A CONSTRUCTION OF A QUOTIENT TENSOR CATEGORY

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Abstract. Let \( f : G \to A \) be a surjective homomorphism of transitive groupoid schemes and let \( L \) denote the kernel of \( f \). The exact sequence of groupoid schemes \( 1 \to L \to G \to A \to 1 \) induces a sequence of functors between the categories of finite representations of these groupoid schemes \( \text{Rep}_f(A) \to \text{Rep}_f(G) \to \text{Rep}_f(L) \). We show that the category \( \text{Rep}_f(L) \) is a quotient category of \( \text{Rep}_f(G) \) by \( \text{Rep}_f(A) \) in an appropriate sense. We also generalize this setting to the framework where the tensor categories are not necessarily Tannaka categories (i.e. not of the form \( \text{Rep}_f(G) \) for some groupoid scheme \( G \)), where we show under certain assumption the uniqueness of the quotient tensor category.

Introduction

Let \( \mathcal{T} \) be a Tannaka category over \( k \) with fiber functor \( \omega \) to \( \text{vect}_K \), where \( K \supset k \) and let \( \mathcal{S} \) be a full tensor subcategory of \( \mathcal{T} \) which is closed under taking sub- and quotient objects. The natural inclusion \( \mathcal{S} \to \mathcal{T} \) induces a surjective homomorphism of \( k \)-groupoids acting upon \( K \)

\[ G(\mathcal{T}) \to G(\mathcal{S}) \tag{0.1} \]

In [4] it is shown that the kernel of this homomorphism is a discrete \( K \)-groupoid which can therefore be identified with a \( K \)-group (scheme). Let \( L \) denote this \( K \)-group, \( \mathcal{Q} \) the category of its (finite dimensional) representation and \( q : \mathcal{T} \to \mathcal{Q} \) the restriction functor. For the sequence of functors

\[ \mathcal{S} \to \mathcal{T} \to \mathcal{Q} \tag{0.2} \]

it is shown that

(i) An object of \( \mathcal{T} \) is isomorphic (in \( \mathcal{T} \)) to an object from \( \mathcal{S} \) iff its image under \( q \) is trivial (i.e. isomorphic to the direct sum of copies of the unit object) in \( \mathcal{Q} \).

(ii) Each object in \( \mathcal{Q} \) is isomorphic (in \( \mathcal{Q} \)) to a subobject of the image under \( q \) of an object from \( \mathcal{T} \).

The problem we want to address in this work is to give an abstract description of the quotient category \( \mathcal{Q} \). This question was posed by P. Deligne in connection with our description of the representation category of \( L \) given in [4]. While considering this problem we realize that, with some “technical assumptions”, one can assume \( \mathcal{T} \) merely to be a rigid tensor category. On the other hand, as already noticed by Milne in [5] for the existence of a quotient \( \mathcal{Q} \)
the category $\mathcal{S}$ is necessarily a Tannaka category. Let us start by the definition of a (normal) quotient category of a rigid tensor category $\mathcal{T}$.

A (normal) $K$-quotient of $\mathcal{T}$ is by definition a pair $(\mathcal{Q}, q : \mathcal{T} \rightarrow \mathcal{Q})$ consisting of a $K$-linear rigid tensor category $\mathcal{Q}$ and $k$-linear exact tensor functor $q$ (the $k$-linear structure over $\mathcal{Q}$ is induced from the inclusion $k \subset K$), such that:

(i) for an object $X \in \mathcal{T}$ the largest trivial subobject of $q(X)$ is isomorphic to the image under $q$ of a subobject of $X$;

(ii) each object of $\mathcal{Q}$ is isomorphic to a subobject of $q(X)$ with $X \in \mathcal{T}$.

Given a $K$-quotient $(\mathcal{Q}, q)$ of $\mathcal{T}$, let $\mathcal{S}$ denote the full subcategory of $\mathcal{T}$ consisting of those objects of $\mathcal{T}$ whose images in $\mathcal{Q}$ are trivial (i.e. isomorphic to a direct sum of the unit object). Thus $\mathcal{S}$ is a tensor subcategory and is closed under taking sub- and quotient objects. We shall call $\mathcal{S}$ the invariant subcategory with respect to the quotient $(\mathcal{Q}, q)$. $\mathcal{S}$ is a Tannaka category with fiber functor to $\text{vect}_K$ (Lemma 3.3).

For each object $X \in \mathcal{T}$ let $X_S$ denote the maximal subobject of $X$ which is isomorphic to an object of $\mathcal{S}$. We refer to 4.11, 5.3, 5.7 for the condition that $\mathcal{T}$ is flat over $\mathcal{S}$ and over $K$. Our main results are:

Let $\mathcal{T}$ be a rigid tensor category over $k$ and $\mathcal{S}$ be a full tensor subcategory which is closed under taking sub- and quotient objects.

(i) Assume that $\mathcal{S}$ is a Tannaka category with fiber functor $\omega$ to $\text{vect}_K$, and $\mathcal{T}$ is flat over $K$ and over $\mathcal{S}$. Then if a $K$-quotient $(\mathcal{Q}, q)$ of $\mathcal{T}$ by $\mathcal{S}$ exists, it is equivalent to the category, whose objects are triples $(X, Y, f \in \omega(X^\vee \otimes Y)_S)$, and morphisms are appropriately defined. Consequently $(\mathcal{Q}, q)$ is uniquely determined, up to a tensor equivalence.

(ii) If $\mathcal{T}$ is a Tannaka category, then it is flat over any full tensor subcategory which is closed under sub- and quotient objects, and the quotient of $\mathcal{T}$ with respect to such a subcategory exists.

The work is organized as follows. We first recall some basic fact about an exact sequence of algebraic group schemes $1 \rightarrow L \rightarrow G \rightarrow A \rightarrow 1$. The construction recalled here will be generalized in the subsequent sections. In section 2 we first define the kernel $L$ of a morphism of transitive groupoid schemes $f : G \rightarrow A$ and provide some basic properties of $L$, for instance, the transitivity. Then using the kernel we describe the base changes of a groupoid scheme. After that we provide a description of the representation category of $L$ generalize the one mentioned in section 1 for group schemes. In section 3 we introduce the notion of quotient tensor category of a rigid tensor category $\mathcal{T}$ by a Tannaka subcategory $\mathcal{S}$ which is closed under taking sub- and quotient objects. In sections 4 and 5 we give a description of this category. Section 4 is devoted to the case when $\mathcal{S}$ is a neutral Tannaka category and section 5 is devoted to the general case. Results of section 2 show in particular that $\text{Rep}_f(L)$ (the category of finite dimensional representation of $L$) is the quotient of $\text{Rep}_f(G)$ by $\text{Rep}_f(A)$. A consequence of this result which might be useful is a criterion 5.11 for a sequence of groupoid schemes to be exact. Unfortunately
our description of the quotient category depends on an assumption about the flatness (of \( T \) over \( S \) and \( K \)), which we cannot check when \( T \) is not Tannaka category. Nevertheless we believe that the assumptions have the potential to hold true. To this end some open questions are mentioned in 5.12.

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1. Preliminaries

Consider a homomorphism \( f : G \to A \) of affine group schemes (not necessarily of finite type) over a field \( k \), which we shall call groups for short. Let \( L \) denote the kernel of \( f \), which is then a normal subgroup of \( G \). We collect here some known information on the relationship between these groups.

1.1. We shall use the notation \( \mathcal{O}(G) \) to denote the function algebra over \( G \), which is a Hopf \( k \)-algebra. The reader is referred to [6] for details on the structure of \( \mathcal{O}(G) \). We shall use the following notations for defining the structure maps on \( \mathcal{O}(G) \):

- \( m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G) \) for the multiplication;
- \( u : k \to \mathcal{O}(G) \) for the unit element map;
- \( \epsilon : \mathcal{O}(G) \to k \) for the counit map (which is induced from the unit element \( e \) of \( G \));
- \( \Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \) for the coproduct (which is induced from the product on \( G \));
- \( \iota \) for the antipode map (which is induced from the inverse element map on \( G \)).

We shall adopt the Sweedler notation for the coproduct:

\[
\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}
\]

1.2. The category \( \text{Rep}(G) \) of \( k \)-linear representation of \( G \) is equivalent to the category \( \mathcal{O}(G)\text{-Comod} \) of \( \mathcal{O}(G) \) comodules, which consists of pairs \( (V, \rho_V : V \to V \otimes \mathcal{O}(G)) \), where \( V \) is a vector space and \( \rho_V \) is a \( k \)-linear map satisfying
the following commutative diagrams:

\[
\begin{array}{ccc}
V & \xrightarrow{\rho_V} & V \otimes \mathcal{O}(G) \\
\downarrow{\rho_V} & & \downarrow{\rho_V \otimes \text{id}} \\
V \otimes \mathcal{O}(G) & \xrightarrow{\text{id} \otimes \Delta} & V \otimes V \otimes \mathcal{O}(G)
\end{array}
\]

\[
\begin{array}{ccc}
V & \xrightarrow{\rho_V} & V \otimes \mathcal{O}(G) \\
\downarrow{\text{id}} & & \downarrow{\text{id} \otimes \epsilon} \\
V & \xrightarrow{\rho_V \otimes \text{id}} & V \otimes \mathcal{O}(G)
\end{array}
\]

The coproduct \( \Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G) \) defines a right coaction (as well as a left coaction) of \( \mathcal{O}(G) \) on itself, this coaction corresponds to the right (resp. the left) regular representation of \( G \) in \( \mathcal{O}(G) \). The terminologies: \( G \)-equivariant and \( \mathcal{O}(G) \)-colinear are equivalent.

A homomorphism \( f : G \to A \) is the same as a homomorphism of Hopf algebras \( f^* : \mathcal{O}(A) \to \mathcal{O}(G) \). The fundamental theorem of algebraic group theory claims that \( \mathcal{O}(G) \) is faithfully flat over its subalgebra \( f^*(\mathcal{O}(A)) \), cf. [3].

1.3. Let \( q : L \to G \) be the kernel of \( f \). We notice that \( L \) can be defined as the fiber product over \( A \) of \( G \) with \( \text{Spec} (k) \), where the morphism \( e : \text{Spec} (k) \to A \) is given by the unit element of \( A \). Thus we have

\[
(1.1) \quad \mathcal{O}(L) \cong \mathcal{O}(G) \otimes_{\mathcal{O}(A)} k
\]

The homomorphism \( q^* : \mathcal{O}(G) \to \mathcal{O}(L) \) is just the projection \( \mathcal{O}(G) \to \mathcal{O}(G) \otimes_{\mathcal{O}(A)} k \) obtained by tensoring \( \mathcal{O}(G) \) with the map \( \epsilon : \mathcal{O}(A) \to k \). We shall assume from now on that \( \mathcal{O}(A) \) is a Hopf subalgebra of \( \mathcal{O}(G) \).

1.4. The homomorphism \( q : L \to G \) induces a tensor functor \( \text{res}^q : \text{Rep}(G) \to \text{Rep}(L) \), which restricts a representation of \( G \) in a vector space \( V \) through \( q \) to a representation of \( L \). The functor \( \text{res}^q : \text{Rep}(G) \to \text{Rep}(L) \) admits a right adjoint, which is the induced representation functor \( \text{ind}_q : \text{Rep}(L) \to \text{Rep}(G) \), that is we have a functorial isomorphism

\[
(1.2) \quad \text{Hom}_L(\text{res}^q(V), U) \cong \text{Hom}_G(V, \text{ind}_q(U)), \quad V \in \text{Rep}(G), U \in \text{Rep}(L)
\]

The functoriality yields a canonical \( L \)-linear map

\[
\varepsilon_U : \text{res}^q \text{ind}_q(U) \to U
\]

The functor \( \text{ind}_q \) can be explicitly computed in terms of the invariant space functor \((-)^L\). We prefer here the following Hopf algebraic description, which will be exploited in the next sections.

For an \( L \)-representation \( U \), denote by \( U \square_{\mathcal{O}(L)} \mathcal{O}(G) \) the equalizer of the following maps

\[
(1.3) \quad U \otimes \mathcal{O}(G) \xrightarrow{\text{id} \otimes \Delta} U \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \xrightarrow{\text{id} \otimes q^* \otimes \text{id}} U \otimes \mathcal{O}(L) \otimes \mathcal{O}(G)
\]

\( U \square_{\mathcal{O}(L)} \mathcal{O}(G) \) is called the cotensor product over \( \mathcal{O}(L) \) of \( U \) with \( \mathcal{O}(G) \). Then we have a functorial isomorphism

\[
(1.4) \quad \text{ind}_q(U) \cong U \square_{\mathcal{O}(L)} \mathcal{O}(G)
\]
For the cotensor product there is the following key isomorphism first considered by Takeuchi [7]

\begin{equation}
(U \square_{\mathcal{O}(L)} \mathcal{O}(G)) \otimes_{\mathcal{O}(A)} \mathcal{O}(G) \cong U \otimes \mathcal{O}(G),
\end{equation}

\(u \otimes g \otimes h \mapsto \sum_{(u)} u_{(0)} \otimes u_{(1)} gh,\)

which together with the faithful flatness of \(\mathcal{O}(G)\) over \(\mathcal{O}(A)\) (cf. 1.2) shows in particular that the functor \(\text{ind}_q = (-) \square_{\mathcal{O}(L)} \mathcal{O}(G)\) is faithfully exact.

A direct consequence of (1.5) is the following isomorphism

\begin{equation}
k \square_{\mathcal{O}(L)} \mathcal{O}(G) \cong \mathcal{O}(A)
\end{equation}

For a representation \(V\) of \(G\), we have the following \(G\)-equivariant isomorphism

\begin{equation}
V \otimes \mathcal{O}(G) \rightarrow (V) \otimes \mathcal{O}(G) = \mathcal{O}(G)^{\oplus \dim_k V},
\end{equation}

\(v \otimes h \mapsto \sum_{(v)} v_{(0)} \otimes v_{(1)} h\)

Therefore, considering \(V\) as an \(\mathcal{O}(L)\)-comodule, (1.6) and (1.7) imply a \(G\)-equivariant isomorphism (where \(G\) acts diagonally on the right object)

\begin{equation}
V \square_{\mathcal{O}(L)} \mathcal{O}(G) \cong V \otimes \mathcal{O}(A)
\end{equation}

In other words we have \(\text{ind}_q \text{res}^q(V) \cong V \otimes \mathcal{O}(A)\). Note that \(\mathcal{O}(A)\) acts on \(V \otimes \mathcal{O}(A)\) through the action on the second component.

1.5. In general, for any \(L\)-representation \(U\) there exists an \(\mathcal{O}(A)\) module structure \(\mu_U : \mathcal{O}(A) \otimes \text{ind}_q(U) \rightarrow \text{ind}_q(U)\) on \(\text{ind}_q(U)\), induced from the inclusion of \(\mathcal{O}(A)\) in \(\mathcal{O}(G)\). The action \(\mu_U\) is \(G\)-equivariant where \(G\) acts diagonally on \(\mathcal{O}(A) \otimes \text{ind}_q(U)\). The functor \(\text{ind}_q\) thus factors through a functor to the category \(\text{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}\) of the so-called \((\mathcal{O}(G)-\mathcal{O}(A))\)-Hopf modules. By definition, an \((\mathcal{O}(G)-\mathcal{O}(A))\)-Hopf module is a \(k\)-vector space \(M\) together with a coaction \(\rho_M\) of \(\mathcal{O}(G)\) and an action \(\mu_M\) of \(\mathcal{O}(A)\) which are compatible in the sense that \(\mu_M\) is \(\mathcal{O}(G)\)-colinear where \(\mathcal{O}(G)\) coacts diagonally on \(M \otimes \mathcal{O}(A)\). The category \(\text{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}\) is in fact a tensor category with respect to the tensor product over \(\mathcal{O}(A)\). It follows from the various isomorphisms above that \(\text{ind}_q\) defines an equivalence of tensor categories between \(\text{Rep}(L)\) and \(\text{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}\). In particular, the isomorphism in (1.8) is \(\mathcal{O}(A)\)-linear.

The equivalence mentioned above can be reformulated in the following more geometric language: there exists an equivalence between \(L\)-representations and \(G\)-equivariant vector bundles over \(A\). This was pointed out to the author by P. Deligne.

1.6. A new consequence of the classical facts in 1.1-1.6 is the following, cf. [4]. It follows from the faithful exactness of \(\text{ind}_q\) that the canonical homomorphism \(\varepsilon_U : \text{res}^q \text{ind}_q(U) \rightarrow U\) is surjective. Assume that \(U\) has finite dimension over \(k\) then we can find a finite dimensional \(G\)-subrepresentation \(V\) of \(\text{ind}_q(U)\) which still maps surjectively on \(U\). Thus \(U\) is a quotient of the restriction to \(L\) of a finite dimensional representation of \(G\). Consequently, \(U\) can also be embedded
in to the restriction to $L$ of a finite dimensional $G$-representation. Thus each finite dimensional representation $U$ of $L$ can be put (in a non-canonical way) in to a sequence $\text{res}^q(V) \xrightarrow{\pi} U \xrightarrow{\iota} \text{res}^q(W)$, where $V, W$ are finite dimensional $G$-representations. In other words, $U$ is equivalent as an $L$-representation to the image of an $L$-linear map $g : \text{res}^q(V) \rightarrow \text{res}^q(W)$. Using the equivalence between $\text{Rep}(L)$ and $\text{Mod}^{O(G)}_{O(A)}$ and the isomorphism (1.8) we can show that such $g$ are in a 1-1 correspondence with morphisms $\tilde{g} : V \otimes O(A) \rightarrow W \otimes O(A)$ in $\text{Mod}^{O(G)}_{O(A)}$. Since $\tilde{g}$ is $O(A)$ linear, it is uniquely determined by a $G$-equivariant map $f : V \rightarrow W \otimes O(A)$.

Let $Q$ be the category, whose objects are triples $(V, W, f : V \rightarrow W \otimes O(A))$, where $V, W$ are finite dimensional representations of $G$ and $f$ is $G$-equivariant, and whose morphisms are defined in an adequate way. Composing $f$ with the morphism $\text{id} \otimes \epsilon : W \otimes O(A) \rightarrow W \otimes k \cong W$ which is $L$-linear. We define a functor $Q \rightarrow \text{Rep}(L)$, sending a triple $(V, W, f)$ to the image of $f_0$. It follows from the discussion of this paragraph that this functor is an equivalence of categories between $Q$ and the category $\text{Rep}_f(L)$ of finite dimensional representations of $L$.

2. Groupoids

2.1. Groupoids and their representations. We refer to [2, §1.14] for the definition of an affine groupoid scheme, which we shall call here simply groupoid. Fix a field $k$ and let $K$ be another field containing $k$. A $k$-groupoid acting upon $\text{Spec} K$ will usually be denoted like $G^K_k$. $G^K_k$ is called transitive if it acts transitively upon $\text{Spec} K$, which means that $G^K_k$ is flat over $\text{Spec} K \times_k \text{Spec} K$ with respect to the source and target map $(s, t) : G^K_k \rightarrow \text{Spec} K \times_k \text{Spec} K$. The category of $K$-representation of $G^K_k$ is denoted by $\text{Rep}(G^K_k)$, its subcategory of finite dimensional (over $K$) representations is denoted by $\text{Rep}_f(G^K_k)$.

2.2. Homomorphisms of groupoids. Assume we are given the following field extensions

\begin{equation}
(2.1) 
k_0 \subset k \subset K_0 \subset K
\end{equation}

and transitive groupoids $G = G^K_k$ and $A = A^K_{k_0}$. A $k_0$-morphism $f : G \rightarrow A$ is called a groupoid homomorphism if for any $k$-scheme $S$, $f$ induces a functor

\begin{equation}
(2.2) 
f_S : (G(S), K(S)) \rightarrow (A(S), K_0(S))
\end{equation}

of abstract groupoids, where, on objects, $f_S$ is given by the inclusion of fields

\begin{equation}
(2.3) 
\begin{tikzcd}
\text{Spec} K \arrow{r} \arrow[swap]{d}{a} & \text{Spec} K_0 \arrow{d}{f_S(a)} \\
& S
\end{tikzcd}
\end{equation}
and on morphism $f_S$ is given by

$$\phi \quad f_S(\phi)$$

That is, we have the following diagram

$$\begin{array}{ccc}
S & \xrightarrow{f_S} & A \\
\downarrow & & \downarrow \\
Spec K \times_k Spec K & \xrightarrow{(f_S(a), f_S(b))} & Spec K_0 \times_{k_0} Spec K_0
\end{array}$$

Setting $S = G$, $\phi = id$, we obtain the following commutative diagram

$$\begin{array}{ccc}
G & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
Spec K \times_k Spec K & \xrightarrow{} & Spec K_0 \times_{k_0} Spec K_0
\end{array}$$

Similarly, $f$ should be compatible with the groupoid structure on $G$ and $A$ which are related to each other by this diagram.

The kernel of $f$ is defined as the fiber product

$$\begin{array}{ccc}
L & \longrightarrow & Spec K_0 \\
\downarrow & & \downarrow e \\
G & \xrightarrow{f} & A
\end{array}$$

where $e : Spec K_0 \to A$ is given by the unite element of $A$.

**Lemma 2.3.** $L$ is a $K_0$-groupoid acting transitively upon $Spec K$.

**Proof.** The fiber product of $Spec K_0$ with $Spec K \times_k Spec K$ over $Spec K_0 \times_{k_0} Spec K_0$ is $Spec K \times_{K_0} Spec K$. Therefore there exists a map $L \to Spec K \times_{K_0} Spec K$. We show that $L$ is transitive over $Spec K \times_{K_0} Spec$. According to [2, Prop. 3.3], this is equivalent to saying that the associated stack $G_{K,L}$ is a gerbes. This last fact was shown in [5, 1.2].

**2.4. Base change.** Using the notion of kernel, we define in this section the “lower” base change for groupoids. Let $G = G^K_k$ be a transitive groupoid and $k \subset k_1 \subset K$ an intermediate field. The projection

$$\begin{array}{ccc}
Spec K \times_k Spec K & \rightarrow & Spec k_1 \times_k Spec k_1
\end{array}$$

induces a homomorphism of groupoids $G^K_k \to Spec k_1 \times_k Spec k_1$. The kernel of this homomorphism is called the $k_1$-diagonal subgroupoid of $G^K_k$ and denoted
by $G^K_{k_1}$

\[
\begin{array}{ccc}
G^K_{k_1} & \longrightarrow & G^K_k \\
\downarrow & & \downarrow \\
\text{Spec } K \times_k \text{Spec } K & \longrightarrow & \text{Spec } k_1 \times_k \text{Spec } k_1
\end{array}
\]

It follows from definition of $G^K_{k_1}$ that the left square in this diagram is cartesian. In case $k_1 = K$, $G^K_K$ is just the usual diagonal subgroup of $G^K_k$.

On the other hand, for any extension $K \subset K_1$, Deligne [2] defines a $k$-groupoid $G^K_{k_1}$ acting (transitively) upon $\text{Spec } K_1$:

\[
\begin{array}{ccc}
G^K_{k_1} & \longrightarrow & G^K_k \\
\downarrow & & \downarrow \\
\text{Spec } K_1 \times_k \text{Spec } K_1 & \longrightarrow & \text{Spec } K \times_k \text{Spec } K
\end{array}
\]

which in our context might be called the “upper” base change. We notice that the category $\text{Rep}(G^K_{k_1})$ is equivalent to the category $\text{Rep}(G^K_k)$ (cf. [2, (3.5.1)]).

Thus, given field extensions $k_0 \subset k \subset K_0 \subset K$ and a homomorphism of transitive groupoids $f : G^K_k \to A^K_{k_0}$. Then $f$ factors through

\[
G^K_k \xrightarrow{f_k} A^K_k \to A^K_{k_0} \to A^K_{k_0}
\]

Since the last map in (2.11) is an injection, in what follows we shall only consider the situation $k_0 = k$, i.e. field extensions $k \subset K_0 \subset K$.

**Lemma 2.5.** Let $q : L^K_{K_0} \to G^K_k$ be the kernel of a homomorphism $f : G^K_k \to A^K_{k_0}$ of transitive groupoids. For any $V \in \text{Rep}(G^K_k)$, the $L^K_{K_0}$-invariant subspace $V^{L^K_{K_0}}$ is a $G^K_k$-subrepresentation of $V$, which is the pull-back through $f$ of a representation of $A^K_{K_0}$.

**Proof.** By making the lower base change we see that $L^K_{K_0}$ is the kernel of $f_K : G^K_K \to A^K_K$, hence is normal in $G^K_K$ as $K$-groups schemes. Thus $V^{L^K_{K_0}} \subset V$ is invariant under the action of $G^K_K$, but this also implies that $V^{L^K_{K_0}}$ is invariant under the action of $G^K_k$. Since $L^K_{K_0}$ can also be considered as the kernel of the morphism $f : G^K_k \to A^K_k$, we see that $A^K_k$ acts on $V^{L^K_{K_0}}$. As we noticed after (2.11), $\text{Rep}(A^K_k)$ is equivalent to $\text{Rep}(A^K_{K_0})$, which means that $V^{L^K_{K_0}}$ is indeed a representation of $G^K_k$ that comes from a representation of $A^K_{k_0}$ by pulling-back through $f$. We therefore conclude that $L^K_{K_0}$ acts trivially on this space. Since $V^{L^K_{K_0}} \subset V^{L^K_{K}}$, these spaces coincide. \( \square \)

**2.6. The function algebra.** We refer to [2, 1.14] for the properties of the function algebra $\mathcal{O} := \mathcal{O}(G^K_k)$ of a groupoid $G^K_k$. See also the appendix to [4]. $\mathcal{O}$ is a $k$-Hopf algebroid acting on $K$. The structure maps are denoted as follows:

- $m : \mathcal{O} \otimes_k \mathcal{O} \to \mathcal{O}$, the multiplication;
We notice that the $K$-for the coproduct on the tensor product is the one for $K$-satisfying morphism $\epsilon$. And for $O$-
In particular, $(O, m, s \otimes t)$ is a $K \otimes_k K$-algebra. The $k$-linear maps $s, t : K \rightarrow O$ induce to structures of $K$-vector space on $O$, making it a $K$-bimodule. We shall assume that the left action of $K$ is given by $t$ and the right one by $s$. Then $(O, \Delta, \epsilon)$ is a $K$-bimodule coalgebra, i.e. $O$ is considered as $K$-bimodule, the tensor product is the one for $K$-bimodules. We adopt Sweedler notation for the coproduct on $O$: $$\Delta(h) = \sum_{(h)} h_{(1)} s \otimes_t h_{(2)}.$$ We notice that the $K$-linearity of $\Delta$ reads $$\Delta(t(\lambda)h s(\mu)) = \sum_{(h)} t(\lambda)h_{(1)} s \otimes_t h_{(2)s(\mu)} \quad h \in O, \lambda, \mu \in K$$ And for $\epsilon$ we have $\epsilon(s(\lambda)ht(\mu)) = s(\lambda)\epsilon(h)t(\mu)$.

The category $Rep(G^K_k)$ of $G^K_k$ representations over $K$ is the same as the category of $O$-comodules, i.e. of pairs $(V, \rho_V)$ of a $K$-vector space $V$ and a morphism $\rho_V : V \rightarrow V \otimes_t O$ satisfying

$$V \xrightarrow{\rho_V} V \otimes_t O \xrightarrow{\rho_V} V \otimes_t O \xrightarrow{\rho_V} V \otimes_t O \xrightarrow{\rho_V} V \otimes_t O \xrightarrow{\rho_V} V \otimes_t O \xrightarrow{\rho_V} V \otimes_t O \xrightarrow{\rho_V} V$$

We note that the $K$-linearity of $\rho_V$ means:

$$\rho_V(\lambda v) = t(\lambda)\rho_V(v).$$

2.7. The induced representation functor. Consider extensions of fields $k \subset K_0 \subset K$ and a homomorphism $f : G^K_k \rightarrow A^K_k$ of groupoids as in 2.2. Let $q : L^K_{K_0} \rightarrow G^K_k$ be the kernel of $f$ and $res^q : Rep(G^K_k) \rightarrow Rep(L^K_{K_0})$ denote the restriction functor. By definition $O(L^K_{K_0}) = O(G^K_k) \otimes_{O(A^K_{K_0})} K_0$, where $O(A^K_{K_0})$ acts on $O(G^K_k)$ through $f^*$, and $q^* : O(G^K_k) \rightarrow O(L^K_{K_0})$ is the projection. Define a morphism

$$\phi : O(G^K_k) \otimes_{O(A^K_{K_0})} O(G^K_k) \rightarrow O(L^K_{K_0})_s \otimes_t O(G)$$
The morphism in (2.14) is an isomorphism and is $\mathcal{O}(G_k^K)$-colinear (i.e. $G_k^K$-equivariant) with respect to the right coaction of $\mathcal{O}(G_k^K)$ on the second tensor component as well as $\mathcal{O}(L_{K_0}^K)$-colinear with respect to the left coaction of $\mathcal{O}(L_{K_0}^K)$ on the first tensor components.

**Proof.** We give the inverse map. We first define a map $\bar{\psi}: \mathcal{O}(G_k^K)_s \otimes \mathcal{O}(G_k^K) \to \mathcal{O}(G_k^K) \otimes \mathcal{O}(A_k^{K_0}) \mathcal{O}(G_k^K)$, $\psi(g \otimes h) = \sum (g) g_{(1)} \otimes \iota(g_{(2)}) h$, where $\iota$ is the antipode of $\mathcal{O}(G_k^K)$. Then we note that this map indeed factors through a map $\psi: \mathcal{O}(L_{K_0}^K)_s \otimes \mathcal{O}(G_k^K) \to \mathcal{O}(G_k^K) \otimes \mathcal{O}(A_k^{K_0}) \mathcal{O}(G_k^K)$ which is the inverse to $\phi$. The second claim is obvious from the definition of $\phi$. $\square$

The functor $\text{res}^q$ possesses a right adjoint which is the induced representation functor, denoted by $\text{ind}_q: \text{Rep}(L_{K_0}^K) \to \text{Rep}(G_k^K)$. For an $\mathcal{O}(L_{K_0}^K)$ comodule $U$ denote $U \square_{\mathcal{O}(L_{K_0}^K)} \mathcal{O}(G_k^K)$ the equalizer of the maps

\[(2.15)\]

\[
\begin{array}{ccc}
U \otimes t \mathcal{O}(G_k^K) & \xrightarrow{id \otimes \Delta} & U \otimes t \mathcal{O}(G_k^K)_s \otimes t \mathcal{O}(G_k^K) \\
& \xrightarrow{} & V \otimes \mathcal{O}(L_{K_0}^K)_s \otimes t \mathcal{O}(G_k^K).
\end{array}
\]

**Proposition 2.9.** Let $(L = L_{K_0}^K, q)$ be the kernel of $f: G = G_k^K \to A = A_k^{K_0}$ as above. Then for $U \in \text{Rep}(L)$, we have

(i) $\text{ind}_q(U)$ is canonically isomorphic to $U \square_{\mathcal{O}(L)} \mathcal{O}(G)$;

(ii) there is an isomorphism

\[(2.16)\]

\[
(U \square_{\mathcal{O}(L)} \mathcal{O}(G)) \otimes \mathcal{O}(A) \mathcal{O}(G) \cong U \otimes t \mathcal{O}(G).
\]

(iii) the functor $\text{ind}_q : \text{Rep}(L) \to \text{Rep}(G)$ is exact (and hence faithfully exact) iff $\mathcal{O}(G)$ is flat (hence faithfully flat over $f^* \mathcal{O}(A)$).

**Proof.** We show that the functor $U \mapsto U \square_{\mathcal{O}(L)} \mathcal{O}(G)$ is right adjoint to the restriction functor $\text{res}^q$, which amounts to

\[
\text{Hom}_G(V, U \square_{\mathcal{O}(L)} \mathcal{O}(G)) \cong \text{Hom}_L(\text{res}(V), U), \quad V \in \text{Rep}(G), U \in \text{Rep}(L).
\]

The map is given by composing a morphism $U \to U \square_{\mathcal{O}(L)} \mathcal{O}(G)$ with the canonical map $\epsilon_U: U \square_{\mathcal{O}(L)} \mathcal{O}(G) \to U$ given by $v \otimes g \mapsto v \epsilon(g)$, where $\epsilon$ denotes the counit of $\mathcal{O}(G)$. The inverse map is given by $f \mapsto (f \otimes \text{id}) \rho_U$. Thus (i) is proved.

To show the isomorphism in (2.16) we first tensor both sides of (2.14) with $U$ from the left and then taking the equalizer as in (2.15). The last claim follows from (2.16) since the right hand side of (2.16) is a faithfully flat functor on $U$. $\square$

**Lemma 2.10.** [4, Lem. 6.2] Let $q: L_K^K \to G_k^K$ be the kernel for a homomorphism $f: G_k^K \to A_k^K$ of transitive groupoids. Then the adjoint functor $\text{ind}_q : \text{Rep}(L) \to \text{Rep}(G)$ is exact. $\square$
Proposition 2.11. Let \( q : L^K_{K_0} \to G^K_k \) be the kernel of the morphism \( f : G^K_k \to A^K_{k_0} \) of transitive groupoids. Then:

(i) the induced representation functor \( \text{ind}_q \) is faithfully exact;
(ii) each finite dimensional representation of \( L \) can be embedded into a representation of \( G \) considered as representation of \( L \).

Proof. We have the following commutative diagram with exact lines:

\[
\begin{array}{c}
\xymatrix{L^K_{K_0} \ar[r]^q & G^K_k \\
& A^K_k 
}
\end{array}
\]

where the right triangle was given in (2.11).

According to 2.10, the adjoint functor \( \text{ind}_{\bar{q}} \) is faithfully exact, hence, according to 2.9, (iii), \( \mathcal{O}(G^K_k) \) is faithfully flat over \( \bar{f}^* \mathcal{O}(A^K_k) \). Since \( \mathcal{O}(A^K_k) \) is faithfully flat over \( \mathcal{O}(A^K_{k_0}) \), we conclude that \( \mathcal{O}(G^K_k) \) is faithfully flat over \( f^* \mathcal{O}(A^K_{k_0}) \), hence \( \text{ind}_q \) is faithfully exact. Thus (i) is proved.

Now (ii) follows from (i) by standard argument, see e.g. [4, Lem. 5.5-5.6]. □

2.12. In the situation of Proposition 2.11, we call a representation \( V \) of \( G^K_k \) a \( K_0/k \)-representation if \( V \) is equipped with a \( k \)-linear homomorphism \( K_0 \to \text{End}_{G^K_k}(V) \). In other words, there exists a structure of \( K_0 \)-vector space on \( V \), which is compatible with the \( k \)-structure and commutes with the actions of \( K \) and \( G^K_k \). In the language of comodules, denote the new action of \( K_0 \) on \( V \) by \( (\lambda, v) \mapsto t(\lambda) v \), then the above assumption amounts to saying that the comodule map \( \rho_V : V \to V \otimes \mathcal{O}(G^K_k) \) satisfies \( \rho_V(t(\lambda) v) = t(\lambda) \rho_V(v) \). Denote the category of \( K_0/k \)-representations by \( \text{Rep}(G^K_k)_{K_0/k} \). We notice that the category \( \text{Rep}(G^K_k)_{K_0/k} \) is a tensor category with respect to the tensor product over \( K_0 \).

Consider the situation of (2.9): \( G^K_{K_0} \xrightarrow{\Delta_{K_0}} G^K_k \to \text{Spec}(K_0) \times_k \text{Spec}(K_0) \). Then for any representation \( W \) over \( G^K_{K_0} \), \( \text{ind}_{\Delta_{K_0}}(W) = W \square_{\mathcal{O}(G^K_{K_0})} \mathcal{O}(G^K_k) \) is a \( K_0/k \)-representation of \( G^K_k \) with the \( K_0 \)-action induced from the \( K_0 \)-action on \( W \).

Lemma 2.13. The functor \( \text{ind}_{\Delta_{K_0}} \) induces an equivalence of tensor categories \( \text{Rep}(G^K_k) \to \text{Rep}(G^K_k)_{K_0/k} \). In particular, the tensor product over \( K_0 \) in \( \text{Rep}(G^K_k)_{K_0/k} \) is flat.
We define the map to be 
\[(2.18)\]
\[
(\phi \otimes \text{id}) : (V \boxtimes \mathcal{O}(G^K_k), \mathcal{O}(G^K_k)) \otimes_{\mathcal{O}(\mathcal{K}_0)} \mathcal{O}(G^K_k) \rightarrow
(V \otimes G_k, W) \mathcal{K}_0 \mathcal{O}(G^K_k), \mathcal{O}(G^K_k)
\]
We define the map to be
\[
\phi : (v \otimes h) \otimes (w \otimes g) \rightarrow \sum_{(v)(w)} (v(0) \otimes w(0)) \otimes v(1)w(1)gh
\]
To see that this defines an isomorphism, using the fact that \(\mathcal{O}(G^K_k)\) is faithfully flat over \(\mathcal{O}(A^K_k)\), if suffices to show that (the cotensor product is over \(\mathcal{K}_0\))
\[
\phi \otimes \text{id} : (V \boxtimes \mathcal{O}(G^K_k)) \otimes_{\mathcal{K}_0} (W \boxtimes \mathcal{O}(G^K_k)) \otimes_{\mathcal{O}(\mathcal{K}_0)} \mathcal{O}(G^K_k) \rightarrow
(V \otimes G_k, W) \mathcal{K}_0 \mathcal{O}(G^K_k), \mathcal{O}(G^K_k)
\]
is an isomorphism. This last fact follows from the isomorphism in (2.16). \(\square\)

2.14. Consider again the situation of 2.11. For a representation \(W\) of \(L^K_{\mathcal{K}_0}\), the \(\mathcal{K}_0\)-structure on \(W\) yields a \(\mathcal{K}_0\)-structure on \(\text{ind}_q(W) = W \boxtimes_{\mathcal{O}(L^K_{\mathcal{K}_0})} \mathcal{O}(G^K_k)\) making it a \(\mathcal{K}_0/k\)-representation of \(G^K_k\). In particular \(f^* \mathcal{O}(A^K_{\mathcal{K}_0}) \cong \text{ind}_q(K)\) is an object of \(\text{Rep}(G^K_k)_{\mathcal{K}_0/k}\); moreover, it is an algebra in this tensor category. To simplify the situation, we shall assume that \(f^*\) is injective, i.e., \(f\) is a surjective homomorphism of groupoids, and identify \(\mathcal{O}(A^K_{\mathcal{K}_0})\) with its image in \(\mathcal{O}(G^K_k)\).

Denote \(\text{Rep}(G^K_k)_{\mathcal{K}_0}/A^K_{\mathcal{K}_0}\) the category of \(\mathcal{O}(A^K_{\mathcal{K}_0})\)-modules in \(\text{Rep}(G^K_k)_{\mathcal{K}_0/k}\). Then this is a tensor category with respect to the tensor product over \(\mathcal{O}(A^K_{\mathcal{K}_0})\).

**Proposition 2.15.** With the assumption of 2.11 and that \(f\) is surjective we have an equivalence of tensor categories \(\text{Rep}(L^K_{\mathcal{K}_0}) \rightarrow \text{Rep}(G^K_k)_{\mathcal{K}_0}/A^K_{\mathcal{K}_0}\) given by the functor \(\text{ind}_q\).

**Proof.** The proof is similar to that of Lemma 2.13. \(\square\)

3. QUOTIENT CATEGORIES

3.1. Tensor categories and Tannaka duality. We refer to [1, §1] for the definition of tensor categories. A tensor category \(\mathcal{C}\) over a field \(k\) is a \(k\)-linear abelian category equipped with a symmetric tensor product (i.e. a symmetric monoidal structure), such that the endomorphism ring of the unit object (always denoted by \(I\)) is isomorphic to \(k\). \(\mathcal{C}\) is called rigid if each object is rigid, i.e. possesses a dual object and each object has finite length (of decomposition series).

A tensor functor between tensor categories is an additive functor that preserves the tensor product as well as the symmetry. A \(K\)-valued fiber functor of a rigid tensor category \(\mathcal{C}\) over \(k\) (\(K \supset k\)) is \(k\)-linear exact tensor functor from \(\mathcal{C}\) to \(\text{Vect}_K\), the category of \(K\)-vector spaces, its image lies automatically in the subcategory \(\text{vect}_K\) of finite dimensional vector spaces. If such a fiber
functor exists, $\mathcal{C}$ is called a Tannaka category. If, more over, $K = k$ then $\mathcal{C}$ is called neutral Tannaka.

For example, $\text{Rep}_f(G^K_k)$, where $G^K_k$ is a transitive groupoid, is a Tannaka category with the fiber functor being the forgetful functor to $\text{vect}_K$. The general Tannaka duality [2, Thm. 1.12] establishes a 1-1 correspondence between rigid tensor categories over $k$ together with a fiber functor to $\text{vect}_K$ and groupoids acting transitively over $\text{Spec} K$.

Assume that $\mathcal{C}$ is rigid over $k$ but not necessarily Tannaka. Let $\text{Ind-C}$ be the Ind-category of $\mathcal{C}$, whose object are directed systems of objects of $\mathcal{C}$, and whose hom-sets are defined as follows:

$$\text{Hom}(X_{i,i \in I}, Y_{j,j \in J}) := \lim_{\leftarrow i} \lim_{\rightarrow j} \text{Hom}_\mathcal{T}(X_i, Y_j)$$

One can also define $\text{Ind-C}$ as the category of left-exact functors from $\mathcal{T}^{\text{op}}$ to $\text{vect}_k$, an object $X$ of $\mathcal{C}$ can then be identified with the functor $\text{Hom}_\mathcal{C}(-, X)$. The natural inclusion $\mathcal{T} \hookrightarrow \text{Ind-} \mathcal{T}$ is exact and full. Further, $\mathcal{T}$ is closed under taking sub- and quotient objects, and each object of $\text{Ind-} \mathcal{T}$ is the limit of its subobjects which are isomorphic to objects from $\mathcal{T}$.

**Definition 3.2.** Let $\mathcal{T}$ be a rigid tensor category over $k$. Let $K \supseteq k$ be a field extension. A (normal) $K$-quotient of $\mathcal{T}$ is a pair $(\mathcal{Q}, q : \mathcal{T} \rightarrow \mathcal{Q})$ consisting of a $K$-linear rigid tensor category $\mathcal{Q}$ and $k$-linear exact tensor functor $q$ (the $k$-linear structure over $\mathcal{Q}$ is induced from the inclusion $k \subset K$), such that:

(i) for an object $X \in \mathcal{T}$ the largest trivial subobject of $q(X)$ is isomorphic to the image under $q$ of a subobject of $X$;

(ii) each object of $\mathcal{Q}$ is isomorphic to a subobject of the image of an object from $\mathcal{T}$ as well as a quotient of the image of an object from $\mathcal{T}$.

Our notion of normal quotient category in case $K = k$ and $\mathcal{T}$ is a Tannaka category is equivalent to Milne’s notion of normal quotient [5]. For convenience we shall omit the term “normal” in the rest of the work.

Given a $K$-quotient $(\mathcal{Q}, q)$ of $\mathcal{T}$, let $\mathcal{S}$ denote the full subcategory of $\mathcal{T}$ consisting of those objects of $\mathcal{T}$ whose images in $\mathcal{Q}$ are trivial (i.e. isomorphic to a direct sum of the unit object). It is easy to see that $\mathcal{S}$ is a tensor subcategory and is closed under taking sub- and quotient objects. We shall call $\mathcal{S}$ the invariant subcategory with respect to the quotient $(\mathcal{Q}, q)$.

**Lemma 3.3.** [5, §2] Let $(\mathcal{Q}, q)$ be a $K$-quotient of $\mathcal{T}$ and $\mathcal{S}$ be the invariant subcategory of $\mathcal{T}$. Then $\mathcal{S}$ is a Tannaka category with a fiber functor to $\text{vect}_K$.

**Proof.** The full subcategory of $\mathcal{Q}$ of trivial subobjects is equivalent to $\text{vect}_K$. The fiber functor is given by $\mathcal{S} \ni X \mapsto q(X) \mapsto \text{Hom}_\mathcal{Q}(I, q(X))$ where $I$ denotes the unit object in $\mathcal{Q}$. Since $q(X)$ is trivial in $\mathcal{Q}$, this functor is a fiber functor. Hence $\mathcal{S}$ is a Tannaka category. \qed
4. Quotient category by a neutral Tannaka subcategory

Let $\mathcal{T}$ be a rigid tensor category over a field $k$. Let $q : \mathcal{T} \to \mathcal{Q}$ be a $k$-quotient of $\mathcal{T}$ and $\mathcal{S}$ be the invariant category as in Definition 3.2 (with $K = k$). According to Lemma 3.3, $\mathcal{S}$ is a neutral Tannaka category with fiber functor given by $\omega(\mathcal{S}) \cong \text{Hom}_{\mathcal{Q}}(I, q\mathcal{S})$. Let us consider the category $\text{vect}_k$ as a full subcategory of $\mathcal{Q}$ by identifying a vector space $V$ with $V \otimes I$ in $\mathcal{Q}$ [1]. Then we can consider the above fiber functor as the restriction of $q$ to $\mathcal{S}$. In other words, we have the following functorial isomorphism
\[
\omega(\mathcal{S}) \otimes_k I = \text{Hom}_{\mathcal{Q}}(I, q\mathcal{S}) \otimes_k I \cong q(\mathcal{S}), \quad S \in \mathcal{S}
\]
by means of which we shall identify $\omega(\mathcal{S})$ with $q(\mathcal{S})$ for $S \in \mathcal{S}$.

4.1. The existence. Assume that $\mathcal{T}$ is a Tannaka category. Thus, there exists a fiber functor $\tilde{\omega} : \mathcal{T} \to \text{vect}_k$ extending the fiber functor $\omega$ and $\mathcal{T} \cong \text{Rep}_f(G^K_k), \mathcal{S} \cong \text{Rep}_f(A^k_k)$ for some groupoid scheme $G^K_k$ and group scheme $A^k_k$. According to Lemma 2.5, Proposition 2.11, $\mathcal{Q} := \text{Rep}_f(L^k_k)$ is a $k$-quotient of $\mathcal{T}$ by $\mathcal{S}$.

4.2. The algebra $\mathcal{O}$. By means of the fiber functor $\omega$, $\mathcal{S}$ is equivalent to the category $\text{Rep}_f(G(\mathcal{S}))$ of finite dimensional $k$-representation of $G(\mathcal{S})$, where $G(\mathcal{S})$ is an affine $k$-group scheme, and Ind-$\mathcal{S}$ is equivalent to $\text{Rep}(G(\mathcal{S}))$. Let $\mathcal{O}$ denote the function algebra of the group $G(\mathcal{S})$. It is a $k$-Hopf algebra. By means of the right regular action of $G(\mathcal{S})$ on $\mathcal{O}$, that is, consider $\mathcal{O}$ as a right comodule on itself by means of the coproduct map, $\mathcal{O}$ can be considered as an object in Ind-$\mathcal{S}$. Since the unit map $u : k \to \mathcal{O}$ and the multiplication map $m : \mathcal{O} \otimes_k \mathcal{O} \to \mathcal{O}$ of $\mathcal{O}$ are compatible with the coproduct and the counit maps, they are also morphisms in Ind-$\mathcal{S}$, hence $(\mathcal{O}, m, u)$ is an algebra in Ind-$\mathcal{S}$ (it is not a Hopf algebra since the coproduct and the counit are not morphisms in Ind-$\mathcal{S}$). For convenience we shall use the same notation for denoting $\mathcal{O}$ as a $k$-vector space or as an object in $\mathcal{S}$ as well as an object of $\mathcal{Q}$ when we consider $\omega$ (the fiber functor of $\mathcal{O}$) as the restriction of $q$ to $\mathcal{S}$.

4.3. The largest $\mathcal{S}$-subobject. For an object $X \in \mathcal{T}$, let $X_S$ denote the largest subobject of $X$ which is isomorphic to an object in $\mathcal{S}$. Since $\mathcal{S}$ is closed under taking sub- and quotient objects, we have the equality
\[
\text{Hom}_T(S, X) = \text{Hom}_S(S, X_S), \quad S \in \mathcal{S}, X \in \mathcal{T}
\]
Thus we have a functor $(-)_S : \mathcal{T} \to \mathcal{S}$ whose definition on hom-sets is just the restriction of morphisms $\text{Hom}_T(X, Y) \mapsto \text{Hom}_T(X_S, Y_S) = \text{Hom}_T(X_S, Y_S)$. Equation (4.2) also shows that this functor is right adjoint to the inclusion functor $\mathcal{S} \hookrightarrow \mathcal{T}$. The functor $(-)_S$ is canonically extended to a functor Ind-$\mathcal{T} \to$ Ind-$\mathcal{S}$ denoted by the same symbol which is also the right adjoint to the inclusion functor Ind-$\mathcal{S} \to$ Ind-$\mathcal{T}$.

Let $X^\vee$ denote the dual object to $X$. The for any $S \in \mathcal{S}$, we have
\[
\text{Hom}_S(X, S) \cong \text{Hom}_S(S^\vee, X^\vee) \cong \text{Hom}_S(S^\vee, (X^\vee)_S) \cong \text{Hom}_S((X^\vee)_S^\vee, S)
\]
Since $X_S \rightarrow X$ is mono, $X^\vee \rightarrow (X^\vee)_S^\vee$ is epi. Thus the largest $\mathcal{S}$-quotient of $X$ is isomorphic to $(X^\vee)_S^\vee$.

**Lemma 4.4.** There is a functorial isomorphism

\[ \text{Hom}_{\text{Ind-}\mathcal{T}}(I, X \otimes \mathcal{O}) \xrightarrow{\cong} \omega(X_S), \quad X \in \mathcal{T}. \]

where $\omega$ is the fiber functor of $\mathcal{S}$ to $\text{vect}_k$.

**Proof.** If $X = S \in \mathcal{S}$ we have the following well-known isomorphism

\[ S \otimes \mathcal{O} \cong \omega(S) \otimes_k \mathcal{O} \]

Indeed, by applying $\omega$ on both side it suffices to exhibit a $G(\mathcal{S})$-linear isomorphism $\omega(S) \otimes_k \mathcal{O} \rightarrow \mathcal{O}^{\dim_k \omega(S)}$, where $G(\mathcal{S})$ as on the source by the diagonal action. Let $\delta : \omega(S) \rightarrow \omega(S) \otimes_k \mathcal{O}$ denote the coaction of $\mathcal{O}$ on $\omega(S)$ induced from the action of $G(\mathcal{S})$, then the following map

\[ \omega(S) \otimes_k \mathcal{O} \xrightarrow{\delta \otimes \text{id}} \omega(S) \otimes_k \mathcal{O} \otimes_k \mathcal{O} \xrightarrow{\text{id} \otimes m} \omega(S) \otimes_k \mathcal{O} \cong \mathcal{O}^{\dim_k \omega(S)} \]

is $G(\mathcal{S})$-linear and bijective with the inverse given by

\[ \mathcal{O}^{\dim_k \omega(S)} \cong \omega(S) \otimes_k \mathcal{O} \xrightarrow{\delta \otimes \text{id}} \omega(S) \otimes_k \mathcal{O} \otimes_k \mathcal{O} \xrightarrow{\text{id} \otimes \text{m}} \omega(S) \otimes_k \mathcal{O} \]

Since $\text{Hom}_{G(\mathcal{S})}(k, \mathcal{O}) \cong k$, we obtain

\[ \text{Hom}_{\text{Ind-}\mathcal{T}}(I, S \otimes \mathcal{O}) \cong \text{Hom}_{G(\mathcal{S})}(k, \mathcal{O}^{\dim_k \omega(S)}) \cong \omega(S) \]

In the general case, let $X^\vee$ be the dual object to $X$, then morphisms $I \rightarrow X \otimes \mathcal{O}$ are in 1-1 correspondence with morphisms $X^\vee \rightarrow \mathcal{O}$, which are in 1-1 correspondence with morphisms from the largest $\mathcal{S}$-quotient of $X^\vee$ to $\mathcal{O}$, since $\mathcal{O}$ is an object of $\text{Ind-}\mathcal{S}$. Notice that the largest $\mathcal{S}$-quotient of $X^\vee$ is isomorphic to $(X_S)^\vee$, as for $X \in \mathcal{T}$ one has $(X^\vee)^\vee \cong X$. Thus we have

\[ \text{Hom}_{\text{Ind-}\mathcal{T}}(I, X \otimes \mathcal{O}) \cong \text{Hom}_{\text{Ind-}\mathcal{T}}(X^\vee, \mathcal{O}) \cong \text{Hom}_{\text{Ind-}\mathcal{T}}((X_S)^\vee, \mathcal{O}) \cong \omega(X_S) \]

Therefore (4.3) is proved. The functoriality of (4.3) is obvious since any morphism $f : X \rightarrow Y$ restricts to a morphism $f : X_S \rightarrow Y_S$. \qed

**Corollary 4.5.** There exists a functorial isomorphism

\[ \text{Hom}_\mathcal{Q}(I, q(X^\vee \otimes Y)) \cong \text{Hom}_{\text{Ind-}\mathcal{T}}(X \otimes Y^\vee, \mathcal{O}) \]

**Proof.** By definition of quotient category, the largest trivial subobject of an object $q(X) \in \mathcal{Q}$ has the form $q(X')$, where $X'$ is a subobject of $X$ (in $\mathcal{T}$). By definition of $\mathcal{S}$, $X'$ is in $\mathcal{S}$ and should be the largest $\mathcal{S}$-subobject of $X$, that is $X' = X_S$. We have

\[ \text{Hom}_\mathcal{Q}(q(X), q(Y)) \cong \text{Hom}_\mathcal{Q}(I, q(X \otimes Y)) \cong \text{Hom}_\mathcal{Q}(I, q((X^\vee \otimes Y)_S)) \]

These isomorphisms together with (4.1) and Lemma 4.4 imply (4.5). \qed

The isomorphism in (4.5) implies the following

\[ \text{Hom}_\mathcal{Q}(q(X), q(Y)) \cong \text{Hom}_{\text{Ind-}\mathcal{T}}(X, Y \otimes \mathcal{O}) \]
Let us denote this map by

\[ f \mapsto \bar{f} \]

By the functoriality of (4.3) we see that, for \( X = Y \),

\[ \overline{id_X} = id_X \otimes u : X \to X \otimes \mathcal{O} \]

where \( u : I \to \mathcal{O} \) is the unit map of \( \mathcal{O} \).

### 4.6. The adjoint functor to \( q \).

The quotient functor \( q \) extends to a functor from \( \text{Ind-} \mathcal{T} \to \text{Ind-} \mathcal{Q} \), denoted by the same symbol \( q \).

\[ q(\lim_{\to} X_i) := \lim_{\to} q(X_i) \]

\( q \) possesses a right adjoint, denoted by \( p \). Indeed, for an object \( U \) of \( \mathcal{Q} \), \( p(U) \) is determined by the condition

\[ \text{Hom}_{\text{Ind-} \mathcal{T}}(X, p(U)) \cong \text{Hom}_{\mathcal{Q}}(q(X), U), \quad \forall X \in \mathcal{T} \]

and for an object \( \lim_{\to} U_i \in \text{Ind-} \mathcal{Q} \),

\[ p(\lim_{\to} U_i) := \lim_{\to} p(U_i) \]

Thus \( p \) satisfies the required functorial isomorphism:

\[ (4.8) \quad \text{Hom}_{\text{Ind-} \mathcal{Q}}(q(X), U) \cong \text{Hom}_{\text{Ind-} \mathcal{T}}(X, p(U)), \quad \forall X \in \text{Ind-} \mathcal{T}, U \in \text{Ind-} \mathcal{Q} \]

For \( X = p(U) \) in (4.8), the identity of \( p(U) \) corresponds to a map \( \varepsilon_U : qp(U) \to U \), and (4.8) given by composing a morphism on the right hand side with \( \varepsilon_U \).

\[ \begin{array}{ccc} q(p(U)) & \xrightarrow{q(f)} & \varepsilon_U \\ \downarrow & & \downarrow \\ q(X) & \xrightarrow{f} & U \end{array} \]

**Lemma 4.7.** For an object \( X \in \text{Ind-} \mathcal{T} \) the object \( pq(X) \) in \( \text{Ind-} \mathcal{T} \) is canonically isomorphic to \( X \otimes \mathcal{O} \) and the map

\[ q(X) \otimes \mathcal{O} \cong qpq(X) \xrightarrow{\xi_X} q(X) \]

is given by \( \text{id}_X \otimes \varepsilon \), where \( \varepsilon \) is the counit of \( \mathcal{O} \) (considering \( \omega \) as the restriction of \( p \) to \( S \)). With respect to the isomorphism \( \text{pq}(X) \cong X \otimes \mathcal{O} \), (4.8) for \( U = q(Y) \) reduces to (4.6).

**Proof.** First assume \( X \in \mathcal{T} \). According to Lemma 4.4 and (4.8) we have, for \( X, Y \in \mathcal{T} \),

\[ (4.9) \quad \text{Hom}_{\text{Ind-} \mathcal{T}}(X, p(Y)) \cong \text{Hom}_{\mathcal{Q}}(qX, qY) \cong \text{Hom}_{\text{Ind-} \mathcal{T}}(X, Y \otimes \mathcal{O}) \]
Since this isomorphism holds for any $X,Y$ we conclude that $p(Y)$ is canonically isomorphism to $Y \otimes O$. The map $\varepsilon_Y$ induces a morphism $q(Y) \otimes O \rightarrow q(Y)$ which will be denoted by the same symbol. Thus we have the following diagram

\[(4.10)\]

\[
\begin{array}{ccc}
q(Y) \otimes O & \xrightarrow{q(f)} & q(Y) \\
\downarrow{\varepsilon_Y} & & \downarrow{\varepsilon_Y} \\
q(Y) & \xrightarrow{f} & q(Y)
\end{array}
\]

Set $X = Y$ in (4.9), then, according to (4.7), the identity on $Y$ is mapped under the second isomorphism of (4.9) to the morphism $\text{id}_Y \otimes u : Y \rightarrow Y \otimes O$. Since the inverse map to the second isomorphism in (4.9) is given by $f \mapsto \text{id} \otimes (\varepsilon_Y)$, we conclude that $\varepsilon_Y(\text{id} \otimes u) = \text{id} \otimes \varepsilon_Y$. Therefore, $\varepsilon_Y = \text{id}_q(Y) \otimes \varepsilon$ (recall that we identify $\varepsilon$ with $q(\varepsilon)$).

For the general case we note that the tensor product in Ind-$\mathcal{T}$ commutes with direct limits, hence

\[
pq(\lim_i X_i) \cong \lim_i (X_i \otimes O) \cong (\lim_i X_i) \otimes O.
\]

□

Corollary 4.8. In the isomorphism (4.6) the composition $q(X) \xrightarrow{f} q(Y) \xrightarrow{g} q(Z)$ corresponds to the morphism

\[
X \xrightarrow{f} Y \otimes O \xrightarrow{q \otimes \text{id}} Z \otimes O \otimes O \xrightarrow{id \otimes m} Z \otimes O
\]

Proof. Since $q$ is a faithful functor, it suffices to check that the outer diagram below commutes.

\[
\begin{array}{ccc}
q(X) & \xrightarrow{q(f)} & q(Y) \otimes O \xrightarrow{q(g) \otimes \text{id}} q(Z) \otimes O \otimes O \\
\downarrow{f} & & \downarrow{\text{id} \otimes \varepsilon} \\
q(Y) & \xrightarrow{g} & q(Z)
\end{array}
\]

The commutativity of the first triangle and the middle square follow from (4.10) and of the right triangle is by the multiplicativity of the counit map $\varepsilon$.

□

Using the same method we can prove the following fact.

Corollary 4.9. Let $f_i : q(X_i) \rightarrow q(Y_i)$, $i = 0,1$ be morphisms in $Q$. Then the morphism $f_0 \otimes f_1$ is given by

\[
\begin{array}{ccc}
X_0 \otimes X_1 & \xrightarrow{f_0 \otimes f_1} & Y_0 \otimes Y_1 \otimes O \\
\end{array}
\]

\[
\begin{array}{ccc}
Y_0 \otimes O \otimes Y_1 \otimes O & \xrightarrow{\tau(23)} & Y_0 \otimes Y_1 \otimes O \otimes O
\end{array}
\]
where the map \( \tau_{(23)} \) interchanges the second and the third tensor terms.

**Proposition 4.10.** Let \( f : q(X) \to q(Y) \) be a morphism in \( Q \). Then

\[
(4.11) \quad \text{p}(f) = X \otimes \mathcal{O} \xrightarrow{f \otimes \text{id}} Y \otimes \mathcal{O} \otimes \mathcal{O} \xrightarrow{\text{id} \otimes m} Y \otimes \mathcal{O}
\]

**Proof.** The morphism \( \text{p}(f) \) fits in to the following commutative square

\[
\begin{array}{ccc}
\text{Hom}_Q(q(Z), q(X)) & \xrightarrow{\cong} & \text{Hom}_{\text{Ind-}T}(Z, \text{pq}(X)) \\
\downarrow \quad f \circ - & & \downarrow \quad \text{p}(f) \circ - \\
\text{Hom}_Q(q(Z), q(Y)) & \xrightarrow{\cong} & \text{Hom}_{\text{Ind-}T}(Z, \text{pq}(X))
\end{array}
\]

and is indeed determined by this square (for all \( Z \in \mathcal{T} \)). Thus, in terms of (4.8) \( \text{p}(f) \) is uniquely determined by the following commuting triangle

\[
\begin{array}{ccc}
Z & \xrightarrow{\bar{g}} & \text{pq}(X) \\
\downarrow \quad \text{P} & & \downarrow \quad \text{p}(f) \\
\text{pq}(Y)
\end{array}
\]

for all \( Z \in \mathcal{T} \), \( g : q(Z) \to q(X) \). According to Corollary 4.8, the morphism on the right hand side of (4.11) satisfies this property, hence is equal to \( \text{p}(f) \). □

Recall that \( \mathcal{O} \) is a commutative algebra in \( \text{Ind-}T \). We can thus consider the category \( \text{Mod}_O \) of \( O \)-modules in \( \text{Ind-}T \), which is an abelian category equipped with a tensor product over \( \mathcal{O} \).

**Definition 4.11.** Let \( (\mathcal{S}, \omega : \mathcal{S} \to \text{vect}_k) \) be a neutral Tannaka subcategory of a rigid tensor category \( \mathcal{T} \) which is closed under taking sub- and quotient objects. Denote by \( \mathcal{O} \) the function algebra over its Tannaka group, viewed as an object in \( \text{Ind-}\mathcal{S} \subset \text{Ind-}T \). The category \( \mathcal{T} \) is said to be flat over \( \mathcal{S} \) if for any \( \mathcal{O} \)-linear morphism \( f : X \otimes \mathcal{O} \to Y \otimes \mathcal{O} \) in \( \text{Ind-}T \), \( X, Y \in \mathcal{T} \), the kernel of \( f \) is flat with respect to tensor product in \( \text{Mod}_O \).

**Theorem 4.12.** Let \( (\mathcal{Q}, q) \) be a \( k \)-quotient category of a tensor category \( \mathcal{T} \) over \( k \) and denote by \( \mathcal{S} \) the corresponding invariant category. Let \( \mathcal{O} \) denote the function algebra over the Tannaka group of \( \mathcal{S} \). Let \( \text{p} \) be the right adjoint to the quotient functor \( q \). Then:

1. For any object \( U \in \mathcal{Q} \), there exists a morphism \( \mu_U : \text{p}(U) \otimes \mathcal{O} \to \text{p}(U) \) making \( \text{p}(U) \) an \( \mathcal{O} \)-module.
2. Assume that \( \mathcal{T} \) is flat over \( \mathcal{S} \). Then \( \text{p} \) is a tensor functor from \( \mathcal{Q} \) to the category of \( \mathcal{O} \)-modules with the tensor product being the tensor product over \( \mathcal{O} \). Consequently, \( \text{p} \) is exact.
Proof. Using (4.8), we define $\mu_U$ as the unique morphisms in $\text{Ind-}T$ making the following diagram commutative

$$
\begin{array}{ccc}
\text{qp}(U) & \xleftarrow{\varepsilon_U} & \text{qp}(U) \\
\downarrow & & \downarrow \\
\text{qp}(U) \otimes \mathcal{O} & \xrightarrow{\mu_U} & U
\end{array}
$$

This definition is functorial hence the action of $\mu$ commutes with morphisms in $\mathcal{Q}$. The associativity of this action can also be checked by this method. Thus $p$ factors through a functor to $\text{Mod-}\mathcal{O}$, denoted by the same notation.

For objects in $\mathcal{Q}$ of the form $q(X)$, $pq(X) \cong X \otimes \mathcal{O}$, we see that the multiplication map on $\mathcal{O}$ makes the following diagram commutative

$$
\begin{array}{ccc}
\text{id} \otimes \mu_X & \xrightarrow{\varepsilon_{q(X)}} & \text{id} \otimes \epsilon \\
\downarrow & & \downarrow \\
\text{q}(X) \otimes \mathcal{O} \otimes \mathcal{O} & \xrightarrow{\mu_X} & \text{q}(X)
\end{array}
$$

Thus, the action of $\mathcal{O}$ on $p(X) \cong X \otimes \mathcal{O}$ is induced from the action of $\mathcal{O}$ on itself.

Assume now that $\mathcal{T}$ is flat over $\mathcal{S}$. We want to show that $p$ is a monoidal functor from $\mathcal{Q}$ to $\mathcal{O}$-modules. This is so for objects of the form $q(X)$, according to Corollary 4.9. For an arbitrary object $U$ of $\mathcal{Q}$, there exists objects $X,Y$ in $\mathcal{T}$ and a morphism $f : q(X) \to q(Y)$ such that $U = \ker f$. Since the functor $p$ is left exact (being a right adjoint functor), $p(U)$ is isomorphic to the kernel of $p(f)$. Now, for $f_i : q(X_i) \to q(Y_i)$ we have $p(f_1 \otimes f_2) = p(f_1) \otimes_{\mathcal{O}} p(f_2)$. Since for $U_i := \ker f_i$ are flat over $\mathcal{O}$ by assumption, we have an $\mathcal{O}$-linear isomorphism

$$
p(U_1) \otimes_{\mathcal{O}} p(U_2) \cong p(U_1 \otimes U_2)
$$

Thus $p$ is a tensor functor to $\mathcal{O}$-modules. Since $\mathcal{Q}$ is rigid with the endomorphism ring of the unit object isomorphic to $k$, we conclude that $p$ is an exact functor, cf. [2, 2.10]. □

4.13. A description of the quotient. We define a category $\mathcal{P}$, whose objects are triples

$$
(X,Y,f : X \to Y \otimes \mathcal{O})
$$

where $X,Y$ are objects of $\mathcal{T}$ and $f$ is a morphism in $\text{Ind-}T$. The morphisms $f : X \to Y \otimes \mathcal{O}$ defines an $\mathcal{O}$-linear morphism $\hat{f}$

$$
\hat{f} : X \otimes \mathcal{O} \overset{f \otimes \text{id}}{\to} Y \otimes \mathcal{O} \otimes \mathcal{O} \overset{\text{id} \otimes m}{\to} Y \otimes \mathcal{O}
$$

The image of $\hat{f}$ is thus an $\mathcal{O}$-module.

We define morphisms in $\mathcal{P}$ between $(X_i,Y_i,f_i)$, $i = 0,1,2$ as $\mathcal{O}$-linear morphisms $\phi : \text{im}\hat{f}_0 \to \text{im}\hat{f}_1$. Thus we obtain a category $\mathcal{P}$ which is $k$-linear and
additive. Further, the direct sum of objects in $\mathcal{P}$ exists:

$$\text{(4.13)} \quad (X_0, Y_0, f_0) \oplus (X_1, Y_1, f_1) := (X_0 \oplus X_1, Y_0 \oplus Y_1, f_0 \oplus f_1).$$

The tensor structure on $\mathcal{P}$ is defined as follows.

$$\text{(4.14)} \quad (X_0, Y_0, f_0) \otimes (X_1, Y_1, f_1) := (X_0 \otimes X_1, Y_0 \otimes Y_1, (\text{id} \otimes m)(f_0 \otimes f_1))$$

The unit object is $I = (I, I, u : I \to \mathcal{O})$. Finally, we define a functor $q' : \mathcal{T} \to \mathcal{P}$ sending $X$ to $q'(X) = (X, X, \text{id} \otimes u : X \to X \otimes \mathcal{O})$. It is easy to see that $q'$ is a $k$-linear, additive tensor functor.

**Proposition 4.14.** Let $\mathcal{T} \to \mathcal{Q}$ be $k$-quotient and $\mathcal{S}$ be the corresponding invariant subcategory. Assume that $\mathcal{T}$ is flat over $\mathcal{S}$. Then the category $\mathcal{P}$ constructed above is equivalent to $\mathcal{Q}$.

**Proof.** We construct a functor $F : \mathcal{P} \to \mathcal{Q}$ such that $Fq' = q$. For any $X, Y \in \mathcal{T}$, according to 4.9, morphisms $f : X \to Y \otimes \mathcal{O}$ in $\mathcal{T}$ are in 1-1 correspondence with morphisms $f_\mathcal{Q} : q(X) \to q(Y)$ in $\mathcal{Q}$. This allows us to define the image of an object $U = (X, Y, f : X \to Y \otimes \mathcal{O}) \in \mathcal{Q}'$ as the image of $f_\mathcal{Q}$ in $\mathcal{Q}'$.

According to Theorem 4.12, $p$ is exact. Hence for any morphism $f : q(X) \to q(Y)$ in $\mathcal{Q}$ with $U = \text{im} f$ one has

$$p(U) \cong \text{im}(f)$$

Consequently, there is a 1-1 correspondence between morphisms $U \to V$ in $\mathcal{Q}$ and morphisms between $p(U) \to p(V)$ in $\mathcal{T}$ which are $\mathcal{O}$-linear. This allows us to define $F$ on morphisms and to check that $F$ is a fully faithful functor.

Further it is easy to see that the image of $F$ is essential in $\mathcal{Q}$. Thus $F$ is an equivalence. $\square$

As a consequence of the above theorem and Proposition 2.15 we have the following.

**Corollary 4.15.** Let $f : G^K_k \to A^K_k$ be a homomorphism of transitive groupoids and $L^K_k$ be the kernel of $f$. Then $\text{Rep}_f(L^K_k)$ is equivalent to the category whose objects are triples $(U, V, f : U \to V \otimes \mathcal{O}(A))$, where $U, V \in \text{Rep}_f(G)$, $f$ is $G$-equivariant ($G$ acts diagonally on $V \otimes \mathcal{O}(A)$).

**Remark 4.16.** There is an alternative description of objects of $\mathcal{Q}$, proposed by Deligne. Notice that morphisms $f : X \to Y \otimes \mathcal{O}$ are in 1-1 correspondence with morphisms $f^\mathcal{O} : I \to X^\vee \otimes Y \otimes \mathcal{O}$. As noticed in 4.4, such a morphism $f^\mathcal{O}$ corresponds to an element of $\omega((X^\vee \otimes Y)_S)$. Thus objects of $\mathcal{Q}$ can be characterized as triples $(X, Y, f \in \omega((X^\vee \otimes Y)_S))$.

5. **Quotient category by a not necessary neutral Tannaka subcategory**

5.1. **Base change for tensor category.** Let $\mathcal{T}$ be a tensor category over $k$. Assume that for a field extension $k \subset K$, a $K$-quotient $\mathcal{T}(K)$ of $\mathcal{T}$ exists such that the invariant subcategory is the trivial subcategory of $\mathcal{T}$, i.e. equivalent.
to \texttt{vect}_k. \mathcal{T}_{(K)} is called the tensor category over \( K \) obtained from \( \mathcal{T} \) by base change. Denote the quotient functor by \( -_{(K)} : X \mapsto X_{(K)} \). Thus, the largest trivial subobject \( X_{(K)}^{\text{triv}} \) of \( X_{(K)} \) in \( \mathcal{T}_{(K)} \) is isomorphic to the image of the largest trivial object \( X^{\text{triv}} \) of \( X \) in \( \mathcal{T} \). Hence

\[
\text{Hom}_{\mathcal{T}_{(K)}}(I, X_{(K)}) = \text{Hom}_{\mathcal{T}_{(K)}}(I, X_{(K)}^{\text{triv}}) \\
\approx \text{Hom}_{\mathcal{T}}(I, X^{\text{triv}}) \otimes_k K \\
\approx \text{Hom}_{\text{Ind}-\mathcal{T}}(I, X^{\text{triv}}) \otimes_k K \\
= \text{Hom}_{\text{Ind}-\mathcal{T}}(I, X) \otimes_k K
\]

On the other hand, we have an isomorphism

\[
(5.1) \quad \text{Hom}_{\mathcal{T}}(X, Y) \otimes_k K \cong \text{Hom}_{\text{Ind}-\mathcal{T}}(X, Y \otimes_k K)
\]

given explicitly as follows. Fix a basis \( \{e_i, i \in I\} \) of \( K \) over \( k \). An element \( f \in \text{Hom}_{\mathcal{T}}(X, Y) \otimes_k K \), represented as \( f = \sum_{i \in I_0} f_i \otimes e_i \), for a certain finite subset \( I_0 \subset I \), is mapped to the morphism

\[
X \xrightarrow{\Delta} \bigoplus_{i \in I_0} X_i \xrightarrow{\oplus f_i} \bigoplus_{i \in I_0} Y_i \hookrightarrow Y \otimes_k K
\]

where \( X_i \) (resp. \( Y_i \)) are copies of \( X \) (resp. \( Y \)) and the last inclusion in given by the chosen basis of \( K \).

Extend \( -_{(K)} \) to a functor \( \text{Ind}-\mathcal{T} \rightarrow \text{Ind}-\mathcal{T}_{(K)} \) and let \( -^{K} : \text{Ind}-\mathcal{T}_{(K)} \rightarrow \text{Ind}-\mathcal{T} \) denote the adjoint functor to \( -_{(K)} \).

**Lemma 5.2.** For \( X \in \mathcal{T} \) holds: \((X_{(K)})^K \cong X \otimes_k K \) and the map \( \varepsilon_X \) is given by \( \varepsilon_X : X_{(K)} \otimes_k K \rightarrow X_{(K)} \otimes_k K \cong X_{(K)} \).

Other claims similar to those of Corollaries 4.8, 4.9 and of Prop. 4.10 also hold.

By identifying \( K \) with \( K \otimes_k I \), we can consider \( K \) as an algebra in \( \text{Ind}-\mathcal{T} \) and consider the category \( \text{Ind}-\mathcal{T}_K \) of \( K \)-modules. Then the functor \( -^{K} \) factors through a functor to \( \text{Ind}-\mathcal{T}_K \), denoted by the same symbol.

**Definition 5.3.** We say that \( \mathcal{T} \) is flat over \( K \) if for any \( X, Y \in \mathcal{T} \) and \( K \)-linear morphism \( f : X \otimes_k K \rightarrow Y \otimes_k K \) in \( \text{Ind}-\mathcal{T}_K \), the kernel of \( f \) is flat with respect to the tensor product over \( K \).

Consequently we have the following theorem, which is analogous to Theorem 4.12.

**Theorem 5.4.** Assume that \( \mathcal{T} \) is flat over a field \( K \supset k \) and that the base change \( \mathcal{T}_{(K)} \) exists. Then, for an object \( U \in \mathcal{T}_{(K)} \), there exists a map \( \mathfrak{p}(U) \otimes_k K \rightarrow \mathfrak{p}(U) \) making \( \mathfrak{p}(U) \) a \( K \)-module. Further \( \mathfrak{p} \) is a \( K \)-linear tensor functor from \( \mathcal{T}_{(K)} \) to the category \( \text{Ind}-\mathcal{T}_K \) of \( K \)-modules in \( \mathcal{T} \). Consequently \( -^{K} \) is exact.

**Corollary 5.5.** Assume that \( \mathcal{T} \) is flat over \( K \) and that \( \mathcal{T}_{(K)} \) exists. Then \( \mathcal{T}_{(K)} \) is equivalent to the category of triples \((X, Y, f : X \rightarrow Y \otimes_k K), X, Y \in \mathcal{T},\)
$f \in \text{Mor} \text{ Ind-} \mathcal{T}$, and morphism defined as in section 4. Consequently, any $k$-linear tensor functor $\omega$ from $\mathcal{T}$ to a $K$-linear tensor category $\mathcal{C}$ factors through a $K$-linear tensor $\omega_K$ from $\mathcal{T}(K)$ to $\mathcal{C}$.

**Proof.** The first claim is proved analogously as in the proof of Proposition 4.14. The second claim is a consequence of the first one. In deed, the functor $\omega$ induces a $K$-linear map $\omega \otimes K : \text{Hom}_\mathcal{T}(X,Y) \otimes_k K \to \text{Hom}_\mathcal{C}(\omega(X),\omega(Y))$

Thus, considering $f$ as an element of $\text{Hom}_\mathcal{T}(X,Y) \otimes_k K$ by means of (5.1), we can define $\omega_K$ on an object $(X,Y,f : X \to Y \otimes K)$ as the image of $(\omega \otimes K)f : \omega(X) \to \omega(Y)$. □

**Corollary 5.6.** Let $\mathcal{T}$ be a Tannaka category over $k$ with fiber functor to $\text{vect}_K$, $K \supset k$ and $G^K_k$ the corresponding Tannaka groupoid. Let $k \subset K_0 \subset K$ be an intermediate field. Then the $K_0$-base change of $\mathcal{T}$ exists: $\mathcal{T}_{K_0} \cong \text{Rep}_f(G^K_{K_0})$.

**Proof.** This follows from Proposition 2.15 for $A = \text{Spec} K_0 \times_k \text{Spec} K_0$ and Theorem 5.4. □

The category $\mathcal{T}_K$ for $K$ a finite extension of $k$ was studied by Deligne-Milne [1, Sect. 3].

5.7. **The description of the quotient category.** Assume now that $(\mathcal{Q},q)$ is a $K$-quotient of $\mathcal{T}$ with $\mathcal{S}$ being the invariant category and assume that the base change $\mathcal{T}(K)$ of $\mathcal{T}$ exists and that $\mathcal{T}$ is flat over $K$. By Corollary 5.5, $q$ factors through a $K$-linear tensor functor $q_K : \mathcal{T}(K) \to \mathcal{Q}$, which can easily shown to be a quotient functor of tensor categories over $K$ and the invariant category of $q_K$ is equivalent to $\mathcal{S}_K$.

Let $\mathcal{O}$ denote the function algebra of the Tannaka groupoid $A^K_k$ of $\mathcal{S}$. Consider $\mathcal{O}$ as an object of $\text{Ind-} \mathcal{S}$ by means of the right regular coaction $\Delta : \mathcal{O} \to \mathcal{O} \otimes_s \mathcal{O}$. Then $\mathcal{O}$ is a $K$-object in the sense of 5.1 where $K$ acts through the map $t$. Further $\mathcal{O}$ is a $K$-algebra, i.e. an algebra in $\text{Ind-} \mathcal{T}_K$, which in this case means that $\mathcal{O}$ is an algebra over $K \otimes K$ through the map $s \otimes t$ in the usual sense. We define by $\text{Mod}_{\mathcal{O},K}$ the category of $\mathcal{O}$-modules in $\text{Ind-}\mathcal{T}_K$, whose objects are thus $K$-object equipped with an action of $\mathcal{O}$. This is a tensor category with respect to the tensor product over $\mathcal{O}$.

**Definition 5.8.** With the assumption of 5.7, $\mathcal{T}$ is said to be flat over $\mathcal{S}$ if for any $\mathcal{O}$-linear morphism $f : X \otimes \mathcal{O} \to Y \otimes \mathcal{O}$, $X,Y \in \mathcal{T}$, the kernel of $f$ is flat with respect to the product over $\mathcal{O}$.

**Theorem 5.9.** Let $\mathcal{T}$ be a rigid tensor category over $k$. Let $(\mathcal{S},\omega : \mathcal{S} \to \text{vect}_K)$ be a Tannaka subcategory, which is closed under taking sub- and quotient objects. Assume that

(i) $\mathcal{T}$ is flat over $K$ and the base change $\mathcal{T}(K)$ exists;

(ii) $\mathcal{T}$ is flat over $\mathcal{S}$ and a $K$-quotient $(\mathcal{Q},q)$ of $\mathcal{T}$ by $\mathcal{S}$ exists.
Then the right adjoint functor $p$ to $q$ induces an exact tensor functor to the category $\text{Mod}_{O,K}$.

**Proof.** Denote by $O_K$ the function algebra of $A^K_{K}$ - the diagonal subgroup of $A^K_{k}$, where $A^K_{k}$ is the Tannaka groupoid of $S$. Let $p_K$ denote the right adjoint functor to $q_K : \text{Ind-}T(K) \rightarrow \text{Ind-}Q$. Then according to 4.12, $q_K$ is an exact tensor functor to the category of $O_K$-modules in $\text{Ind-}T(K)$.

On the other hand, since $S \cong \text{Rep}_f(A^K_k), S(K) \cong \text{Rep}(A^K_k)$, the functor $-^K$ in fact the functor $\text{ind}\Delta_{K}$ in the situation of 2.12. Thus $\text{Ind-}T(K) \rightarrow \text{Ind-}Q$. This isomorphism gives us a morphism

$$(O_K)^K \cong O_K \square O_K \cong O$$

We don’t know if this morphism is an isomorphism for an arbitrary pair $U, V \in T(K)$, but we know it is for $U, V$ in the image of $q_K$, thanks to the assumption that $T$ is flat over $S$. In deed, for $X \in T$, we have $pq(X) \cong X \otimes O$, using the method as in the proof of 4.12 we deduce the required isomorphism. □.

**Corollary 5.10.** Let $f : G^K_k \rightarrow A^K_{K_0}$ be a homomorphism of transitive groupoids and $L^K_{K_0}$ be the kernel of $f$. Then $\text{Rep}_f(L^K_{K_0})$ is equivalent to the category whose objects are triples $(U, V, f : U \rightarrow V \otimes O(A))$, where $U, V \in \text{Rep}_f(G)$, $f$ is $G$-equivariant ($G$ acts diagonally on $V \otimes O(A)$).

Since morphisms $f : X \rightarrow Y \otimes O$ are in 1-1 correspondence with morphism $f^\#: I \rightarrow X^\vee \otimes Y \otimes O$. As noticed in 4.4, such a morphism $f^\#$ corresponds to an element of $\omega((X^\vee \otimes Y)_S)$. Thus objects of $Q$ can be characterized as triples $(X, Y, f \in \omega((X^\vee \otimes Y)_S))$.

Another consequence of the above theorem which may be useful in checking whether a sequence of groupoids $L^K_{K_0} \rightarrow G^K_k \rightarrow A^K_{K_0}$ is exact is the following

**Corollary 5.11.** Assume we are given a sequence of homomorphisms of groupoids $L^K_{K_0} \rightarrow G^K_k \rightarrow A^K_{K_0}$ with $fq$ trivial. Then the sequence is exact in the sense that $q$ is the kernel of $f$ iff the representation categories of these groups satisfy the condition (i) and (ii) of Definition 3.2

**Remark 5.12.** The following questions are open to us:

1. Is $T$ always flat over $S$, $K$?
2. Assume that $T$ is flat over $S$ (and $K$), does a quotient exist, in particular, does the base change to $K$ exist?
3. Assume that $T$ is flat over $S$ (and $K$), does the functor $p$ induce an equivalence of tensor categories between $\text{Ind-}Q$ and $\text{Mod}_{O,K}$?

Finally we mention a related question raised by Deligne, namely whether the the fundamental group of $T$ is flat over $S$, see [2] for definition of the fundamental group of a rigid tensor category.
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