

ON A RATIONALITY QUESTION IN THE GROTHENDIECK RING OF VARIETIES

HÉLÈNE ESNAULT AND ECKART VIEHWEG

ABSTRACT. We discuss elementary rationality questions in the Grothendieck ring of varieties for the quotient of a finite dimensional vector space over a characteristic 0 field by a finite group.

1. INTRODUCTION

Let k be a field. One defines the *Grothendieck group of varieties* $K_0(\text{Var}_k)$ over k [7, Definition 2.1] to be the free abelian group generated by k -schemes modulo the subgroup spanned the scissor relations

$$[X] = [X \setminus Z] + [Z]$$

where $Z \subset X$ is a closed subscheme. The product

$$[X \times_k Y] = [X] \cdot [Y]$$

for two k -schemes makes it a commutative ring, with unit $1 = [\text{Spec } k]$. As the underlying topological space of the complement $X \setminus X_{\text{red}}$ is empty, $[X] = [X_{\text{red}}]$. This justifies the terminology “varieties” rather than “schemes”.

In characteristic 0, first examples of 0-divisors in this ring were shown to exist by Poonen [8]. He constructed two abelian varieties A, B over \mathbb{Q} such that

$$([A] - [B]) \cdot ([A] + [B]) = 0$$

but with $\iota^*[A] \neq \iota^*[B]$ for all field extensions $\mathbb{Q} \xrightarrow{\iota} k$. The main tool to distinguish those two classes relies ultimately on a deep insight in the structure of birational morphisms, gathered in the *Weak Factorization Theorem* [1]. It implies both the presentation of $K_0(\text{Var}_k)$ as the free group generated by smooth projective varieties modulo the blow up relation [2] and the isomorphism $K_0(\text{Var}_k)/\langle \mathbb{L} \rangle \xrightarrow{\cong} \mathbb{Z}[SB]$ [4]. Here \mathbb{L} is the class of the affine line \mathbb{A}^1 over k , $\langle \mathbb{L} \rangle$ is the ideal spanned by it, $\mathbb{Z}[SB]$ is the free abelian group on stably birational classes of projective smooth k -varieties, endowed with the ring structure stemming from the product of varieties over k . So there are no relations in $\mathbb{Z}[SB]$ and this allows to recognize certain classes. Of course this does not help in understanding \mathbb{L} , and the question whether or not \mathbb{L} is a 0-divisor remains open.

Date: August 19, 2009.

Partially supported by the DFG Leibniz Preis, the SFB/TR45, the ERC Advanced Grant 226257.

Later Kollár [3] used $\mathbb{Z}[SB]$ to distinguish in characteristic 0 the $K_0(\text{Var}_k)$ -classes of non-trivial Severi-Brauer varieties from trivial ones. Rökaeus [9] and Nicaise [7], using in addition specialization of $K_0(\text{Var}_k)$ from k to a finite fields, studied 0-divisors which are classes of 0-dimensional varieties, in particular those of the form $\text{Spec } K$ for a non-trivial field extension of a number field k . This indicates that one can not expect “descent”. For two k -varieties X and Y the equality

$$[X \times_k \text{Spec } K] = [Y \times_k \text{Spec } K] \in K_0(\text{Var}_K)$$

implies, by the projection formula, that

$$[X] \cdot [\text{Spec } K] = [Y] \cdot [\text{Spec } K] \in K_0(\text{Var}_k),$$

but this does not imply the equality $[X] = [Y] \in K_0(\text{Var}_k)$.

For applications of the Grothendieck ring, it is of importance to understand the class of quotients $[X/G]$ where X is a variety and G is a finite group acting on it. In [5, Lemma 5.1], Looijenga shows that if k is an algebraically closed field of characteristic 0, and if G is a finite abelian group acting linearly on a finite dimensional k -vector space V , then

$$(1.1) \quad [V/G] = \mathbb{L}^{\dim_k V} \in K_0(\text{Var}_k).$$

In fact the formula (1.1), as well as its proof, remain valid if k is any field of characteristic 0 containing the $|G|$ -th roots of 1. However the condition that G be abelian seems to be essential, although we do not know an explicit counterexample.

In this note, we discuss possible simple generalizations of Looijenga’s formula in various ways. Our first result is the following.

Lemma 1.1. *Let G be a finite abelian group with quotient $G \rightarrow \Gamma$. Let k be a field of characteristic 0 and let $K \supset k$ be an abelian Galois extension with Galois group Γ . Assume, that the Galois action of Γ on K lifts to a k -linear action of G on a finite dimensional K -vector space V . If, for $N = \exp(G)$, all N -th roots of 1 lie in k , then (1.1) holds, i.e.*

$$[V/G] = \mathbb{L}^{\dim_K V} \in K_0(\text{Var}_k).$$

The condition that k contains the N -th roots of 1 is really necessary. In particular, if one allows the group G to act non-trivially on the ground field, the equation (1.1) is not compatible with descent to smaller ground fields.

Example 1.2. *Assume $k = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt{-1})$, $V = \mathbb{Q}^2 \otimes_{\mathbb{Q}} K$,*

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where the chosen basis of K as a 2-dimensional vector space over \mathbb{Q} is $(1, \sqrt{-1})$. Then

$$[\mathbb{L}^2] \neq [V/G] \in K_0(\text{Var}_{\mathbb{Q}}).$$

If $G \subset GL_k(V)$ is a finite group acting linearly on a finite dimensional vector space V over a characteristic 0 field k , then G acts semi-simply. So as a G -representation, $V = \bigoplus_i V_i \otimes T_i$, where V_i is an irreducible representation with $\text{Hom}_G(V_i, V_j) = \delta_{ij} \cdot k$, and T_i is the trivial representation of dimension m_i equal to the multiplicity of V_i in V . If G is commutative of exponent N and if the N -th roots of 1 lie in k , then $d_i = \dim_k V_i = 1$. Since V_i/G is normal and one dimensional, it is smooth. So the starting point of Looijenga's proof of (1.1) is the simple observation that there is a k -isomorphism $V_i/G \cong V_i$ of k -varieties. The proof of (1.1) then proceeds by stratifying V .

For $d_i \geq 2$, the quotient V_i/G might be singular, thus it can not be isomorphic to V_i , not even over a field extension. Nevertheless, one can show that the formula (1.1) remains true for irreducible two dimensional representations, or after stratifying, whenever all the d_i are 1 or 2 and G is a prime power order cyclic group.

Proposition 1.3. *Let k be a field of characteristic 0 and let V be a finite dimensional k -vector space. Let $G \rightarrow GL_k(V)$ be a linear representation of a finite abelian group.*

- 1) *If $\dim_k V \leq 2$, then (1.1) holds true.*
- 2) *If G is cyclic of prime power order, and if each irreducible subrepresentation V_i has $\dim(V_i) \leq 2$, then (1.1) holds true.*

The main reason for the restriction to $\dim(V_i) \leq 2$ is that in this case $\mathbb{P}(V_i) \cong \mathbb{P}_k^1$ and hence $\mathbb{P}(V_i)/G \cong \mathbb{P}_k^1$ as well. If V is an irreducible representation of dimension $d \geq 3$ a similar statement fails, and we were unable to prove the equation (1.1).

Acknowledgment: We thank Johannes Nicaise for several discussions which motivated this little elementary note. Lemma 1.1 and Example 1.2 were shown in a letter to him dated September 27, 2008. They have been further worked out by Tran Nguyen Khanh Linh and Le Hoang Phuoc in their master thesis in Essen, 2009.

2. PROOF OF LEMMA 1.1

By assumption $G \subset GL_k(V)$ lifts the action of the quotient Γ on K , hence writing

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\varrho} \Gamma \longrightarrow 1,$$

one has $\sigma(\lambda \cdot v) = \gamma(\lambda) \cdot \sigma(v)$, for $\gamma = \varrho(\sigma)$, for all $\lambda \in K$ and for all $v \in V$. In particular H is a subgroup of $GL_K(V)$. This defines the fiber square

$$(2.1) \quad \begin{array}{ccc} V/H & \longrightarrow & \text{Spec } K \\ \downarrow & \square & \downarrow \\ V/G & \longrightarrow & \text{Spec } k. \end{array}$$

By the rationality assumption, $\mu_N(k) \cong_k \mathbb{Z}/N$, for $N = \exp(G)$, and hence the characters of G are k -rational. So writing \hat{H} for the character group of H and $V_\chi(H)$ for the eigenspace with respect to the character χ of H , one has a fortiori the K -eigenspace decomposition

$$V = \bigoplus_{\chi \in \hat{H}} V_\chi(H).$$

Since G is commutative the subspace $V_\chi(H)$ of V is G -invariant.

Now on the geometric side, one proceeds as in Looijenga's Bourbaki lecture [5, Lemma 5.1]. Write

$$V = \prod_{\chi \in \hat{H}} V_\chi(H)$$

as a K -scheme. One sets $V_\chi^\times = V_\chi(H) \setminus \{0\}$ (where here $\{0\} = \text{Spec } K$), and defines the stratification

$$(2.2) \quad V = \bigsqcup_{I \subset \hat{H}} V_I, \quad \text{with} \quad V_I = \prod_{\chi \in I} V_\chi^\times.$$

The \mathbb{G}_m -fibration $V_\chi^\times \rightarrow \mathbb{P}(V_\chi(H))$ is defined over K and G -equivariant. So for $I \subset \hat{H}$ given, the K -morphism

$$V_I \rightarrow \prod_{\chi \in I} \mathbb{P}(V_\chi(H))$$

is a G -equivariant fibration, locally trivial for the Zariski topology. The fibres are isomorphic to $\mathbb{G}_m^{\#I}$.

The group H acts trivially on $\mathbb{P}(V_\chi(H))$, hence it acts trivially on $\prod_{\chi \in I} \mathbb{P}(V_\chi(H))$. On the fibres $\mathbb{G}_m^{\#I}$ the action of H respects the product structure. One obtains an induced K -morphism

$$V_I/H \rightarrow \prod_{\chi \in I} \mathbb{P}(V_\chi(H))$$

which is still a Zariski locally trivial $\mathbb{G}_m^{\#I}$ -fibration. Since $V_\chi(H)$ is invariant under the action of G the fiber square (2.1) induces fiber squares

$$\begin{array}{ccccc} V_I/H & \longrightarrow & \prod_{\chi \in I} \mathbb{P}(V_\chi(H)) & \longrightarrow & \text{Spec } K \\ \downarrow & & \square & & \downarrow \\ V_I/G & \longrightarrow & \prod_{\chi \in I} \mathbb{P}(V_\chi(H))/G & \longrightarrow & \text{Spec } k. \end{array}$$

In particular, $\mathbb{P}(V_\chi(H))/G$ is a k -form of $\mathbb{P}_k^{r_\chi}$ for $r_\chi = \dim_K V_\chi(H) - 1$.

Claim 2.1. The k -form $\mathbb{P}(V_\chi(H))/G$ of $\mathbb{P}_k^{r_\chi}$ is split and $[V_I/H] = [V_I/G]$ in $K_0(\text{Var}_k)$.

Proof. The second assertion follows from the first one. So we have to show that $\mathbb{P}(V_\chi(H))/G$ has a k -rational point.

By assumption k contains the N -th roots of 1 for $N = \exp(G)$. Hence $V_\chi(H)$, regarded as a k -vector space, has a G -eigenvector v . Then the line $\langle v \rangle_K$ defines a point $c \in \mathbb{P}(V_\chi(H))(K)$. Since the action of G on $K(c) = K$ factors through the Galois action of Γ on $K(c)$, the image of c lies in $(\mathbb{P}(V_\chi(H))/G)(k)$. \square

By (2.2) this finishes the proof of Lemma 1.1. \square

3. VERIFICATION OF THE PROPERTIES IN EXAMPLE 1.2

In the standard basis e_1, e_2 of \mathbb{Q}^2 and the basis $(1, \sqrt{-1})$ of K/k , we write

$$\sigma : (x_1 + iy_1)e_1 + (x_2 + iy_2)e_2 \mapsto (-x_1 + iy_1)e_2 + (x_2 - iy_2)e_1,$$

As σ is k -linear, it leaves the origin of V invariant, thus acts on $V^\times = V \setminus \{0\}$. One has $\sigma^2 = -\text{Id}$ and this defines the extension

$$0 \longrightarrow H := \langle \sigma^2 \rangle \longrightarrow G \longrightarrow \Gamma := \langle \gamma \rangle \longrightarrow 0$$

$$\text{with } \Gamma = \langle \gamma \rangle \cong \mathbb{Z}/2 = \text{Aut}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}), \quad \text{and } \gamma(\sqrt{-1}) = -\sqrt{-1}.$$

Thus one has the fiber square

$$\begin{array}{ccc} V/H & \longrightarrow & \text{Spec } K \\ \downarrow & \square & \downarrow \\ V/G & \longrightarrow & \text{Spec } \mathbb{Q} \end{array}$$

The \mathbb{G}_m -bundle $V^\times \rightarrow \mathbb{P}_K^1$ is compatible with the G -action. The subgroup H acts trivially on \mathbb{P}_K^1 while σ acts via

$$\bar{\sigma} : (x_1 + iy_1 : x_2 + iy_2) \mapsto (x_2 - iy_2 : -x_1 + iy_1).$$

This yields the fiber square

$$\begin{array}{ccccc} V^\times/H & \longrightarrow & \mathbb{P}_K^1 & \longrightarrow & \text{Spec } K \\ \downarrow & & \square & & \downarrow \\ V^\times/G & \longrightarrow & \mathbb{P}_K^1/G & \longrightarrow & \text{Spec } \mathbb{Q} \end{array}$$

Claim 3.1. \mathbb{P}_K^1/G is a genus 0 curve over \mathbb{Q} without a rational point.

Proof. Indeed, a rational point is a fixpoint of \mathbb{P}_K^1 under $\bar{\sigma}$. But the equation for a fixpoint is precisely

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = 0, \quad \text{with } (x_1, x_2, y_1, y_2) \neq (0, 0, 0, 0).$$

So over \mathbb{Q} there are no solutions. \square

Corollary 3.2. $\mathbb{L}^2 = [\mathbb{A}_\mathbb{Q}^2] \neq [V/G] \in K_0(\text{Var}_\mathbb{Q})$.

Proof. The origin $x_1 = x_2 = y_1 = y_2 = 0$ in V is a fixpoint under G . Thus

$$[V/G] = [V^\times/G] + [\text{Spec } \mathbb{Q}].$$

On the other hand, $V^\times \rightarrow \mathbb{P}_K^1$ is the total space of the \mathbb{G}_m -bundle to the invertible sheaf $\mathcal{O}_{\mathbb{P}_K^1}(-1)$, while $V^\times/H \rightarrow \mathbb{P}_K^1$ is the total space of the \mathbb{G}_m -bundle to the invertible sheaf $\mathcal{O}_{\mathbb{P}_K^1}(-2) = \pi^*\mathcal{L}$, where $\mathcal{L} \in \text{Pic}(\mathbb{P}_K^1/G)$. So $V^\times/G \rightarrow \mathbb{P}_K^1/G$ is the \mathbb{G}_m -bundle to the invertible sheaf \mathcal{L} . One concludes

$$[V/G] - [\text{Spec } \mathbb{Q}] = [V^\times/G] = [\mathbb{G}_m] \cdot [\mathbb{P}_K^1/G] \in K_0(\text{Var}_\mathbb{Q}).$$

On the other hand, one also has

$$\mathbb{L}^2 - [\text{Spec } \mathbb{Q}] = [\mathbb{A}_\mathbb{Q}^2 \setminus \{0\}] = [\mathbb{G}_m] \cdot [\mathbb{P}_\mathbb{Q}^1] \in K_0(\text{Var}_\mathbb{Q}).$$

If $[V/G]$ was equal to \mathbb{L}^2 in $K_0(\text{Var}_\mathbb{Q})$, then one would have the relation $[V^\times/G] = [\mathbb{A}_\mathbb{Q}^2 \setminus \{0\}]$ in $K_0(\text{Var}_\mathbb{Q})$, thus the relation

$$\Phi([V^\times/G]) = \Phi([\mathbb{P}_K^1/G]) = \Phi([\mathbb{A}_\mathbb{Q}^2 \setminus \{0\}]) = \Phi([\mathbb{P}_\mathbb{Q}^1]) \quad \text{in } \mathbb{Z}[SB],$$

where $\Phi : K_0(\text{Var}_\mathbb{Q}) \rightarrow \mathbb{Z}[SB]$ maps the class $[X]$ of a smooth projective \mathbb{Q} -variety X to its stably birational equivalence class.

This however contradicts Claim 3.1, as the existence of a rational point is compatible with the stably birational equivalence on smooth varieties over an infinite field.

For sake of completeness let us recall the proof of this well known fact. If $\tau : V \rightarrow W$ is a birational map between two smooth projective varieties, and τ is well defined near $v \in V(k)$, then $\tau(v)$ is well defined and lies in $W(k)$. Else one blows up v . This yields an exceptional divisor $\mathbb{P}^{\dim_k V - 1}$. Since τ is well defined outside of codimension ≥ 2 , and since k is infinite, there are rational points on it on which τ is defined and one repeats the argument. \square

4. PROOF OF PROPOSITION 1.3

We first show 1). If V has k -dimension ≤ 2 , we write the G -equivariant stratification $V = \{0\} \sqcup V^\times$. Furthermore, the projection $V^\times \rightarrow \mathbb{P}(V)$ is G -equivariant as well. Looijenga's argument shows here

$$[V^\times/G] = [\mathbb{G}_m] \cdot [\mathbb{P}(V)/G] \in K_0(\text{Var}_k).$$

On the other hand, either

$$\mathbb{P}(V) = \text{Spec } k = \mathbb{P}(V)/G \quad \text{or} \quad \mathbb{P}(V)/G \cong_k \mathbb{P}_k^1 \cong_k \mathbb{P}(V).$$

Adding up, one finds $[V/G] = \mathbb{L}^2 \in K_0(\text{Var}_k)$.

We now show 2). Instead of the decomposition $V = \bigoplus_{i=1}^r V_i \otimes T_i$ of V as a direct sums of irreducible G representations, considered in the introduction, we will drop the condition that $\text{Hom}_G(V_i, V_j) = \delta_{ij} \cdot k$ and choose a decomposition $V = \bigoplus_{i=1}^m V_i$ as a direct sum of irreducible representations. As usual we consider V as a variety and write

$$(4.1) \quad V = \prod_{i=1}^m V_i.$$

The monodromy group, that is the image of G in $GL_k(V)$ is still a p -order cyclic group. So we may assume

$$(4.2) \quad G \subset GL_k(V)$$

in the discussion.

Claim 4.1. There is a direct factor V_i of (4.1) such that $G \subset GL_k(V_i)$.

Proof. Since a p -power order cyclic group G contains a unique p -order cyclic subgroup $C(G)$, if $\{1\} \neq K_i := \text{Ker}(G \rightarrow GL_k(V_i))$, then $C(G) = C(K_i) \subset K_i$. We conclude by (4.2). \square

We now change the notation: we set $U = V_i$, $W = \bigoplus_{j \neq i} V_j$ with V_i constructed in Claim 4.1. So $V = U \oplus W$ equivariantly. We assume that the dimension of U is 2. If this is 1, the argument simplifies enormously and we don't detail. We define the G -equivariant stratifications

$$(4.3) \quad \begin{aligned} U &= \{0\} \sqcup D^\times \sqcup U^{(2)} \\ V &= (\{0\} \times_k W) \sqcup (D^\times \times_k W) \sqcup (U^{(2)} \times_k W). \end{aligned}$$

The strata are defined as follows. Write $\langle \sigma \rangle = G$. Let $F(T) \in k[T]$ be the minimal polynomial of σ . Since U is irreducible, $F(T)$ is also the characteristic polynomial. This defines the quadratic extension

$$(4.4) \quad K = k[T]/(F(T)) \hookrightarrow \text{End}_k(U)$$

identifying $\langle \sigma \rangle$ with the multiplicative subgroup of K^\times spanned by T . The linear map $\sigma \otimes K \in GL(U \otimes K)$ has two conjugate eigenlines. Then $D = \{0\} \sqcup D^\times \subset U$

the k -irreducible curve defined by the union of the two lines. Further $U^{(2)} = U \setminus D$. By definition, G acts fixpoint free on $U^{(2)}$.

Claim 4.2. $[(U^{(2)} \times_k W)/G] = [(U^{(2)}/G) \times_k W] = [U^{(2)}/G] \cdot \mathbb{L}^2 \in K_0(\text{Var}_k)$.

Proof. One has a G -equivariant projection $q : (U^{(2)} \times_k W)/G \rightarrow U^{(2)}/G$. Since $G \subset GL_k(U)$, for all points $x \in U^{(2)}$ with residue field $\kappa(x) \supset k$, one has $q^{-1}(x) \cong_{\kappa(x)} W \otimes_k \kappa(x)$. By construction, one has a fiber square

$$(4.5) \quad \begin{array}{ccc} U^{(2)} \times_k W & \longrightarrow & (U^{(2)} \times_k W)/G \\ \downarrow & \square & \downarrow q \\ U^{(2)} & \longrightarrow & U^{(2)}/G. \end{array}$$

Since $U^{(2)} \rightarrow U^{(2)}/G$ is étale, q defines a local system in $H_{\text{ét}}^1(U^{(2)}/G, G_W)$ where G_W is the image of G in $GL_k(W)$. Then $(U^{(2)} \times_k W)/G$ is the total space of the torsor in $H_{\text{ét}}^1(U^{(2)}/G, GL_k(W))$ induced by $G_W \hookrightarrow GL_k(W)$. By flat descent [6, Lemma 4.10],

$$H_{\text{ét}}^1(U^{(2)}/G, GL_k(W)) = H_{\text{Zar}}^1(U^{(2)}/G, GL_k(W)).$$

Thus $(U^{(2)} \times_k W)/G$, as the total space of a vector bundle, is Zariski locally trivial. We conclude

$$(4.6) \quad [(U^{(2)} \times_k W)/G] = [U^{(2)}/G] \cdot [W] = [U^{(2)}/G] \cdot \mathbb{L}^2 \in K_0(\text{Var}_k). \quad \square$$

So using (4.3) and Claim 4.2, we see

$$(4.7) \quad [V] - [V/G] = ([W] - [W/G]) + ([D^\times \times_k W] - [(D^\times \times_k W)/G]) + ([U^{(2)}] - [U^{(2)}/G]) \cdot \mathbb{L}^2.$$

Claim 4.3. The action of G in $\text{Spec } K$ is trivial.

Proof. As already observed, σ acts via multiplication by T , thus $\sigma(T) = T^2$. Writing $F(T) = T^2 - aT + b \in k[T]$, then $\sigma(1) = T$, $\sigma(T) = aT - b$, thus $\sigma\left(\frac{T}{1}\right) = a - \frac{b}{T} = \frac{T}{1}$. \square

We can now analyze the second difference in (4.7). One has the G -equivariant fibration $D^\times \times_k W \rightarrow \text{Spec } K \times_k W$, which is a \mathbb{G}_m -bundle. So

$$[D^\times \times_k W] = [\mathbb{G}_m] \cdot [\text{Spec } K] \cdot [W].$$

By Claim 4.3, one has

$$[(D^\times \times_k W)/G] = [\mathbb{G}_m] \cdot \text{Spec } K \cdot [W/G].$$

We conclude

$$(4.8) \quad [D^\times \times_k W] - [(D^\times \times_k W)/G] = [\mathbb{G}_m] \cdot [\text{Spec } K] \cdot ([\mathbb{L}^2] - [W/G]).$$

We now analyze the third difference in (4.7). One has the G -equivariant fibration $U^{(2)} \rightarrow \mathbb{P}(U) \setminus \text{Spec } K$, which is a \mathbb{G}_m -bundle. So

$$[U^{(2)}] = [\mathbb{G}_m] \cdot ([\mathbb{P}(U)] - [\text{Spec } K]).$$

Since $\mathbb{P}(U)/G$ is k -isomorphic to \mathbb{P}_k^1 and G acts trivially on $\text{Spec } K$, one has

$$(4.9) \quad [U^{(2)}/G] = [\mathbb{G}_m] \cdot ([\mathbb{P}(U)/G] - [\text{Spec } K]) = \\ [\mathbb{G}_m] \cdot ([\mathbb{P}(U)] - [\text{Spec } K]) = [U^{(2)}] \in K_0(\text{Var}_k).$$

Summing up, (4.7) reads

$$(4.10) \quad [V] - [V/G] = \mathbb{L} \cdot [\text{Spec } K] \cdot ([\mathbb{L}^2] - [W/G]).$$

Now W has one less irreducible factor than V . We argue by induction on the number of irreducible factors, applying 1) to start the induction. This finishes the proof. \square

REFERENCES

- [1] Abramovich, D.; Karu, K.; Matsuki, K.; Włodarczyk, J.: *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002), 531–572.
- [2] Bittner, F.: *The universal Euler characteristic for varieties of characteristic zero*, Compositio math., **140**(4) (2004), 1011–1032.
- [3] Kollár, J.: *Conics in the Grothendieck ring*, Adv. Math., **198** (1) (2005), 27–35.
- [4] Larsen, M., Lunts, V. A.: *Motivic measures and stable birational geometry*, Mosc. Math. J., **3**(1) (2003), 85–95.
- [5] Looijenga, E.: *Motivic measures*, Séminaire Bourbaki 1999/2000, Astérisque **276** (2002), 267–297.
- [6] Milne, J.: *Étale cohomology*, Princeton Mathematical Series **33** (1980), Princeton University Press.
- [7] Nicaise, J.: *A trace formula for varieties over a discretely valued field*, preprint.
- [8] Poonen, B.: *The Grothendieck ring of varieties is not a domain*, Math. Res. Let. **9** (4) (2002), 493–498.
- [9] Rökaeus, K.: *The computation of the classes of some tori in the Grothendieck ring of varieties*, 2007, arXiv:0708.4396.

UNIVERSITÄT DUISBURG-ESSEN, MATHEMATIK, 45117 ESSEN, GERMANY
E-mail address: esnault@uni-due.de

UNIVERSITÄT DUISBURG-ESSEN, MATHEMATIK, 45117 ESSEN, GERMANY
E-mail address: viehweg@uni-due.de