal{V}(A_{s-1}N_s, I_s) \subset \mathbb{P}^{2s}(\text{no } DV_s,\ A_v,\ A_s)$. Collecting (103), (104), (106) (108) and (109) together, we obtain

$$\operatorname{gr}_i^W H_c^q(U) \cong \operatorname{gr}_i^W H_c^{q-2}(U'), \quad i = 0, 1, \tag{110}$$

where U is defined by (100) and $q \geq 2s + 1$.
If $s = 1$, go to *the Last Step*.

When we come to *Step 3* with some s, $n_i \leq s \leq n_i + l_i - 2$, $i \not\equiv t \bmod 2$, we must apply this step $s - n_i - 1$ times with $N_s = N_s^+$ and then one more time with $N_s = N_s^-$. After this, we are in a new situation.

Step 4. Suppose that the entry a_{ss} of M_{GZZ} is B_s and $a_{s+1\,s+1} = C_{s+1}$. This means that $s = n_i - 1$ for some $i \not\equiv t \bmod 2$. Denote by DV_s the dropped variables that are all the variables appearing in I_{n-1-s}^{s+1} but A_s. Again, we want to compute $H_c^q(U)$ for $q \geq 2s + 1$ and $U \subset \mathbb{P}^{2s+2}$ defined by

$$U := \mathcal{V}(A_sN_{s+1})\backslash\mathcal{V}(A_sN_{s+1}, I_{s+1}), \tag{111}$$

where $N_{s+1} = -I_{s+1} + A_sI_s - L_{s+1}$. Define $U_1 := \mathcal{V}(A_s)\backslash\mathcal{V}(A_s, I_{s+1})$, $U_2 := \mathcal{V}(N_{s+1})\backslash\mathcal{V}(N_{s+1}, I_{s+1})$. This covering gives us an exact sequence

$$\begin{aligned}\longrightarrow H_c^{q-1}(U_1) \oplus H_c^{q-1}(U_1) \longrightarrow H_c^{2s}(U_3) \longrightarrow \\ H_c^q(U) \longrightarrow H_c^q(U_1) \oplus H_c^q(U_1) \longrightarrow,\end{aligned} \tag{112}$$

where $U_3 := U_1 \cap U_2$. The polynomials in the definition of U_1 do not depend on A_m. Moreover, we can rewrite

$$\begin{aligned} N_{s+1} = & - I_{s+1} + A_s I_s - L_{s+1} = -B_s I_s + A_{s-1}^2 I_{s-1} + A_s I_s + \\ & A_{s-1} L_s = (A_s - B_s) I_s + A_{s-1}^2 I_{s-1} + A_{s-1} L_s \end{aligned} \tag{113}$$

and see that N_{s+1} depends neither on B_s nor on A_s but on the difference $A_s - B_s$. After the change of variables $B_s := A_s - B_s$, the polynomial N_{s+1} becomes independent of A_s. Applying *h.i.* to U_1 and U_2, and then applying the functors gr_i^W to the sequence (112), we get

$$\operatorname{gr}_i^W H_c^q(U) \cong \operatorname{gr}_i^W H_c^{q-1}(U_3), \quad i = 0, 1. \tag{114}$$

Now, $U_3 \subset \mathbb{P}^{2s+2}(\text{no } DV_s)$ is given by the system

$$\begin{cases} A_s = 0 \\ N_{s+1} = 0 \\ I_{s+1} \neq 0 \end{cases} \Leftrightarrow \quad \begin{cases} A_s = 0 \\ I_{s+1} + L_{s+1} = 0 \\ I_{s+1} \neq 0. \end{cases} \tag{115}$$

We eliminate the variable A_s and consider an open $U_4 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$ defined by the last two conditions, then

$$H_c^{q-1}(U_3) \cong H_c^{q-1}(U_4). \tag{116}$$

Define $T := \mathcal{V}(I_{s+1} + L_{s+1}), Y := \mathcal{V}(I_{s+1} + L_{s+1}, I_{s+1})$ in $\mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$. We can write an exact sequence

$$\longrightarrow H_{prim}^{q-2}(T) \longrightarrow H_{prim}^{q-2}(Y) \longrightarrow H_c^{q-1}(U_4) \longrightarrow H_{prim}^{q-1}(T) \longrightarrow . \tag{117}$$

Similar to (113), one has $I_{s+1} + L_{s+1} = B_s I_s - A_{s-1}^2 I_{s-1} - A_{s-1} L_s$. Let $T_1 := T \cap \mathcal{V}(I_s) = \mathcal{V}(I_s, A_{s-1}^2 I_{s-1} + A_{s-1} L_s) \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$. For $p = q-1$ and $p = q-2$ we write an exact sequence

$$\longrightarrow H_c^p(T \backslash T_1) \longrightarrow H_{prim}^p(T) \longrightarrow H_{prim}^p(T_1) \longrightarrow . \tag{118}$$

On $T \backslash T_1$ we can express B_s and get an isomorphism $T \backslash T_1 \cong \mathbb{P}^{2s} \backslash \mathcal{V}(I_s)$ with $\mathbb{P}^{2s}(\text{no } DV_s,\ A_s,\ B_s)$. The polynomial I_s does not depend on A_m. Furthermore, the defining polynomials of T_1 are independent of B_s. We apply *h.i.* to $T \backslash T_1$ and T_1, and apply gr_i^W to the sequence (118). We get

$$\operatorname{gr}_i^W H_{prim}^p(T) = 0, \quad i = 0, 1, \ \ p = q-1, q-2. \tag{119}$$

We return to $Y = \mathcal{V}(L_{s+1}, I_{s+1}) = \mathcal{V}(A_{s-1} L_s, B_s I_s - A_{s-1}^2 I_{s-1})$. One can write an exact sequence

$$\to H_{prim}^{q-3}(Y_1) \to H_c^{q-2}(Y \backslash Y_1) \to H_{prim}^{q-2}(Y) \to H_{prim}^{q-2}(Y_1) \to, \tag{120}$$

where $Y_1 := Y \cap \mathcal{V}(I_s) = \mathcal{V}(I_s, A_{s-1}L_{s-1}, A_{s-1}I_{s-1}) \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$. The last three polynomials are independent of B_s. After applying *h.i.* and taking gr_i^W, the sequence (120) implies

$$\mathrm{gr}_i^W H^{q-2}_{prim}(Y) \cong \mathrm{gr}_i^W H^{q-2}_c(Y \backslash Y_1), \quad i = 0, 1. \tag{121}$$

Now, $Y \backslash Y_1 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$ is defined by the system

$$\begin{cases} A_{s-1}L_s = 0 \\ B_s I_s - A_{s-1}^2 I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{122}$$

We can express B_s from the second equation, we get an isomorphism $Y \backslash \hat{Y}_1 \cong U'$, where

$$U' := \mathcal{V}(A_{s-1}L_s) \backslash \mathcal{V}(A_{s-1}L_s, I_s) \subset \mathbb{P}^{2s}\ (\text{no } DV_s, A_s, B_s). \tag{123}$$

Finally, combining (114), (116), (117),(119) and (121), we get

$$\mathrm{gr}_i^W H^q_c(U) \cong \mathrm{gr}_i^W H^{q-2}(Y) \cong \mathrm{gr}_i^W H^{q-2}_c(U'), \ \ i = 0, 1, \tag{124}$$

for U and U' defined by (111) and (123) respectively, and $q \geq 2s + 1$.

Now, if $s = 1$, go to *the Last Step.* If $a_{s-1\,s-1} = C_{s-1}$, we go to *Step 6.* Otherwise do the next step.

Step 5. Consider an entry $a_{ss} = B_s$ of M_{GZZ} such that $a_{s+1s+1} = B_{s+1}$. With other words, s satisfies the condition $n_i \leq s \leq n_i + l_{i+1} - 2$ for some $i \equiv t \bmod 2$. Let $U \subset \mathbb{P}^{2s+2}(\text{no } DV_s)$ be defined by

$$U := \mathcal{V}(A_s L_{s+1}) \backslash \mathcal{V}(A_s L_{s+1}, I_{s+1}), \tag{125}$$

and denote by DV_s all the variables appearing in I^{s+1}_{n-1-s}. As usual, we try to compute $H^q_c(U)$ for $q \geq 2s + 1$. Define $U_1 := \mathcal{V}(A_s) \backslash \mathcal{V}(A_s, I_{s+1})$, $U_2 := \mathcal{V}(L_{s+1}) \backslash \mathcal{V}(L_{s+1}, I_{s+1})$ in $\mathbb{P}^{2s+2}$. One can write an exact sequence

$$\begin{aligned} &\longrightarrow H^{q-1}_c(U_1) \oplus H^{q-1}_c(U_2) \longrightarrow H^{q-1}_c(U_3) \longrightarrow \\ &\qquad\qquad H^q_c(U) \longrightarrow H^q_c(U_1) \oplus H^q_c(U_2) \longrightarrow \end{aligned} \tag{126}$$

where $U_3 := U_1 \cap U_2$. The defining polynomials of U_1 do not depend on A_m, thus *Theorem B* ($N = 2s + 2$, $k = 1$, $t = 1$) implies $H^i_c(U_1) = 0$ for $i < 2s + 2$. Since $L_{s+1} = -A_{s-1}L_s$ and I_{s+1} are independent of A_s, and I_{s+1} is also independent of A_m, we can apply *h.i.* to U_1 and U_2. After applying gr_i^W to the sequence (126), one gets

$$\mathrm{gr}_i^W H^q_c(U) \cong \mathrm{gr}_i^W H^{q-1}_c(U_3), \quad i = 0, 1. \tag{127}$$

We eliminate A_s, which is zero along U_3, and get an isomorphism

$$U_3 \cong U_4 := \mathcal{V}(L_{s+1}) \backslash \mathcal{V}(L_{s+1}, I_{s+1}) \tag{128}$$

with $U_4 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$. Defining $T := \mathcal{V}(L_{s+1})$, $Y := \mathcal{V}(L_{s+1}, I_{s+1})$ in $\mathbb{P}^{2s+1}$, we get an exact sequence

$$\to H^{q-2}_{prim}(T) \to H^{q-2}_{prim}(Y) \to H^{q-1}_c(U_4) \to H^{q-1}_{prim}(T) \to . \tag{129}$$

Since L_{s+1} is independent of B_s, *h.i.* applied to T implies

$$\mathrm{gr}^W_i H^{q-1}_c(U_4) \cong \mathrm{gr}^W_i H^{q-2}_{prim}(Y), \quad i = 0, 1. \tag{130}$$

Let $Y_1 := Y \cap \mathcal{V}(I_s) = \mathcal{V}(L_s, I_{s+1}, I_s)$. One has an exact sequence

$$\to H^{q-3}_{prim}(Y_1) \to H^{q-2}_c(Y \backslash Y_1) \to H^{q-2}_{prim}(Y) \to H^{q-2}_{prim}(Y_1) \to . \tag{131}$$

Since $Y_1 = \mathcal{V}(L_{s+1}, B_s I_s - A^2_{s-1} I_{s-1}, I_s) = \mathcal{V}(A_{s-1} L_s, I_s, A_{s-1} I_{s-1})$, the defining polynomials forget B_s. Applying *h.i.* to Y_1, and gr^W_i to the sequence (131), one gets

$$\mathrm{gr}^W_i H^{q-2}_{prim}(Y) \cong \mathrm{gr}^W_i H^{q-2}_c(Y \backslash Y_1), \quad i = 0, 1. \tag{132}$$

The open subscheme $Y \backslash Y_1 \subset Y$ is given by the system

$$\begin{cases} A_{s-1} L_{s-1} = 0 \\ B_s I_s - A^2_{s-1} I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{133}$$

Expressing B_s from the second equation and projecting from the point where all the variables but B_s are zero, we get an isomorphism $Y \backslash Y_1 \cong U'$, where

$$U' := \mathcal{V}(A_{s-1} L_s) \backslash \mathcal{V}(A_{s-1} L_s, I_s) \subset \mathbb{P}^{2s}. \tag{134}$$

Collecting now (127),(128),(130) and (132) together, we obtain

$$\mathrm{gr}^W_i H^q_c(U) \cong \mathrm{gr}^W_i H^{q-2}_c(U'), \quad i = 0, 1 \tag{135}$$

for U and U' defined by (125) and (134) respectively, and $q \geq 2s + 1$.
If $s = 1$, go to *the Last Step.*

After repeating a suitable number of times *Step 5*, we come to the following situation.

Step 6. Suppose that the entry a_{ss} of the matrix M_{GZZ} is C_s and $a_{s+1s+1} = B_{s+1}$. This happens when $s = n_i - 1$ for some $i \equiv t \bmod 2$. For C_s we have

$$C_s = A_v + A_s \pm A_{s-1} \tag{136}$$

with "+" only when $l_i = 1$. Let $U \subset \mathbb{P}^{2s+2}(\text{no } DV_s)$ be defined by

$$U := \mathcal{V}(A_s L_{s+1}) \backslash \mathcal{V}(A_s L_{s+1}, I_{s+1}), \tag{137}$$

and denote by DV_s all the variables appearing in I^{s+1}_{n-1-s}. As in the previous case, we define $U_1 := \mathcal{V}(A_s)\backslash\mathcal{V}(A_s, I_{s+1})$, $U_2 := \mathcal{V}(L_{s+1})\backslash\mathcal{V}(L_{s+1}, I_{s+1})$ and write an exact sequence

$$\begin{aligned} \longrightarrow H^{q-1}_c(U_1) \oplus H^{q-1}_c(U_2) \longrightarrow H^{q-1}_c(U_3) \longrightarrow \\ H^q_c(U) \longrightarrow H^q_c(U_1) \oplus H^q_c(U_2) \longrightarrow \end{aligned} \tag{138}$$

where $U_3 := U_1 \cap U_2$ and $q \geq 2s+1$. For this step we have

$$\begin{aligned} L_{s+1} &= A_v I_s - A_{s-1} L_s, \\ I_{s+1} &= C_s I_s - A^2_{s-1} I_{s-1}, \end{aligned} \tag{139}$$

so the polynomials defining U_1 and U_2 are independent of A_m and A_s respectively. We apply *h.i.* to U_1 and U_2, and the sequence (138) yields

$$\mathrm{gr}^W_i H^q_c(U) \cong \mathrm{gr}^W_i H^{q-1}_c(U_3), \quad i = 0, 1. \tag{140}$$

Eliminating A_s, which is zero on U_3, we get an isomorphism

$$U_3 \cong U_4 := \mathcal{V}(L_{s+1})\backslash\mathcal{V}(L_{s+1}, I_{s+1}) \tag{141}$$

with $U_4 \subset \mathbb{P}^{2s+1}(\text{no } DV_s,\ A_s)$. Denoting by I'_{s+1} the polynomial I_{s+1} after setting $A_s = 0$, we define $T := \mathcal{V}(L_{s+1})$ and $Y := \mathcal{V}(L_{s+1}, I'_{s+1})$ in $\mathbb{P}^{2s+1}$. One gets an exact sequence

$$\to H^{q-2}_{prim}(T) \to H^{q-2}_{prim}(Y) \to H^{q-1}_c(U_4) \to H^{q-1}_{prim}(T) \to . \tag{142}$$

Motivated by (139), we define $T_1 := T \cap \mathcal{V}(I_s) = \mathcal{V}(I_s, A_{s-1}L_s)$. One can write an exact sequence

$$\longrightarrow H^p_c(T\backslash T_1) \longrightarrow H^p_{prim}(T) \longrightarrow H^p_{prim}(T_1) \longrightarrow \tag{143}$$

where $p = q-1$ or $p = q-2$. On $T\backslash T_1$ we can express A_v from the equation $L_{s+1} = 0$. Projecting from the point where all the variables but A_v are zero, we get an isomorphism $T\backslash T_1 \cong \mathbb{P}^{2s}\backslash\mathcal{V}(I_s)$. The polynomial I_s does not depend on A_m. Furthermore, the polynomials defining T_1 are independent of A_v. Applying *h.i.* to T_1 and $T\backslash T_1$, and then applying the functors $\mathrm{gr}\,_i^W$ to the sequence (143), we obtain

$$\mathrm{gr}\,_0^W H^p_{prim}(T) = \mathrm{gr}\,_1^W H^p_{prim}(T) = 0, \quad p = q-1, q-2. \tag{144}$$

Let $Y_1 := \mathcal{V}(L_{s+1}, I'_{s+1}, I_s) = \mathcal{V}(A_{s-1}L_s, A_{s-1}I_{s-1}, I_s) \subset Y$. One has an exact sequence

$$\to H^{q-3}_{prim}(Y_1) \to H^{q-2}_c(Y\backslash Y_1) \to H^{q-2}_{prim}(Y) \to H^{q-2}_{prim}(Y_1) \to \tag{145}$$

The defining polynomials of Y_1 do not depend on A_v. Using *h.i.* and $\mathrm{gr}\,_i^W$, we get

$$\mathrm{gr}\,_i^W H^{q-2}_{prim}(Y) \cong \mathrm{gr}\,_i^W H^{q-2}_c(Y\backslash Y_1), \quad i = 0, 1. \tag{146}$$

The open subscheme $Y\backslash \hat{Y}_1 \subset Y$ is defined by the system

$$\begin{cases} L_{s+1} = 0 \\ I'_{s+1} = 0 \\ I_s \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_v I_s - A_{s-1}L_s = 0 \\ (A_v \pm A_{s-1})I_s - A^2_{s-1}I_{s-1} = 0 \\ I_s \neq 0. \end{cases} \tag{147}$$

We can express A_v from the first and second equation and this expressions must be equal. So we define $N_s := \pm I_s + A_{s-1}I_{s-1} - L_s$ with "$-$" only when $l_i = 1$. The expression for A_v and the natural projection from the point where all the variables but A_v are zero yield an isomorphism $Y\backslash \hat{Y}_1 \cong U'$, where

$$U' := \mathcal{V}(A_{s-1}N_s)\backslash \mathcal{V}(A_{s-1}N_s, I_s) \subset \mathbb{P}^{2s}. \tag{148}$$

Collecting (140), (141), (144) and (146) together, one gets

$$\mathrm{gr}\,_i^W H^q_c(U) \cong \mathrm{gr}\,_i^W H^{q-2}_c(U'), \quad i = 0, 1, \tag{149}$$

with U and U' defined by (137) and (148) respectively.

If $s = 1$, go to *the Last Step.* If $a_{s-1\,s-1} = B_{s-1}$, return to *Step 4* with $N_s = N_s^-$; return to *Step 3* with $N_s = N_s^+$ otherwise.

the Last Step. Recall that $l_1 > 1$. In the case $t \equiv 0 \bmod 2$ we have come from *Step 4* or *Step 5*. The matrix looks like

$$\mathcal{M}_{GZZ} = \begin{pmatrix} B_0 & A_0 & \vdots & A_m \\ A_0 & B_1 & \vdots & 0 \\ \dots & \dots & \ddots & \vdots \\ A_m & 0 & \dots & \ddots \end{pmatrix}. \tag{150}$$

We are interested in $H^q(U)$ for $q \geq 1$, where $U := \mathcal{V}(A_0L_1)\backslash\mathcal{V}(A_0L_1, I_1) = \mathcal{V}(A_0A_m)\backslash\mathcal{V}(A_0A_m, B_0) \subset \mathbb{P}^2(A_0 : A_m : B_0)$. One has an exact sequence

$$\begin{aligned}\longrightarrow H^0_{prim}(\mathcal{V}(A_0A_m)) \longrightarrow H^0_{prim}(\mathcal{V}(A_0A_m, B_0)) \longrightarrow & H^1_c(U) \longrightarrow \\ H^1(\mathcal{V}(A_0A_m)) \longrightarrow H^1(\mathcal{V}(A_0A_m, B_0)) \longrightarrow & H^2_c(U) \longrightarrow \\ H^2(\mathcal{V}(A_0A_m)) \longrightarrow H^2(\mathcal{V}(A_0A_m, B_0)) \longrightarrow & .\end{aligned} \tag{151}$$

The varieties $\mathcal{V}(A_0A_m)$ and $\mathcal{V}(A_0A_m, B_0)$ are the union of two intersected lines and a double point. The sequence implies $H^1_c(U) \cong \mathbb{Q}(0)$ and $H^2_c(U) = Q(-1)^{\oplus 2}$. So, finishing our computation, we write

$$\mathrm{gr}\,^W_i H^q_c(U) = \begin{cases} Q(0), & q = 1, i = 0 \\ 0, & q = 1, i = 1 \\ 0, & q > 1, i = 0, 1. \end{cases} \tag{152}$$

In the opposite case, when $t \not\equiv 0 \bmod 2$, the matrix looks like

$$\mathcal{M}_{GZZ} = \begin{pmatrix} B_0 & A_0 & \vdots & A_m \\ A_0 & C_1 & \vdots & A_{m-1} \\ \dots & \dots & \ddots & \vdots \\ A_m & A_{m-1} & \dots & \ddots \end{pmatrix}, \tag{153}$$

and we had come from *Step 3* or *Step 6*. We deal with $U \subset \mathbb{P}^2(A_0 : A_m : B_0)$ defined by

$$\begin{aligned} U := \mathcal{V}(A_0N_1)\backslash\mathcal{V}(A_0N_1, I_1) = \\ \mathcal{V}(A_0(B_0 + A_0 - A_m))\backslash\mathcal{V}(A_0(B_0 + A_0 - A_m), B_0). \end{aligned} \tag{154}$$

Changing the variables $A_m := B_0 + A_0 - A_m$, we come to the situation above, and we again obtain the same result as (152).

We have constructed a sequence of schemes $U = U^0, U^1, \dots, U^{n-4} = U_3$ (see (97)) such that $\mathrm{gr}\,^W_i H^q_c(U^s) \cong \mathrm{gr}\,^W_i H^{q-2}_c(U^{s-1})$, for $0 \leq s \leq n - 3$, $i = 0, 1$, $U^s \subset \mathbb{P}^{2s+2}$, and $q = q(s) \geq 2s + 1$. By (98), we obtain

$$\begin{aligned} \mathrm{gr}\,^W_i H^{r-4}_{prim}(V) \cong \mathrm{gr}\,^W_i H^{r-5}_c(U_3) \cong \\ \mathrm{gr}\,^W_i H^{r-5}_c(U^{n-4}) \cong \dots \cong \mathrm{gr}\,^W_i H^q_c(U^0), \quad i = 0, 1, \end{aligned} \tag{155}$$

where $q = r - 5 - 2(n - 4) = r - 2n + 3 \geq 2n - 2 - 2n + 3 = 1$. Hence,

$$\mathrm{gr}\,^W_i H^{r-4}_{prim}(V) = \begin{cases} Q(0), & r = 2n - 2, i = 0 \\ 0, & r = 2n - 2, i = 1 \\ 0, & r > 2n - 2, i = 0, 1. \end{cases} \tag{156}$$

Using the isomorphisms

$$\mathrm{gr}\,_i^W H^{2n-2}(X) \cong \mathrm{gr}\,_{i-4}^W H^{2n-6}(V)(-2), \quad i = 0, \ldots, 5. \tag{157}$$

(see (e34)), we finally get

$$\mathrm{gr}\,_i^W H^r_{prim}(X) = \begin{cases} Q(-2), & r = 2n-2, i = 4 \\ 0, & r = 2n-2, i = 0, \ldots, 3, 5 \\ 0, & r > 2n-2, i = 0, \ldots, 5. \end{cases} \tag{158}$$

□

Recall that ZZ_5 is a primitively log divergent graph and the smallest graph in the zigzag series (see, for example, [BrKr], sect. 1) which is not isomorphic to a WS_n graph for some n.

Theorem 4.7
Let $X \subset \mathbb{P}^9$ be the hypersurface associated to ZZ_5 (see Example 2.5 *for the definition), then*

$$H^8_{prim}(ZZ_5) \cong \mathbb{Q}(-2). \tag{159}$$

Proof. We use the same stratification as for an arbitrary GZZ graph, but now we need to apply the more strong vanishing theorems A and B to compute the whole cohomology of the middle degree rather than just the graded pieces.

In the case ZZ_5, we have $n = 5$ and $r = 2n - 2 = mid = 8$, and we start with the same localization sequence as in (62). Applying *Theorem* $B(N = 8,\ k = 0,\ t = 2)$ to $\mathbb{P}^8 \backslash \mathcal{V}(I_4) \cong X \backslash \mathcal{V}(I_5, I_4)$ (I_4 independent of A_4 and A_5), we get $H^i(X \backslash \mathcal{V}(I_5, I_4)) = 0$ for $i < 10$. Thus the sequence (62) implies

$$H^8(X) \cong H^8(\mathcal{V}(I_5, I_4)) \cong H^6(\mathcal{V}(I_4, G_4))(-1), \tag{160}$$

where the variety on the right hand side lives in $\mathbb{P}^8$(no B_4) (compare with (70)). Since I_3 is independent of A_5 and A_4 (after $C_2 := A_2$), *Theorem B* $(N = 6,\ k = 0,\ t = 2)$ implies $H^i(\mathbb{P}^6 \backslash \mathcal{V}(I_3)) = H^i(\mathcal{V}(I_4, G_4) \backslash \hat{V}) = 0$ for $i < 8$, where $\hat{V} := \mathcal{V}(I_4, I_3, G_4) \subset \mathbb{P}^8$(see (74)). The sequence (71) yields an isomorphism $H^6(\mathcal{V}(I_4, G_4)) \cong H^6(\hat{V})$. Combining with (160) and (76), we get

$$H^8(X) \cong H^4(V)(-2), \tag{161}$$

where $V := \mathcal{V}(I_4, I_3, G_4') \subset \mathbb{P}^7$(no B_4, A_4) (compare with (76)). The same formula holds for the cohomology of the middle degree of arbitrary GZZ. Now we get rid of B_3. One has the localization sequence

$$\to H^1(T)(-1) \to H^4_c(V \backslash \hat{T}) \to H^4_{prim}(V) \to H^2_{prim}(T)(-1) \to \tag{162}$$

for $T = V \cap \mathcal{V}(G_3) = \mathcal{V}(I_3, A_2I_2, G_3, A_2A_5B_1) \subset \mathbb{P}^6$(no $B_4,\ A_4,\ B_3$). By *Theorem B*, the cohomology to the left dies. Considering the localization sequence for $T \cap \mathcal{V}(A_2) \subset T$, one can see that

$$H^2_{prim}(T) \cong H^2_c(T \backslash T \cap \mathcal{V}(A_2)). \tag{163}$$

The scheme $T \backslash T \cap \mathcal{V}(A_2)$ is defined by

$$\begin{cases} I_3 = A_2I_2 = 0 \\ G_3 = A_2A_5B_1 = 0 \\ A_2 \neq 0 \end{cases} \Leftrightarrow \begin{cases} I_3 = I_2 = 0 \\ G_3 = A_5B_1 = 0 \\ A_2 \neq 0. \end{cases} \tag{164}$$

Define $R := \mathcal{V}(I_3, I_2, G_3, A_5B_1)$ and $R_1 = S \cap \mathcal{V}(A_2)$ in $\mathbb{P}^6$(no $B_4,\ A_3,\ B_3$). We get an exact sequence

$$\to H^1(R) \to H^1(R_1) \to H^2_c(T \backslash T \cap \mathcal{V}(A_2)) \to H^2_{prim}(R) \to \tag{165}$$

By *Theorem 1.5*,

$$\begin{aligned} R = \mathcal{V}(I_3, I_2, G_3, A_5B_1) = \mathcal{V}(I_3, I_2, A_5^2(B_1C_2 - A_1^2), A_5B_1) = \\ \mathcal{V}(C_2I_2 - A_1^2B_0, I_2, A_5A_1, A_5B_1) = \mathcal{V}(A_1B_0, I_2, A_5A_1, A_5B_1). \end{aligned} \tag{166}$$

The last polynomials are independent of A_2 and A_4, *Theorem A* ($N = 6$, $k = 4$, $t = 2$) implies $H^i_{prim}(R) = 0$ for $i < 4$. Moreover, the defining polynomials of R_1 do not depend on A_4, thus the cohomology $H^1(R_1)$ also vanishes. By (162), (163) (165), we obtain

$$H^4_{prim}(V) \cong H^4_c(V \backslash \hat{T}) \tag{167}$$

(compare with (82)). The polynomials I_3 and A_2I_2 are independent of A_5, by *Theorem A* ($N = 6$, $k = 2$, $t = 1$) for $\mathcal{V}(I_3, A_2I_2)$, the sequence (85) implies

$$H^4_c(V \backslash \hat{T}) \cong H^4_c(U_1) \cong H^3_{prim}(\mathcal{V}(I_3, A_2I_2, G_3)). \tag{168}$$

After the change of variables ($C_2 := A_2$) the defining polynomials of $\hat{S} := \mathcal{V}(I_3, I_2, G_3) \subset \mathbb{P}^6$ become independent of A_4. By *Theorem A* ($N = 6$, $k = 3$, $t = 1$), the sequence (87) yields

$$H^3(\mathcal{V}(I_3, A_2I_2, G_3)) \cong H^3_c(U_2). \tag{169}$$

Solving the equations of U_2' (see (93)) on A_4, one gets $U_2' \cong W \backslash Z$, where $W := \mathcal{V}(A_1I_2 + A_5A_0A_1 - A_1^2B_0) = \mathcal{V}(A_1N_2)$ and $Z = W \cap \mathcal{V}(I_2)$. Collecting (161), (167), (168), (169) and (91), we obtain

$$H^4_{prim}(V) \cong H^4_c(V \backslash \hat{T}) \cong H^3_c(U_2) \cong H^3_c(W \backslash Z). \tag{170}$$

Consider an exact sequence

$$\begin{aligned}\to H^2_c(\mathcal{V}(A_1)\backslash\mathcal{V}(A_1,I_2)) \oplus H^2_c(W_1\backslash Z_1) \to H^2_c(\mathcal{V}(A_1)\cap W_1\backslash Z_1) \to \\ H^3_c(W\backslash Z) \to H^3_c(\mathcal{V}(A_1)\backslash\mathcal{V}(A_1,I_2)) \oplus H^3_c(W_1\backslash Z_1) \to,\end{aligned} \tag{171}$$

with $W_1 = \mathcal{V}(N_2)$ and $Z_1 = W_1 \cap \mathcal{V}(I_2)$ in $\mathbb{P}^4$. Immediately, $H^2_c(W_1\backslash Z_1) = 0$. Moreover, one has the localisation sequence

$$\to H^2_{prim}(W_1) \to H^2_{prim}(Z_1) \to H^3_c(W_1\backslash Z_1) \to H^3(W_1) \to . \tag{172}$$

The term on the left hand side vanishes, and for the right hand side we consider the localization sequence for $W_1 \cap \mathcal{V}(A_0) \subset W_1$. It is easy to see that $W_1 \cap \mathcal{V}(A_0)$ is a cone over double point while its complement is isomorphic to $\mathbb{A}^3$. Thus, the sequence implies $H^3(W_1) = 0$. By (172), $H^3_c(W_1\backslash Z_1) \cong H^2_{prim}(Z_1)$. We can consider the localization sequence for $Z_1 \cap \mathcal{V}(B_0) \subset Z_1$. It turns out that $Z_1 \cap \mathcal{V}(B_0) \cong \mathbb{P}^2$ and $Z_1\backslash Z_1 \cap \mathcal{V}(B_0) \cong \mathbb{A}^2$. Hence, $H^3_c(W_1\backslash Z_1) \cong H^2_{prim}(Z_1) = 0$. We return to the sequence (171). *Theorem B* ($N = 4$, $k = 1$, $t = 1$) implies $H^i_c(\mathcal{V}(A_1)\backslash\mathcal{V}(A_1,I_2)) = 0$ for $i < 4$. Thus, the sequence yields

$$H^3_c(W\backslash Z) \cong H^2_c(\mathcal{V}(A_1)\cap W_1\backslash Z_1). \tag{173}$$

Forgetting A_1, we get $\mathcal{V}(A_1) \cap W_1\backslash Z_1 \cong W_2\backslash Z_2$ with the latter living in $\mathbb{P}^3(A_0 : A_5 : B_0 : B_1)$. Stratification further gives us no result, so some geometrical argument must be involved at this step.

The variety $W_2 \subset \mathbb{P}^3(A_0 : A_5 : B_0 : B_1)$ is a smooth quadric. Up to a change of variables W_2 is the image of Segre imbedding. More precisely, $W_2 = Im(\gamma)$ for

$$\gamma : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 : (a : b), (c : d) \mapsto (ac : ac - bd : ad : bc). \tag{174}$$

Now, $Z_2 \subset W_2 \subset \mathbb{P}^3$ is defined by $Z_2 := \mathcal{V}(A_0A_5, B_0B_1 - A_0^2)$. So Z_2 is a union of 3 components $\ell_1 \cup \ell_2 \cup \ell_3$, where ℓ_1 and ℓ_2 coincide with the lines $\gamma(\{\infty\} \times \mathbb{P}^1)$ and $\gamma(\mathbb{P}^1 \times \{\infty\})$ respectively, and ℓ_3 is a zero of a nontrivial section of $\mathcal{O}(1,1)$. If

$$S\backslash(\ell_1 \cup \ell_2) \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus (\mathbb{P}^1 \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1) = \mathbb{A}^2 \tag{175}$$

has affine coordinates b, d, then $\ell_3 \cap \mathbb{A}^2$ has defining ideal $\langle 1 - bd \rangle$, so is isomorphic to $\mathbb{G}_m$. Thus we get $W_2\backslash Z_2 \cong \mathbb{A}^2\backslash\mathbb{G}_m$. Now it follows that

$$H^2_c(W_2\backslash Z_2) \cong H^2_c(\mathbb{A}^2\backslash\mathbb{G}_m) \cong H^1_c(\mathbb{G}_m) \cong \mathbb{Q}(0). \tag{176}$$

By (161), (170) and (173), we finally get

$$H^8_{prim}(X) \cong H^3_c(W\backslash Z)(-2) \cong \mathbb{Q}(-2). \tag{177}$$

□

Theorem 4.8
Let X the hypersurface associated to a GZZ and $q = p^k$ some prime power. Then

$$\#X(\mathbb{F}_q) \equiv 1 + q + 2q^2 \mod q^3. \tag{178}$$

Proof. We use the same stratification as in the proof of *Theorem 4.6* but translate everything into another language. Now we need ℓ-adic cohomology. The *h.i.* for some $\hat{Y}$, $H^q(\hat{Y}) \cong H^{q-2}(Y)(-1)$ will now mean "Eigenvalues (EV) of F^k on $H^q(\hat{Y})$ are that ones on $H^{q-2}(Y)$ multiplied by q", where F is the geometric Frobenius element (see [Ka] p. 26). This is because $EV(F^k|H^2_c(\mathbb{A}^1)) = \{q\}$. Eigenvalues are a priory living in $\bar{\mathbb{Z}}$ (see [De4], 3.3.4). Assume now that we have an exact sequence like

$$\longrightarrow H^{q-2}_c(U)(-1) \longrightarrow H^q(Z) \longrightarrow H^q(T) \longrightarrow H^{q-1}_c(U)(-1) \longrightarrow \tag{179}$$

with $T \subset Z \subset \mathbb{P}^N$ and $Z\backslash T$ a cone over U. This is the example of a sequence which we usually deal with in the proof above. Instead of applying the gr^W_i functors, we take the eigenvalues of the action F^k. One sees that an eigenvalue of $F^k|H^q(Z)$ is that of $F^k|H^q(T)$ or in $EV(F^k|H^{q-2}_c(U)(-1)) = q \cdot EV(F^k|H^{q-2}_c(U))$. For the sequence (179) we can write

$$EV(F^k|H^q(Z)) = EV(F^k|H^q(T)) \mod q \cdot \bar{\mathbb{Z}}. \tag{180}$$

We play the same game in each situation where we use gr^W_i functors in the proof of *Theorem 4.6*. We get the following result: all EV of $F^k|H^{mid}_{prim}(X)$ are in $q^3 \cdot \bar{\mathbb{Z}}$ but one is exactly q^2; all EV of $F^k|H^r_{prim}(X)$ are in $q^3 \cdot \bar{\mathbb{Z}}$ if $r > mid$. Finally, we use Grothendick-Lefschetz trace formula and get

$$\begin{aligned}\#X(\mathbb{F}_q) &= \sum_{i=0}^{4n-4} (-1)^i Tr(F^k|H^i(X_\Gamma)) = 1 + q + \ldots + q^{2n-2} + \\ &+ \sum_{i=2n-2}^{4n-4} (-1)^i Tr(F^k|H^i_{prim}(X_\Gamma)) \equiv 1 + q + 2q^2 \mod q^3.\end{aligned} \tag{181}$$

□

5 De Rham class for GZZ(n,2)

Fix some $n \geq 2$ and define $\Gamma = \Gamma_n := GZZ(n,2)$. This graph has $2n+6$ edges and $h_1(\Gamma) = n+3$. Let $X_n \subset \mathbb{P}^{2n+5}$ be the graph hypersurface associated to Γ_n. By the results of the previous section, one has an inclusion

$$\mathbb{Q}(-2) \hookrightarrow H^{2n+4}_{prim}(X_n) \cong H^{2n+5}_c(\mathbb{P}^{2n+5}\backslash X). \tag{182}$$

Hence, we get $\dim H^{2n+5}_{DR}(\mathbb{P}^{2n+5}\backslash X_n) \neq 0$. We do not know that this cohomology group is one-dimensional in general. In this section we consider

$$\eta = \eta_n = \frac{\Omega_{2n+5}}{\Psi^2_{\Gamma_n}} \in \Gamma(\mathbb{P}^{2n+5}, \omega(2X_n)) \tag{183}$$

(see (24)) and show that $[\eta_n] \neq 0$ in $H^{2n+5}(\mathbb{P}^{2n+5}\backslash X_n)$. We strongly follow Section 12, [BEK], where the computations for WS_n were done.

Lemma 5.1
Let $U = \operatorname{Spec} R$ be a smooth, affine variety and $0 \neq f, g \in R$. Define $Z := \mathcal{V}(f,g) \subset U$. We have a map of complexes

$$\left(\Omega^*_{R[1/f]}/\Omega^*_R\right) \oplus \left(\Omega^*_{R[1/g]}/\Omega^*_R\right) \xrightarrow{\gamma} \left(\Omega^*_{R[1/fg]}/\Omega^*_R\right) \tag{184}$$

Then the de Rham cohomology with supports $H^*_{Z,DR}(U)$ can by computed by the cone of γ shifted by -2.

Remark 5.2
The direct computation shows that C^* is quasi-isomorphic to the cone of

$$\left(\Omega^*_{R[1/f]}/\Omega^*_R\right) \xrightarrow{\Delta} \left(\Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}\right). \tag{185}$$

For the application, we use $U := \mathbb{P}^{2n+5}\backslash X_n$. Recall that the matrix of Γ_n looks like

$$\mathcal{M}_{\Gamma_n} = \begin{pmatrix} B_0 & A_0 & 0 & \vdots & 0 & 0 & 0 & A_{n+3} \\ A_0 & B_1 & A_1 & \vdots & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_6 & \vdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & B_{n-1} & A_{n-1} & 0 & 0 \\ 0 & 0 & 0 & \vdots & A_{n-1} & C_n & A_n & A_{n+2} \\ 0 & 0 & 0 & \vdots & 0 & A_n & B_{n+1} & A_{n+1} \\ A_{n+3} & 0 & 0 & \vdots & 0 & A_{n+2} & A_{n+1} & B_{n+2} \end{pmatrix}. \tag{186}$$

Define $a_i := \frac{A_i}{A_{n+3}}$, $b_i := \frac{B_i}{A_{n+3}}$ and $c_n := \frac{C_n}{A_{n+3}} = a_{n+2} + a_{n-1} - a_n$. (We will see that the forms we work with have no poles along $A_{n+3} = 0$.) Let

$$i_j = \frac{I_j}{A_{n+3}^j}, \quad g_{n+2} = \frac{G_{n+2}}{A_{n+3}^{n+3}}. \tag{187}$$

Set $f = i_{n+2}$, $g = i_{n+1}$. The equation $b_{n+2}i_{n+2} - g_{n+2} = 0$ defines X_n on $A_{n+3} \neq 0$, then $b_{n+2}i_{n+2} - g_{n+2}$ is invertible on $U = \mathbb{P}^{2n+5} \backslash X$. Thus g_{n+2} is invertible on $\mathcal{V}(f)$. The element

$$\begin{aligned} \beta = -db_0 \wedge \ldots \wedge db_{n-1} \wedge db_{n+1} \wedge \\ \wedge da_0 \wedge \ldots \wedge da_{n+2} \frac{1}{g_{n+2}i_{n+2}} \Big(\frac{g_{n+2}}{b_{n+2}i_{n+2} - g_{n+2}} \Big) \end{aligned} \tag{188}$$

is defined in $\Omega^{2n+4}_{R[1/f]}/\Omega^{2n+4}_R$. It satisfies

$$d\beta = \eta = \frac{db_0 \wedge \ldots \wedge db_{n-1} \wedge db_{n+1} \wedge db_{n+2} \wedge da_0 \wedge \ldots \wedge da_{n+2}}{(b_{n+2}i_{n+2} - g_{n+2})^2}. \tag{189}$$

By Corollary 1.4, $I_{n+1}G_{n+2} \equiv (L_{n+2})^2 \mod I_{n+2}$, thus

$$\begin{aligned} i_{n+1}g_{n+2} \equiv &(a_{n+1}i_{n+1} - a_{n+2}a_n i_n + \\ &(-1)^{n-1}a_n a_{n-1} \ldots a_1 a_0)^2 \mod i_{n+2}. \end{aligned} \tag{190}$$

We also use

$$i_k = b_{k-1}i_{k-1} - a_{k-2}^2 i_{k-2} \tag{191}$$

for $k = n+2$ or $k < n+1$. We now compute in $\Omega^*_{R[1/fg]}/\Omega^*_{R[1/g]}$ and get

$$\begin{aligned} \beta = \frac{di_{n+2}}{i_{n+2}} \wedge \frac{da_{n+2}}{g_{n+2}i_{n+1}} \wedge db_0 \wedge \ldots \wedge db_{n-1} \wedge \\ \wedge da_0 \wedge \ldots \wedge da_{n+1} \cdot \Big(1 - \frac{b_{n+2}i_{n+2}}{b_{n+2}i_{n+2} - g_{n+2}}\Big) = \\ - d\Big(\frac{1}{a_{n+1}i_{n+1} - a_{n+2}a_n i_n + (-1)^{n-1}a_n \ldots a_0} \cdot \frac{di_{n+2}}{i_{n+2}} \wedge \nu \Big), \end{aligned} \tag{192}$$

where

$$\nu := \frac{da_{n+2}}{i_{n+1}} \wedge db_0 \wedge \ldots \wedge db_{n-1} \wedge da_0 \wedge \ldots \wedge da_n. \tag{193}$$

Using the equality

$$i_{n+1} = c_n i_n - a_{n-1}^2 i_{n-1} = (a_{n+2} + a_{n-1} - a_n)i_n - a_{n-1}^2 i_{n-1} \tag{194}$$

and (191), we get

$$\nu = \frac{di_{n+1}}{i_{n+1}} \wedge \frac{db_{n-1}}{i_n} \wedge db_{n-2} \wedge \ldots \wedge db_0 \wedge da_0 \wedge \ldots \wedge da_n. \\ \frac{di_{n+1}}{i_{n+1}} \wedge \frac{di_n}{i_n} \wedge \ldots \wedge \frac{di_2}{i_2} \wedge \frac{db_0}{b_0} \wedge da_0 \wedge \ldots \wedge da_n. \tag{195}$$

By (192) one has $\beta = d\theta$ with

$$\theta := -\frac{1}{a_{n+1}i_{n+1} - a_{n+2}a_n i_n + (-1)^{n-1}a_n \ldots a_0} \cdot \frac{di_{n+2}}{i_{n+2}} \wedge \nu. \tag{196}$$

Both β and θ have no poles along $A_{n+3} = 0$. Thus the pair

$$(\beta, \theta) \in H^{2n+5}_{Z,DR}(U) \tag{197}$$

(see Remark 5.2) represents a class mapping to $[\eta_n] \in H^{2n+5}_{DR}(\mathbb{P}^{2n+5} \backslash X_n)$, where $Z := \mathcal{V}(I_{n+2}, I_{n+1})$.

Lemma 5.3
The natural map

$$H^{2n+5}_Z(\mathbb{P}^{2n+5} \backslash X_n) \longrightarrow H^{2n+5}(\mathbb{P}^{2n+5} \backslash X_n) \tag{198}$$

is injective.

Proof. The proof repeats that of *Lemma 12.3. in [BEK]* because the first two reduction steps for all GZZ yield the same result (see 161).

□

Theorem 5.4
Let X_n be the graph hypersurface for $\Gamma_n = GZZ(n, 2)$ and let $[\eta_n] \in H^{2n+5}_{DR}(\mathbb{P}^{2n+5} \backslash X)$ be the de Rham class of (183). Then $[\eta_n] \neq 0$.

Proof. The proof is almost the same as that of Theorem 12.4, [BEK]. We have lifted the class $[\eta_n]$ to a class $(\beta, \eta) \in H^{2n+5}(\mathbb{P}^{2n+5} \backslash X)$, see (197). By Lemma 5.3, it is enough to show that $(\beta, \eta) \neq 0$. We localize an the generic point of Z and compute further in the function field of Z. Consider the long denominator of β in (192):

$$D := a_{n+1}i_{n+1} - a_{n+2}a_n i_n + (-1)^{n-1}a_n \ldots a_0. \tag{199}$$

On $\mathcal{V}(i_{n+2}, i_{n+1})$ we have

$$\begin{cases} b_{n+1}i_{n+1} - a_n^2 i_n = 0 \\ i_{n+1} = 0 \end{cases} \Rightarrow \begin{cases} a_n i_n = 0 \\ i_{n+1} = 0, \end{cases} \tag{200}$$

thus both the left and the middle summand of D vanish. Now it follows that as the class in the function field of Z, the class (β, η) is represented by

$$\pm d\log(i_n) \wedge \ldots \wedge d\log(i_1) \wedge d\log(a_0) \wedge \ldots \wedge d\log(a_n) \qquad (201)$$

This is a nonzero multiple of

$$d\log(b_{n-1}) \wedge \ldots \wedge d\log(b_0) \wedge d\log(a_0) \wedge \ldots \wedge d\log(a_n), \qquad (202)$$

so is non-zero as a form. The Deligne theory of MHS yields that the vector space of logarithmic forms injects into de Rham cohomology of the open on which those forms are smooth (see (3.1.5.2) in [De2]). Thus the form above is nonzero.

□

Corollary 5.5
Let X be the graph hypersurface for $\Gamma = ZZ_5$. Then the class of η defined in (183) spans $H^9_{DR}(\mathbb{P}^9 \backslash X)$.

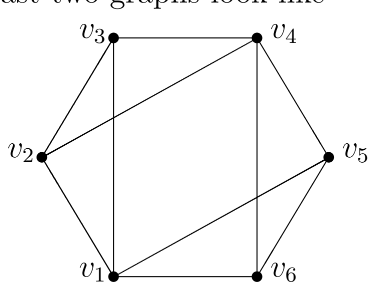
6 Gluings

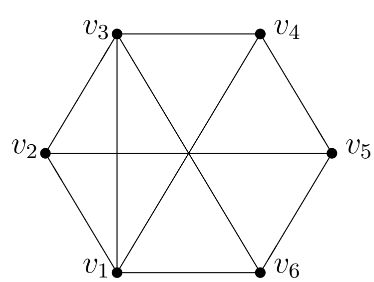
Analysing the different possibilities for the adjacency matrix, one can easy classify the primitively divergent graphs with small number of edges.

Theorem 6.1
Let Γ be a primitively divergent graph with $E(\Gamma) = 2n$ and $n \leq 5$. Then for Γ we have one of the following possibilities

- n=3, *then* $\Gamma \cong WS_3$.
- n=4, *then* $\Gamma \cong WS_4$.
- n=5, *then* Γ *is isomorphic to the one of the following graphs* WS_5, ZZ_5, XX_5 *or* ST_5.

The last two graphs look like





We cannot say anything important about the graph hypersurface of the (strange) graph ST_5 on the cohomological level, but the graph XX_5 motivates us to the following interesting construction.

Definition 6.2
Let Γ and Γ' be two graphs, choose two edges $(u,v) \in E(\Gamma)$ and $(u',v') \in E(\Gamma')$. We define the graph $\Gamma \times \Gamma'$ as follows. We drop the edges (u,v) and (u',v'), and identify vertices u with u' and v with v'. We say also that $\Gamma \times \Gamma'$ is the *gluing* of Γ and Γ' along edges (u,v) and (u',v').

Example 6.3
The graph XX appeared in the *Theorem 6.1* is isomorphic to $WS_3 \times WS_3$.

It is not easy to verify whether the graph is primitively log divergent or not. Nevertheless, we can construct new primitively log divergent graphs from the existing one's by the gluing operation.

Theorem 6.4
The gluing $\Gamma \times \Gamma'$ of two primitively log divergent graphs Γ and Γ' (along edges (u,v) and (u',v')) is again primitively log divergent.

Proof. Suppose that Γ and Γ' have $2n$ and $2m$ edges respectively, then $h_1(\Gamma)$ and $h_1(\Gamma') = m$. We can chose a basis $\{\gamma_1,\dots,\gamma_n\}$ of $H_1(\Gamma,\mathbb{Z})$ such that the edge (u,v) only appears in γ_n. Indeed, we take any basis $\{\gamma_1,\dots,\gamma_{n-1}\}$ of $H_1(\Gamma\backslash\{(u,v)\},\mathbb{Z})$ and define γ_n to be any loop containing (u,v), then $\{\gamma_1,\dots,\gamma_n\}$ form a basis of $H_1(\Gamma,\mathbb{Z})$. Similarly, we choose a basis $\delta_1,\dots,\delta_m$ such that the only appearance of (u',v') is in δ_m. It follows that the loops $\{\gamma_1,\dots,\gamma_{n-1},\delta_1,\dots,\delta_{m-1},\gamma_n\times\delta_m\}$ form a basis of $H_1(\Gamma\times\Gamma',\mathbb{Z})$. Thus, $|E(\Gamma\times\Gamma')| = 2n+2m-2 = 2h_1(\Gamma\times\Gamma')$ and $\Gamma\times\Gamma'$ is logarithmically divergent.

To prove that $\Gamma\times\Gamma'$ is primitively log divergent, we consider a proper subgraph $\Gamma_0 \subset \Gamma\times\Gamma'$ and define Γ_1 (respectively Γ_2) to be the graph $\Gamma_0 \cap \Gamma \cup \{(u,v)\}$ (respectively $\Gamma_0 \cap \Gamma' \cup \{(u',v')\}$). Because the graphs Γ and Γ' are primitively log divergent, for the subgraphs $\Gamma_1 \subset \Gamma$ and $\Gamma_2 \subset \Gamma'$ the inequalities

$$|E(\Gamma_1)| \le 2h_1(\Gamma_1) \quad \text{and} \quad |E(\Gamma_2)| \le 2h_1(\Gamma_2) \tag{203}$$

hold, and the inequalities become strict if subgraphs are proper. Since Γ_0 is the proper subgraph, at least one of the subgraphs Γ_1, Γ_2 is proper. Thus we get

$$|E(\Gamma_1)| + |E(\Gamma_2)| < 2(h_1(\Gamma_1) + h_1(\Gamma_2)) \tag{204}$$

The number of edges of Γ_0 equals $|E(\Gamma_0)| = |E(\Gamma_1)| + |E(\Gamma_2)| - 2$ and one has an inequality $h_1(\Gamma_1) + h_1(\Gamma_2) - 1 \leq h_1(\Gamma_0)$ which becomes an equality if the operation of adding (u, v) to $\Gamma_0 \cap \Gamma$ (or that of (u', v') to $\Gamma_0 \cap \Gamma'$) increases the Betti number. The inequality 204 implies $|E(\Gamma_0)| < 2h_1(\Gamma_0)$. Thus, every proper subgraph of $\Gamma \times \Gamma'$ is convergent.

□

Corollary 6.5
Every gluing Γ of finitely many GZZ graphs (along any pair of edges) is primitively log divergent.

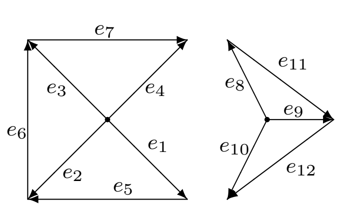
Our goal here is to analyse the middle dimensional (Betti) cohomology of hypersurfaces associated to graphs $WS_n \times WS_3$ for $n \geq 3$. The gluing for $WS_n \times WS_3$ goes along some two boundary edges (not spokes).

Theorem 6.6
Let X be the graph hypersurface for the graph $WS_n \times WS_3$, $n \geq 3$. For the cohomology of the middle degree $H^{mid}(X)$, one has

$$gr_6^W(H_{prim}^{mid}(X)) = \mathbb{Q}(-3) \quad \text{and} \quad gr_8^W(H_{prim}^{mid}(X)) = \mathbb{Q}(-4)^{\oplus d}, \tag{205}$$

where $d = 0$, 1 or 2, and all other $\mathrm{gr}_i^W = 0$. If $n = 3$, then $d = 0$.

Proof. Fix $n \geq 3$ and consider the graph WS_n. We orient the spokes (v_0, v_i) as exiting the center v_0 and label them with e_1 through e_n. The boundary edges (v_i, v_{i+1}) (modulo n) are denoted by e_{n+i} and are oriented exiting v_i. Now we rename the last edge $e_{2n} =: e$, play the same game with the graph WS_3, shifting the numeration of edges by $2n - 1$, and glue WS_n with WS_3 along e and e_{2n+5}. Denote the resulting graph by Γ. To show the way of constructing the tables and the matrices associated to this gluing, we restrict to the case $WS_4 \times WS_3$.



	1	2	3	4	5	6	7	8	9	10	11	12
1	1	−1	0	0	1	0	0	0	0	0	0	0
2	0	1	−1	0	0	1	0	0	0	0	0	0
3	0	0	1	−1	0	0	1	0	0	0	0	0
4	1	0	0	−1	0	0	0	1	0	−1	0	0
5	0	0	0	0	0	0	0	1	−1	0	1	0
6	0	0	0	0	0	0	0	0	1	−1	0	1

The matrix $\mathcal{M}_\Gamma$ has two "blocks" coming from the matrices of WS_n and WS_3 intersected by one element which becomes dependent.

$$\mathcal{M}_\Gamma(A,B) = \begin{pmatrix} B_0 & A_0 & 0 & ... & 0 & 0 & A_{n+1} & 0 & 0 \\ A_0 & B_1 & A_1 & ... & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & B_2 & ... & 0 & 0 & 0 & 0 & 0 \\ \multicolumn{9}{c}{\dotfill} \\ 0 & 0 & 0 & ... & B_{n-3} & A_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & ... & A_{n-3} & B_{n-2} & A_{n-2} & 0 & 0 \\ A_{n+1} & 0 & 0 & ... & 0 & A_{n-2} & C_{n-1} & A_{n-1} & A_{n+2} \\ 0 & 0 & 0 & ... & 0 & 0 & A_{n-1} & B_n & A_n \\ 0 & 0 & 0 & ... & 0 & 0 & A_{n+2} & A_n & B_{n+1} \end{pmatrix} \quad (206)$$

where $C_{n-1} := A_{n+1} + A_{n-2} + A_{n-1} + A_{n+2}$. We work with the graph hypersurface $X := \det \mathcal{M}_\Gamma \subset \mathbb{P}^{2n+3}(A,B)$, and the cohomology to compute is $H^{mid}(X) = H^{2n+2}(X)$. We write $I_{n+2} = B_{n+1}I_{n+1} - G_{n+1}$. One has the localization sequence

$$\longrightarrow H_c^{2n+2}(U) \longrightarrow H^{2n+2}(X) \longrightarrow \\ H^{2n+2}(\mathcal{V}(I_{n+2}, I_{n+1})) \longrightarrow H_c^{2n+3}(U) \longrightarrow, \quad (207)$$

where $U := X \backslash \mathcal{V}(I_{n+2}, I_{n+1}) \subset \mathbb{P}^{2n+3}(A,B)$. For dimensional reasons, we have the vanishing of the term on the right hand side. For the rightmost term one easy gets

$$H_c^{2n+3}(U) \cong H_c^{2n+3}(\mathbb{P}^{2n+2} \backslash \mathcal{V}(I_{n+1})) \cong \\ H_c^{2n+1}(\mathbb{P}^{2n+1} \backslash \mathcal{V}(I_{n+1}))(-1) \cong H^{2n}(\mathcal{V}(I'_{n+1}))(-1). \quad (208)$$

Prime indicates that we made the change of coordinates $C_{n-1} := A_{n+2}$. We consider an exact sequence

$$\longrightarrow H_c^{2n}(T_0) \longrightarrow H_{prim}^{2n}(\mathcal{V}(I'_{n+1})) \longrightarrow H_{prim}^{2n}(\hat{T}) \longrightarrow, \quad (209)$$

where $\hat{T} := \mathcal{V}(I'_{n+1}, I'_n)$ and $T_0 := \mathcal{V}(I'_{n+1}) \backslash \hat{T}$ in $\mathbb{P}^{2n+1}$ (no B_{n+1}, A_n). Since $I'_{n+1} = B_n I'_n - A_{n-1}^2 I_{n-1}$, one gets $T_0 \cong \mathbb{P}^{2n} \backslash \mathcal{V}(I'_n)$ and *Theorem B*$(N = 2n,\ k = 0,\ t = 1)$ implies $H_c^{2n}(T_0) = 0$. For $\hat{T}$, one gets $H_{prim}^{2n}(\hat{T}) \cong H_{prim}^{2n-2}(T)(-1)$ with $T := \mathcal{V}(I'_n, A_{n-1}I_{n-1}) \subset \mathbb{P}^{2n}$(no B_{n+1}, A_n, B_n).

Consider $T_1 := T \cap \mathcal{V}(I_{n-1})$, $T_{00} = T \backslash T_1$, and the localization sequence

$$\longrightarrow H_c^{2n-2}(T_{00}) \longrightarrow H_{prim}^{2n-2}(T) \longrightarrow H_{prim}^{2n-2}(T_1) \longrightarrow \quad (210)$$

The polynomials I'_n and I_{n-1} are independent of A_{n-1}. Thus *Theorem A* $(N = 2n,\ k = 2,\ t = 1)$ implies $H_{prim}^{2n-2}(T_1) = 0$. Solving the equation

$I'_n = C_{n-1}I_{n-1} - G_{n-1}$ on C_{n-1}, one gets $T_{00} \cong \mathbb{P}^{2n-1}\backslash\mathcal{V}(I_{n-1})$. By *Theorem B* ($N = 2n-1$, $k = 0$, $t = 2$), one obtains $H_c^{2n-2}(T_{00}) = 0$.

The exact sequence (210) implies $H_{prim}^{2n+2}(T) = 0$. The sequence (209) an isomorphisms in (208) yield $H_c^{2n+3}(U) \cong H_{prim}^{2n}(\mathcal{V}(I'_{n+1}))(-1) = 0$. We return to the sequence (207) and get an isomorphism

$$H^{2n+2}(X) \cong H^{2n+2}(\mathcal{V}(I_{n+2}, I_{n+1})) \cong H^{2n}(\mathcal{V}(I_{n+1}, G_{n+1}))(-1). \tag{211}$$

We consider $\hat{V} := \mathcal{V}(I_{n+1}, G_{n+1}, I_n) \subset \mathcal{V}(I_{n+1}, G_{n+1}) \subset \mathbb{P}^{2n+2}$ we write an exact sequence

$$\begin{aligned} &\longrightarrow H_c^{2n}(U_1) \longrightarrow H^{2n}(\mathcal{V}(I_{n+1}, G_{n+1})) \longrightarrow \\ &\qquad\qquad H^{2n}(\hat{V}) \longrightarrow H_c^{2n+1}(U_1) \longrightarrow, \end{aligned} \tag{212}$$

where $U_1 := \mathcal{V}(I_{n+1}, G_{n+1})\backslash\hat{V}$. Similar as in (74) and (75), we get $H_c^q(U_1) \cong H_c^{q-2}(\mathbb{P}^{2n-1}\backslash\mathcal{V}(I'_n))(-1) \cong H_{prim}^{q-3}(\mathcal{V}(I'_n))(-1)$ for $q = 2n$, $2n+1$. Note that $\mathcal{V}(I'_n)$ is exactly the graph hypersurface for WS_n. Thus the sequence (212) simplifies to

$$0 \longrightarrow H^{2n}(\mathcal{V}(I_{n+1}, G_{n+1})) \longrightarrow H^{2n}(\hat{V}) \longrightarrow \mathbb{Q}(-3) \longrightarrow . \tag{213}$$

As in (76), one has

$$H^{2n}(\hat{V}) \cong H^{2n-2}(V)(-1), \tag{214}$$

where $V := \mathcal{V}(I_n, I_{n+1}, A_{n+2}B_nI_{n-1}) \subset \mathbb{P}^{2n+1}$ (no B_{n+1}, A_n).

Now we attack V. Rewriting $V = \mathcal{V}(I_n, A_{n+2}B_nI_{n-1}, A_{n-1}I_{n-1})$, we define $V_1 := \mathcal{V}(I_n, B_n, A_{n-1}I_{n-1})$ and $V_2 := \mathcal{V}(I_n, A_{n+2}I_{n-1}, A_{n-1}I_{n-1})$ in $\mathbb{P}^{2n+1}$(no B_{n+1},A_n). Consider an exact sequence

$$\begin{aligned} &\longrightarrow H^{2n-3}(V_1) \oplus H^{2n-3}(V_2) \longrightarrow H^{2n-3}(V_3) \longrightarrow \\ &\qquad H_{prim}^{2n-2}(V) \longrightarrow H_{prim}^{2n-2}(V_1) \oplus H_{prim}^{2n-2}(V_2) \longrightarrow \end{aligned} \tag{215}$$

with $V_3 := V_1 \cap V_2 = \mathcal{V}(I_n, B_n, A_{n+2}I_{n-1}, A_{n-1}I_{n-1})$. The defining polynomials of V_2 are independent of B_n. *Theorem A* ($N = 2n+1$, $k = 3$, $t = 1$) implies $H_{prim}^i(V_2) = 0$ for $i \leq 2n-2$. Considering $V_{11} := \mathcal{V}(I_n, B_n, I_{n-1}) \subset V_1$ and the associated localization sequence, one can also prove $H_{prim}^i(V_2) = 0$ for $i \leq 2n-2$. Thus (215) implies

$$H_{prim}^{2n-2}(V) \cong H^{2n-3}(V_3). \tag{216}$$

Define $V_{31} := V_3 \cap \mathcal{V}(I_{n-1}) = \mathcal{V}(I_n, B_n, I_{n-1}) \subset V_3 \subset \mathbb{P}^{2n+1}$ (no B_{n+1}, A_n). By *Theorem A* ($N = 2n+1$, $k = 3$, $t = 0$), $H_{prim}^i(V_{31}) = 0$ for $i \leq 2n-3$. Thus, the localization implies

$$H^{2n-3}(V_3) \cong H_c^{2n-3}(V_3\backslash V_{31}). \tag{217}$$

The subscheme $V_3\backslash V_{31}$ is defined by the system

$$\begin{cases} A_{n-1}I_{n-1} = I_n = 0 \\ B_n = A_{n+2}I_{n-1} = 0 \\ I_{n-1} \neq 0 \end{cases} \Leftrightarrow \begin{cases} A_{n-1} = I_n = 0 \\ B_n = A_{n+2} = 0. \\ I_{n-1} \neq 0 \end{cases} \tag{218}$$

Define $Y := \mathcal{V}(I_n)$, $S := \mathcal{V}(I_n, I_{n-1})$ in $\mathbb{P}^{2n-2}$(no DV_5), where DV_5 denotes the set of the dropped variables $\{B_{n+1}, A_n, B_n, A_{n+2}, A_{n-1}\}$. This gives us an exact sequence

$$0 \longrightarrow H^{2n-4}_{prim}(S) \longrightarrow H^{2n-3}(V_3\backslash V_{31}) \longrightarrow H^{2n-3}(Y) \longrightarrow \tag{219}$$

After rewriting $S = \mathcal{V}(I_{n-1}, G_{n-1})$, we notice that S is exactly the variety which appears in the first reduction step of the case of WS_n, and we know that

$$H^{2n-4}(S) \cong \mathbb{Q}(-1). \tag{220}$$

The computation of $H^{2n-3}(Y)$ is less easy. The polynomial I_n is similar to the polynomial associated to WS_n with the only difference that C_{n-1} is not independent and is equal $A_{n+1} + A_{n-2}$. We start from the upper left corner of the matrix and write $I_n = B_0 I^1_{n-1} - \widetilde{G}_{n-1}$. Consider $\hat{Y}_1 := \mathcal{V}(I_n, I^1_{n-1}) \subset Y \subset \mathbb{P}^{2n-2}$ (no DV_5). One has

$$\to H^{2n-3}_c(Y\backslash\hat{Y}_1) \to H^{2n-3}(Y) \to H^{2n-3}(\hat{Y}_1) \to H^{2n-2}_c(Y\backslash\hat{Y}_1) \to . \tag{221}$$

For $Y\backslash\hat{Y}_1$ one gets

$$H^q_c(Y\backslash\hat{Y}_1) \cong H^{q-2}_c(\mathbb{P}^{2n-4}\backslash\mathcal{V}(I^1_{n-1})(-1) \cong H^{q-3}_{prim}(\mathcal{V}(I'^1_{n-1}))(-1), \tag{222}$$

for $q = 2n-3, 2n-2$ ($C_{n-1} := A_{n+1}$). We see that I^1_{n-1} is a determinant of a 3-diagonal matrix studied in *Lemma 11.12*, [BEK], and its cohomology of middle degree is trivial. So $H^q_{prim}(\mathcal{V}(I'^1_{n-1})) = 0$ and the sequence (221) and *h.i.* yield

$$H^{2n-3}(Y) \cong H^{2n-5}(Y_1)(-1), \tag{223}$$

where $Y_1 := \mathcal{V}(I_{n-1}, G_{n-1}) \subset \mathbb{P}^{2n-3}$(no DV_5, B_0). Consider an exact sequence

$$\to H^{2n-5}_c(Y_1\backslash\hat{Y}_2) \to H^{2n-5}(Y_1) \to H^{2n-5}(\hat{Y}_2) \to H^{2n-4}_c(Y_1\backslash\hat{Y}_2) \to, \tag{224}$$

where $\hat{Y}_2 := \mathcal{V}(I^1_{n-1}, \widetilde{G}_{n-1}, I^2_{n-2})$. Similar to (74) (while interchanging rows and columns), projecting further, and changing the coordinates ($C_{n-1} :=$

A_{n+1}), one gets $Y_1\backslash\hat{Y}_2 \cong \mathcal{V}(I^1_{n-1})\backslash\mathcal{V}(I^1_{n-1}, I^2_{n-2}) \cong \mathbb{P}^{2n-5}\backslash\mathcal{V}(I'^2_{n-2})$. By the same argument as in (222), we obtain

$$H^q_c(Y_1\backslash\hat{Y}_2) \cong H^{q-2}_c(\mathbb{P}^{2n-6}\backslash\mathcal{V}(I'^2_{n-2}))(-1) \cong H^{q-3}_{prim}(\mathcal{V}(I'^2_{n-2})) = 0, \quad (225)$$

for $q \leq 2n-4$. After projecting further and using *Theorem 1.5*, we derive the following isomorphism from the sequence (224):

$$H^{2n-5}(Y_1) \cong H^{2n-7}(Y_2)(-1), \quad (226)$$

where $Y_2 := \mathcal{V}(I^1_{n-1}, A_{n+1}I^1_{n-2}, I^2_{n-2}) \subset \mathbb{P}^{2n-4}$(no DV_5, B_0, A_0). Now we see that if $n = 3$ then we get the vanishing

$$H^{2n-7}(Y_2) = 0 \quad \text{when } \ n = 3. \quad (227)$$

From now on we assume that $n \geq 4$. Define $Y_{21} := \mathcal{V}(I^1_{n-1}, I^1_{n-2}, I^2_{n-2})$, $Y_{22} := \mathcal{V}(I^1_{n-1}, A_{n+1}, I^2_{n-2}) = \mathcal{V}(A_{n+1}, I^2_{n-2}, A_1I^3_{n-3})$ and $Y_3 := Y_{21} \cap Y_{22}$. One has an exact sequence

$$\begin{aligned} &\longrightarrow H^{2n-8}_{prim}(Y_{21}) \oplus H^{2n-8}_{prim}(Y_{22}) \longrightarrow H^{2n-8}_{prim}(Y_3) \longrightarrow \\ &\qquad H^{2n-7}(Y_2) \longrightarrow H^{2n-7}(Y_{21}) \oplus H^{2n-7}(Y_{22}) \longrightarrow . \end{aligned} \quad (228)$$

By *Theorem A*($N = 2n-4$, $k = 3$, $t = 1$), we obtain $H^q(Y_{22}) = 0$ for $q = 2n-8$, $2n-7$. The variety Y_{21} (after $C_{n-1} := A_{n+1}$) is exactly the variety appeared in the proof of the WS_n case (was called Z_{n-1}, see *Theorem 11.9*, [BEK]). Thus $H^{2n-7}(Y_{21}) \cong \mathbb{Q}(0)$ and the sequence (228) simplifies to

$$0 \longrightarrow H^{2n-8}_{prim}(Y_3) \longrightarrow H^{2n-7}(Y_2) \longrightarrow \mathbb{Q}(0) \longrightarrow, \quad (229)$$

where $Y_3 := \mathcal{V}(A_{n+1}, I^1_{n-1}, I^1_{n-2}, I^2_{n-2})$. We change the notation and consider $Z := \mathcal{V}(I^1_{n-1}, I^1_{n-2}, I^2_{n-2}) \subset \mathbb{P}^{2n-5}$(no DV_8), where $DV_8 := DV_5 \cup \{B_0, A_0, A_{n+1}\}$. To abuse the notation, we write I^i_j for I^i_j after setting $A_{n+1} = 0$ (so, $C_{n-1} = A_{n-2}$). We are interested in $H^{2n-8}_{prim}(Z)$, the computation goes now in the same direction as in *Theorem 11.9*,[BEK]. Consider $Z_1 := \mathcal{V}(I^1_{n-1}, I^1_{n-2})$ and $Z_2 := \mathcal{V}(I^1_{n-1}, I^2_{n-2})$ in $\mathbb{P}^{2n-5}$(no DV_8), then $Z = Z_1 \cap Z_2$. We write an exact sequence

$$\begin{aligned} &\longrightarrow H^{2n-8}_{prim}(Z_1) \oplus H^{2n-8}_{prim}(Z_2) \longrightarrow H^{2n-8}_{prim}(Z) \longrightarrow \\ &\qquad H^{2n-7}(\bar{Z}) \longrightarrow H^{2n-7}(Z_1) \oplus H^{2n-7}(Z_2) \longrightarrow, \end{aligned} \quad (230)$$

where $\bar{Z} := Z_1 \cup Z_2$. Again by *Lemma 11.12*,[BEK], $H^q_{prim}(Z_1) = 0$ for $q = 2n-8$, $2n-7$. Using induction, we can similarly prove $H^q_{prim}(Z_2) = 0$. Thus the sequence above gives us

$$H^{2n-8}_{prim}(Z) \cong H^{2n-7}(\bar{Z}). \quad (231)$$

By Corollary 1.2, we get $\bar{Z} := \mathcal{V}(I^1_{n-1}, I^1_{n-2}I^2_{n-2}) = \mathcal{V}(I^1_{n-1}, S_{n-2})$ with $S_{n-2} = A_1A_2\dots A_{n-2}$. Define $Z_3 := \mathcal{V}(I^1_{n-1}, A_{n-2})$, $Z_4 := \mathcal{V}(I^1_{n-1}, S_{n-3})$, and $Z_5 := Z_3 \cap Z_4$ in $\mathbb{P}^{2n-5}$. One has an exact sequence

$$\longrightarrow H^{2n-8}(Z_5) \longrightarrow H^{2n-7}(\bar{Z}) \longrightarrow \\ H^{2n-7}(Z_3) \oplus H^{2n-7}(Z_4) \longrightarrow H^{2n-7}(Z_5) \longrightarrow \quad (232)$$

Since $I^1_{n-1} = C_{n-1}I^1_{n-2} - A_{n-2}I^1_{n-3}$ with $C_{n-1} = A_{n-2}$, $Z_5 := Z_3 \cap Z_4 = \mathcal{V}(I^1_{n-1}, A_{n-2}, S_{n-3}) = \mathcal{V}(A_{n-2}, S_{n-3})$. The defining polynomials of Z_5 are independent of B_1 and B_2, *Theorem A* ($N = 2n-5$, $k = 2$, $t = 2$) implies $H^i_{prim}(Z_5) = 0$ for $i \le 2n-6$. Moreover, $H^{2n-7}(Z_3) \cong H^{2n-7}(\mathcal{V}(A_{n-2})) = 0$, and the sequence (232) yields

$$H^{2n-7}(\bar{Z}) \cong H^{2n-7}(Z_4) \cong H^{2n-7}(\mathcal{V}(S_{n-3}, I^1_{n-1})). \quad (233)$$

We consider the spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0<\dots<i_p} H^q\big(\mathcal{V}(A_{i_0}, \dots, A_{i_p}, I^1_{n-1})\big) \Rightarrow H^{p+q}\big(\mathcal{V}(S_{n-3}, I^1_{n-1})\big). \quad (234)$$

The only difference to the same situation in the proof of *Theorem 11.9* in [BEK] is that we have $C_{n-1} = A_{n-2}$ instead of B_{n-1}. Analysing this sequence similarly, we obtain

$$H^{2n-7}(Z_4) = \mathbb{Q}(0). \quad (235)$$

By (231) and (233), the sequence (229) simplifies to

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow H^{2n-7}(Y_2) \longrightarrow \mathbb{Q}(0) \longrightarrow, \quad (236)$$

Together with (223) and (226), this gives us the exact sequence

$$0 \longrightarrow \mathbb{Q}(-2) \longrightarrow H^{2n-3}(Y) \longrightarrow \mathbb{Q}(-2) \longrightarrow . \quad (237)$$

Consequently, $H^{2n-3}(Y) \cong \mathbb{Q}(-2)^{\oplus i}$ for $i = 1$ or $i = 2$. Collecting (216),(217) and (220) together, we rewrite the sequence (219) like

$$0 \longrightarrow \mathbb{Q}(-1) \longrightarrow H^{2n-2}_{prim}(V) \longrightarrow H^{2n-3}_{prim}(Y) \longrightarrow . \quad (238)$$

From this, one can describe $H^{2n-2}_{prim}(V)$:

$$\mathrm{gr}\,^W_2(H^{2n-2}_{prim}(V)) = \mathbb{Q}(-1), \;\; \mathrm{gr}\,^W_4(H^{2n-2}_{prim}(V)) = \mathbb{Q}(-2)^{\oplus j} \quad (239)$$

and all other gr^i_W are zero. Here $0 \leq j \leq i$, thus j equals 0, 1 or 2. Using (211), (214) and the sequence (213) we get finally

$$\mathrm{gr}\,^W_6(H^{2n+2}_{prim}(X)) = \mathbb{Q}(-3) \quad \text{and } \mathrm{gr}\,^W_8(H^{2n+2}_{prim}(X)) = \mathbb{Q}(-4)^{\oplus d}, \qquad (240)$$

where $d = 0$, 1 or 2, all other $gr^W_i = 0$, and $n \geq 4$. It remains to see what happens when $n = 3$. By (227), $H^3_{prim}(Y) = 0$, thus $H^4_{prim}(V) \cong \mathbb{Q}(-1)$. By (211) and (213) we get now an exact sequence

$$0 \longrightarrow H^8(X) \longrightarrow Q(-3) \longrightarrow \mathbb{Q}(-4) \longrightarrow . \qquad (241)$$

Applying $\mathrm{gr}\,^W_i$, we see that $H^8(X) \cong \mathbb{Q}(-3)$.

□

Remark 6.7
In physics it is known that the periods of $WS_3 \times WS_3$ and $WS_3 \times WS_4$ are related to $\zeta(3)^2$ and $\zeta(3)\zeta(5)$ resp. For the first case, by duality, we have $H^9(\mathbb{P}^9\backslash X_\Gamma) \cong Q(-6)$, and the (minus) twist 6 coincides with the weight of $\zeta(3)^2$. For the latter, the weight of $\zeta(3)\zeta(5)$ is $3 + 5 = 8$, and our computation shows that $H^{11}(\mathbb{P}^{11}\backslash X_\Gamma) \cong Q(-6)$ has the only nontrivial pieces $Q(-8)$ and $Q(-7)^{\oplus d}$. So, is example $WS_3 \times WS_4$ we see that exactly the minimal graded pies of $H^{mid}(X_\Gamma)$ (or the maximal one of the cohomology of the complement) controls the weight of MZV. This is sort of motivation for studying the minimal graded piece of the cohomology of middle degree for GZZ. And this is also the only piece we can compute.

References

[BEK] Bloch,S., Esnault,H., and Kreimer,D. *On motives associated to graph polynomials.* Commun. Math. Phys. 267, 181-225 (2006).

[BrKr] Broadhurst, D., and Kreimer, D. *Knots and numbers in* Φ^4 *theory to 7 loops and beyond.* Int. J. Mod. Phys. C 6, 519 (1995).

[De2] Deligne, P. *Théorie de Hodge*, II. Publ. Math. IHES 40, 5-58 (1974)

[De3] Deligne, P. *Théorie de Hodge*, III. Publ. Math. IHES 44, 5-78 (1974)

[De4] Deligne, P. *La conjecture de Weil*, I. Publ. Math. IHES 43, 273-307 (1974)

[Es] Esnault, H. *Variation on Artin's vanishing theorem.* Advan. of Math. 198, 715-730 (2005).

[Ka] Kanz,N.M. *Review of ℓ-adic Cohomology*, Proc. of Symp. in Pure Math. 55, 1, 21-30 (1994).

[Mi1] Milne,J.S. *Lectures on Etale cohomology.* http://www.jmilne.org/math/CourseNotes/LEC210.pdf

[SGA7] Séminaire de Géométrie Algébrique. 7 (with Deligne P., and Katz. N.) *Groupes de monodromie en géométrie algebriques* (1967-1968). Lecture Notes in Math, 288, 340, Springer, Heidelberg, 1972-1973.