SURFACE SINGULARITIES DOMINATED BY SMOOTH VARIETIES

HÉLÈNE ESNAULT AND ECKART VIEHWEG

Abstract. We give a version in characteristic $p > 0$ of Mumford’s theorem characterizing a smooth complex germ of surface $(X, x)$ by the triviality of the topological fundamental group of $U = X \setminus \{x\}$.

1. Introduction

Let $(X, x)$ be a 2-dimensional normal complex analytic germ. Let $U = X \setminus \{x\}$. Mumford ([12]) showed the celebrated theorem

**Theorem 1.1** (Mumford). $(X, x)$ if smooth if and only if the topological fundamental group of $U$ is trivial.

This is a remarkable theorem which connects a topological notion to a scheme-theoritic one. His theorem has been a bit refined by Flenner [7] who showed that in fact, the conclusion remains true if one replaces the topological by the étale fundamental group of $U$, that is by its profinite completion. Then one can replace the analytic germ by a complete or henselian germ over an algebraically closed field $k$ of characteristic 0.

If $k$ is an algebraically closed field $k$ of characteristic $p > 0$, Mumford himself observed that the theorem is no longer true. As an example, while in characteristic 0, the singularity $z^2 + xy$ is the quotient of $\hat{\mathbb{A}}^2$, the completion of $\mathbb{A}^2$ at the origin, by the group $\mathbb{Z}/2$ acting via diag(−1, −1), in characteristic 2, it is the quotient of $\hat{\mathbb{A}}^2$ by $\mu_2 = \text{Spec } k[t]/(t^2 - 1)$ acting via diag($t$, $t$). Thus $\pi^\text{et}(U) = \pi^\text{et}(\hat{\mathbb{A}}^2 \setminus \{0\}) = 0$, yet $z^2 + xy$ is not smooth.

Artin asked in [3] whether, if $\pi^\text{et}(U)$ is finite, there is always a finite morphism $\hat{\mathbb{A}}^2 \to X$. He shows this if $(X, x)$ is a rational double point *loc.cit.*.

The purpose of this note is to give an answer to a similar question where one replaces the étale fundamental group by the Nori one. Strictly speaking, Nori in [13, Chapter II] defined his fundamental group-scheme for irreducible reduced schemes endowed with a rational point. But as $U$ has no rational point, one has to modify a tiny bit Nori’s construction to make it work. This is done in subsection 2.2. While the étale fundamental group of $X$ is trivial, Nori’s one

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isn’t. So the right notion of Nori’s fundamental group is a relative one denoted by \( \pi_{\text{loc}}(U, X, x) \) (see Lemma 2.5). Roughly speaking, it measures the torsors on \( U \) under a finite flat \( k \)-group-scheme \( G \) which do not come by restriction from a torsor on \( X \). We show (Theorem 4.2) that if \( \pi_{\text{loc}}^N(U, X, x) \) is finite, then \( (X, x) \) is a rational singularity, and if \( \pi_{\text{loc}}^N(U, X, x) = 0 \), then there is a finite morphism \( f : \hat{k}^2 \to X \).

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2. Local Nori Fundamental Group-scheme

2.1. Nori’s construction. Let \( U \) be a scheme defined over a field \( k \), endowed with a rational point \( u \in U(k) \). In [13, Chapter II] Nori constructed the fundamental group-scheme \( \pi^N(U, u) \). Let \( \mathcal{C}(U, u) \) be the following category. The objects are triples \( (h : V \to U, G, v) \) where \( G \) is a finite \( k \)-group-scheme, \( h \) is a \( G \)-principal bundle and \( v \in V(k) \) with \( h(v) = u \). Recall [13, Chapter I,2.2] that a \( G \)-principal bundle \( h : V \to U \) is a flat morphism, together with a group action \( G \times_k V \to V \) such that \( V \times_k G \to V \times_U V \) is an isomorphism.

The objects of the ind-category \( \mathcal{C}^{\text{ind}}(U, u) \) associated to \( \mathcal{C}(U, u) \) are triples \( (h : V \to U, G, v) \) where \( G = \lim\limits_{\alpha} G_\alpha \) is a prosystem of finite \( k \)-group-schemes \( G_\alpha \), \( h = \lim\limits_{\alpha} h_\alpha, h_\alpha : V_\alpha \to U \), is a pro-\( G \)-principal bundle and \( v = \lim\limits_{\alpha} v_\alpha \in Y(k) \) is a pro-point with \( h(v) = u \). The morphisms are the ind-morphisms \( V_1 \to V_2 \) which are compatible with the principal bundle structure and such that \( f(v_1) = v_2 \).

Then \( (U, u) \) has a fundamental group-scheme \( \pi^N(U, u) \), which is then a \( k \)-profinite group-scheme, if by definition [13, Chapter II, Definition 1] there is a \( (h : W \to U, \pi^N(U, u), w) \in \mathcal{C}^{\text{ind}}(U, u) \) with the property that for any \( (h : V \to U, G, v) \in \mathcal{C}^{\text{ind}}(U, u) \), there is a unique map \( (h : W \to U, \pi^N(U, u), w) \to (h : V \to U, G, v) \) in \( \mathcal{C}^{\text{ind}}(U, u) \).

Nori shows [13, Chapter II, Lemma 1] that if \( G_1, G_2, G_0 \) are three finite \( k \)-group-schemes, \( h_i : V_i \to U \) are \( G_i \)-principal bundles, and \( f_i : V_i \to V_0, i = 1, 2 \) are principal bundle \( U \)-morphisms, then \( V_1 \times_{V_0} V_2 \to Z \) is a principal bundle under \( G_1 \times_{G_0} G_2 \), where \( Z \subseteq U \) is a closed subscheme (no reference to the base point here). Then he shows that \( (U, u) \) has a fundamental group-scheme if and only if \( Z = U \) for all \( (h_i : V_i \to U, G_i, y_i), f_i \in \mathcal{C}(U, u) \) and he concludes [13, Chapter II, Proposition 2] that if \( U \) is reduced and irreducible, then \( (U, u) \) has a fundamental group-scheme.

2.2. Local Nori fundamental group-scheme. Let \( k \) be a field, let \( A \) be a complete normal local \( k \)-algebra with maximal ideal \( \mathfrak{m} \) and residue field \( k \). We
define $X = \text{Spec} A$ and $U = X \setminus \{x\}$, where $x \in X(k)$ is the rational point associated to $m$. So in particular, $U(k) = \emptyset$, and we have to slightly modify Nori’s construction to define the group-scheme of $U$.

Let $G$ be a finite $k$-group-scheme, and let $h : V \to U$ be a $G$-principal bundle. Recall from [15, Corollaire 6.3.2, Proposition 6.3.4] that the integral closure $\hat{h} : Y \to X$ of $h$ is the unique extension $\hat{h} : Y \to X$ of $h$ such that $Y = \text{Spec} B$, $B$ is the integral closure of $A$ in $j_*h_*O_V$, where $j : U \to X$ is the open embedding. Then $\hat{h}$ is finite. In particular, if $h_i : V_i \to U$ are principal bundles under the finite $k$-group-schemes $G_i$, and $f : V_1 \to V_2$ is a $U$-morphism which respects the principal bundle structures, then it extends uniquely to a $X$-morphism $\tilde{f} : Y_1 \to Y_2$, which is then finite. We can now mimic Nori’s construction.

**Definition 2.1.** The objects of the category $C_{\text{loc}}(U, x)$ are triples $(h : V \to U, G, y)$ where $G$ is a finite $k$-group-scheme, $y \in Y(k)$ with $\hat{h}(y) = x$, where $\hat{h} : Y \to X$ is the integral closure of $h$. The morphisms $\text{Hom}((h_1 : V_1 \to U, G_1, y_1) \to (h_2 : V_2 \to U, G_2, y_2))$ consist of $U$-morphisms $f : V_1 \to V_2$ which respect the principal bundle structure and such that $\tilde{f}(y_1) = y_2$.

The objects of the ind-category $C_{\text{ind}}(U, x)$ associated to $C_{\text{loc}}(U, x)$ are triples $(h : V \to U, G, y)$ where $G = \lim_{\alpha} G_{\alpha}$ is a pro-system of finite $k$-group-schemes, $h = \lim_{\alpha} h_{\alpha}, h_{\alpha} : V_\alpha \to U$, is a pro-$G$-principal bundle, and $y = \lim_{\alpha} y_\alpha \in \lim_{\alpha} Y_\alpha(k)$ is a pro-point in the integral closure of $V_\alpha$ mapping to $x$.

One says that $(U, x)$ has a local fundamental group-scheme $\pi^N_{\text{loc}}(U, x)$, which is then a $\text{k}$-profinite group-scheme, if there is a $(h : W \to U, \pi^N_{\text{loc}}(U, x), z) \in C_{\text{ind}}(U, x)$ with the property that for any $(h : V \to U, G, v) \in C_{\text{loc}}(U, x)$, there is a unique map $(h : W \to U, \pi^N_{\text{loc}}(U, x), z) \to (h : V \to U, G, v)$ in $C_{\text{ind}}(U, x)$.

**Proposition 2.2.** If $X$ is reduced and irreducible, then $(U, x)$ has a local fundamental group-scheme $\pi^N_{\text{loc}}(U, x)$.

**Proof.** As explained above, the condition on $X$ implies that if $f_\alpha : (h_{\alpha} : V_{\alpha} \to U, G_{\alpha}, y_{\alpha})$ is a morphism in $C_{\text{loc}}(U, x)$, then $(V_1 \times_{V_0} V_2 \to U, G_1 \times_{G_0} G_2, y_1 \times_{y_0} y_2) \in C_{\text{loc}}(U, x)$, so as in [13, Chapter II,p.87], the pro-system $\lim_{\alpha}(h_{\alpha} : V_{\alpha} \to U, G_{\alpha}, y_{\alpha})$ over all objects $(h_{\alpha} : V_{\alpha} \to U, G_{\alpha}, y_{\alpha})$ of $C_{\text{loc}}(U, x)$ is well defined. So $\pi^N_{\text{loc}}(U, x) = \lim_{\alpha} G_{\alpha}$. \hfill \Box

There is a restriction functor $\rho : C(X, x) \to C_{\text{loc}}(U, x)$ which sends $(h : Y \to X, G, y)$ to its restriction $(h_U : Y \times_X U \to U, G, y)$, as the integral closure of $X$ in $Y \times_X U$ is $Y$. This defines the $k$-group-scheme homomorphism $\rho_* : \pi^N_{\text{loc}}(U, x) \to \pi^N(X, x)$.

**Proposition 2.3.** The homomorphism $\rho$ is faithfully flat.

**Proof.** Faithful flatness of $\rho$ means that if $(h : Y \to X, G, y) \in C(X, x)$ is such that $(Y_U \to G, y) \to (U, \{1\}, x)$ factors through $(\ell : V \to U, H, y) \in C_{\text{loc}}(U, x)$, where
We define \( \sim \) combination \( D \) and relies on Mumford’s basic idea \([12, \text{Section 2}]\) to use a desingularization of a prime number (including \( p \)). Assume \( Y \) and \( \{ y \} \). We now summarize the construction and the elementary properties under \( p > 0 \). We define \( \eta \) as \( \pi \rightarrow U \), it is described as \( Y_U \times_k K \subset Y_U \times_k G \). Thus \( Y \times Z Y \) contains the closure of \( Y_U \times_k K \) in \( Y \times_k G \), that is \( Y \times_k K \). Thus \( Y \times_k K \) consists of connected components of \( Y \times Z Y \) and moreover, if there is another connected component, it lies in \( \{ y \} \times Z Y = \text{Spec} k \). Thus \( Y \times Z Y \cong_k Y \times_k K \) and \( Y \rightarrow Z \) is a \( K \)-torsor. This finishes the proof.

We denote by \( \pi^{et}(U, x) \) the étale proquotient of \( \pi^{N}(U, x) \). From now on, we assume \( k = \overline{k} \). Then \( \pi^{et}(U, x) \) is identified with \( \pi^{et}(U, \eta) \) where \( \eta \rightarrow U \) is a geometric generic point and \( \pi^{et}(U, \eta) \) is Grothendieck’s étale fundamental group. The étale proquotient of \( \pi^{N}(X, x) \) is identified with Grothendieck’s fundamental group based at \( x \), and is trivial by Hensel’s lemma, as \( A \) is complete. If \( \ell \) is a prime number (including \( p \)), we denote by \( \pi^{et, \text{ab}, \ell}(U, x) \) the maximal pro-\( \ell \)-abelian quotient of \( \pi^{et}(U, x) \).

**Definition 2.4.** One defines \( \pi^{N}_{\text{loc}}(U, X, x) = \text{Ker} \left( \pi^{N}_{\text{loc}}(U, x) \overset{\rho}{\to} \pi^{N}(X, x) \right) \).

From the discussion, we see

**Lemma 2.5.** The compositum \( \pi^{N}_{\text{loc}}(U, X, x) \rightarrow \pi^{et}(U, x) \) is surjective. In particular, if \( \pi^{N}_{\text{loc}}(U, X, x) \) is a finite \( k \)-group-scheme, \( \pi^{et}(U, x) \) is a finite group.

3. Construction and elementary properties of the Picard scheme for surface singularities

Let \( k \) be a field, perfect if of characteristic \( p > 0 \), let \( A \) be a complete normal local \( k \)-algebra with maximal ideal \( m \), \( X = \text{Spec} A \) and \( U = X \setminus \{ x \} \), where \( x \in X(k) \) is the rational point associated to \( m \). In [16, Exposé XIII, Section 5] Grothendieck initiated the construction of a pro-system of locally algebraic \( k \)-group-schemes \( G_n \) and a canonical isomomorphism \( G(k) = \text{Pic}(U) \) with \( G(k) = \varprojlim_n G_n(k) \). This construction is performed in [11] (see overview in [9, p. 273]) and relies on Mumford’s basic idea [12, Section 2] to use a desingularization of \( X \), if it exists, so in characteristic 0 or if \( \dim_k X \leq 2 \) if \( k \) has characteristic \( p > 0 \). We now summarize the construction and the elementary properties under the assumptions

1) \( X \) is normal
2) \( \dim_k X = 2 \).

Let \( \sigma : \tilde{X} \rightarrow X \) be a desingularization such that \( \sigma^{-1}(x)_{\text{red}} = \bigcup_i D_i \) is a strict normal crossings divisor and all components \( D_i \) are \( k \)-rational. There is linear combination \( D = \sum_i m_i D_i \) with all \( m_i \geq 1 \) such that \( \mathcal{O}_{\tilde{X}}(-D) \) is relatively ample. We define \( \tilde{X}_n \) to be scheme \( \bigcup_i D_i \) with structure sheaf \( \mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}(-(n + 1)D) \), so
\[ \tilde{X}_0 = D, \text{ and we also define } D_{\text{red}} \text{ with structure sheaf } O_{\tilde{X}}/O_{\tilde{X}}(-\sum_i D_i). \] Then the functors \( \mathcal{P}ic(\tilde{X}_n/k) \) and \( \mathcal{P}ic(D_{\text{red}}/k) \), taken as a Zariski, an étale or a fppf functor, are representable by locally algebraic \( k \)-group-schemes \( \mathcal{P}ic(\tilde{X}_n/k) \) and \( \mathcal{P}ic(D_{\text{red}}/k) \), so \( \mathcal{P}ic(\tilde{X}_n) = \mathcal{P}ic(\tilde{X}_n/k)(k) \), \( \mathcal{P}ic(D_{\text{red}}) = \mathcal{P}ic(D_{\text{red}}/k)(k) \) (see [9, p. 273], [11, Theorem 1.2]). On the other hand, for all \( n \geq 0 \), and all \( k \)-algebras \( R \), one has \( \mathcal{P}ic(\tilde{X}_n \otimes_k R) = H^1(\tilde{X}_n \otimes_k R, O^*) \). As the relative dimension of \( \sigma \) is 1, this implies that the transition homomorphisms \( \mathcal{P}ic(\tilde{X}_{n+1}) \rightarrow \mathcal{P}ic(\tilde{X}_n) \rightarrow \mathcal{P}ic(\tilde{X}_0) \rightarrow \mathcal{P}ic(D_{\text{red}}) \) are all surjective, and that \( \text{Ker}(\mathcal{P}ic(\tilde{X}_{n+1}) \rightarrow \mathcal{P}ic(\tilde{X}_n)) = H^1(\tilde{X}_0, O_{\tilde{X}_0}(-n+1)D) \). Since \(-D\) is a relatively ample divisor on \( \tilde{X} \), there is a \( n_0 \geq 0 \) such that the transition homomorphisms \( \mathcal{P}ic(\tilde{X}_n) \rightarrow \mathcal{P}ic(\tilde{X}_{n_0}) \) are all constant for \( n \geq n_0 \). Since the 1-component \( \mathcal{P}ic^0(D_{\text{red}}) \) of \( \mathcal{P}ic(D_{\text{red}}) \) is a semi-abelian variety, so in particular smooth, and the fibers \( \mathcal{P}ic(\tilde{X}_n) \rightarrow \mathcal{P}ic(D_{\text{red}}) \) are affine [14, p. 9, Corollaire], \( \mathcal{P}ic(\tilde{X}_{n_0}) \) is smooth. One defines

\[
(3.1) \quad \mathcal{P}ic(\tilde{X}) = \mathcal{P}ic(\tilde{X}_{n_0}).
\]

It is thus a locally algebraic smooth \( k \)-group-scheme. It is an extension of \( \bigoplus_i \mathbb{Z}[D_i] \) by its 1-component. Its 1-component \( \mathcal{P}ic^0(\tilde{X}) \subset \mathcal{P}ic(\tilde{X}) \) is an extension of a semi-abelian variety by smooth, connected commutative unipotent algebraic group over \( k \).

Let \( \langle D \rangle \subset \mathcal{P}ic(\tilde{X}) \) be the subgroup-scheme spanned by those divisors with support in \( D \). (In fact, \( \langle D \rangle \) injects into \( \mathcal{P}ic(D_{\text{red}}) \) via the surjection \( \mathcal{P}ic(\tilde{X}) \rightarrow \mathcal{P}ic(D_{\text{red}})) \). It is a discrete subgroup-scheme. One sets

\[
(3.2) \quad \mathcal{P}ic(U) = \mathcal{P}ic(\tilde{X})/\langle D \rangle.
\]

The Zariski tangent space at 1 is

\[
(3.3) \quad H^1(\tilde{X}, O_{\tilde{X}}) = H^1(\tilde{X}_n, O_{\tilde{X}_n}) = \text{Ker}(\mathcal{P}ic(\tilde{X}_n[\epsilon]) \rightarrow \mathcal{P}ic(\tilde{X}_n))
\]

for \( n \geq n_0 \), where \( \tilde{X}_n[\epsilon] := \tilde{X}_n \times_k k[\epsilon]/(\epsilon^2) \). Since \( \mathcal{P}ic(\tilde{X}) \) is smooth,

\[
(3.4) \quad \dim_k H^1(\tilde{X}, O_{\tilde{X}}) = \dim \mathcal{P}ic^0(\tilde{X}) = \mathcal{P}ic^0(U).
\]

The last equality comes from the fact that \( \langle D \rangle \subset \mathcal{P}ic(\tilde{X}) \) is a discrete étale subgroup.

Recall that the surface singularity \((X, x)\) is said to be rational if \( H^1(\tilde{X}, O_{\tilde{X}}) = 0 \). The definition does not depend on the choice of the resolution \( \sigma: \tilde{X} \rightarrow X \) of singularities of \((X, x)\).

One has

**Lemma 3.1.** The following conditions are equivalent.

1. The surface singularity \((X, x)\) is rational.
2. \( \mathcal{P}ic^0(\tilde{X}) = 0 \).
3. \( \mathcal{P}ic(U) \) is finite.
Proof. The equivalence of 1) and 2) is given by (3.4). As \( \langle D \rangle \subset \text{Pic}(\tilde{X}) \) is discrete, the definition (3.2) shows that 3) implies 2). Vice-versa, assume 2) holds. Then \( \text{Pic}(\tilde{X}) \) is a discrete group of finite type. Let \( L \in \text{Pic}(\tilde{X}) \). Since the intersection matrix \( (D_i \cdot D_j) \) is negative definite (but not necessarily unimodular), there is a \( m \in \mathbb{N} \setminus \{0\} \) such that \( L^\oplus m \in \langle D \rangle \subset \text{Pic}(\tilde{X}) \). Thus any \( L \in \text{Pic}(\tilde{X}) \) has finite order in \( \text{Pic}(U) \). Since \( \text{Pic}(\tilde{X}) \) is of finite type, this shows 3).

\[ \square \]

4. The Theorems

Throughout this section, we assume \( k \) to be a field, perfect if of characteristic \( p > 0 \), \( A \) to be a complete normal local \( k \)-algebra with maximal ideal \( m \), of Krull dimension 2 over \( k \). We set \( X = \text{Spec} \, A, U = X \setminus \{x\} \), where \( x \in X(k) \) is the rational point associated to \( m \). We say \( (X, x) \) is a surface singularity over \( k \).

We denote by \( \sigma : \tilde{X} \to X \) a desingularization such that \( \sigma^{-1}(x)_{\text{red}} = \cup_i D_i \) is a strict normal crossings divisor. We define \( H^i(Z, \mathbb{Z}_\ell(1)) := \lim_{\longrightarrow} H^i(Z, \mu_{\ell^n}) \) for a \( k \)-scheme \( Z \).

**Theorem 4.1.** Let \( (X, x) \) be a surface singularity over an algebraically closed field \( k \). The following conditions are equivalent

1. \( H^1(\tilde{X}, \mathbb{Z}_\ell(1)) = 0 \).
2. \( H^1(U, \mathbb{Z}_\ell(1)) = 0 \).
3. There is a prime number \( \ell \), different from \( p \), such that \( \pi_{\text{et}, \text{ab}, \ell}(U, x) \) is finite.
4. For all prime numbers \( \ell \), \( \pi_{\text{et}, \text{ab}, \ell}(U, x) \) is finite and if \( \text{char}(k) = p > 0 \), then \( \pi_{\text{et}, \text{ab}, \ell}(U, x) = 0 \).
5. \( \text{Pic}^0(\tilde{X}) = \text{Pic}^0(U) \) is a smooth, connected commutative unipotent algebraic group-scheme over \( k \).
6. \( D \) is a tree of \( \mathbb{P}^1 \)s.
7. \( \text{Pic}^0(D_{\text{red}}) = 0 \).

**Proof.** We first make general remarks. For any surface singularity, one has the localization sequence

\[ H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1)) \to H^2_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^2(U, \mathbb{Z}_\ell(1)) \to H^3_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^3(\tilde{X}, \mathbb{Z}_\ell(1)). \]

By purity [8, Theorem 2.1.1], the restriction map \( H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1)) \) is injective, and \( H^2_{D_{\text{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) = \oplus_i \mathbb{Z}_\ell \cdot [D_i] \). By base change, \( H^i(\tilde{X}, \mathbb{Z}_\ell(1)) = H^i(D_{\text{red}}, \mathbb{Z}_\ell(1)) \). Thus this group is 0 for \( i \geq 3 \), equal to \( \oplus_i \mathbb{Z}_\ell \cdot [D_i] \) for \( i = 2 \), and equal to \( \text{Pic}(D_{\text{red}})[\ell] \) for \( i = 1 \). In fact, since \( H^2(D_{\text{red}}, \mathbb{Z}_\ell(1)) \) is torsion free, one has \( \text{Pic}(D_{\text{red}})[\ell] = \text{Pic}^0(D_{\text{red}})[\ell] \), where \( 0 \) means of degree 0 on each component \( D_i \).

Furthermore, by definition, the map \( \oplus_i \mathbb{Z}_\ell \cdot [D_i] \to \oplus_i \mathbb{Z}_\ell \cdot [D_i] \) is defined by \( [D_i] \mapsto \oplus_j \text{deg} \mathcal{O}_{D_j}(D_i) \). Since the intersection matrix is definite, the map is injective.
with finite torsion cokernel $\mathcal{T}$. (This cokernel is 0 if and only if the intersection matrix is unimodular). Again by purity, $H^3_{D_{\text{red}}}((\tilde{X}, \mathbb{Z}_\ell(1)) \subset \bigoplus_i H^1(D^0_i, \mathbb{Z}_\ell)$ where $D^0_i = D_i \setminus \cup_{j \neq i} D_i \cap D_j$. In particular, $H^3_{D_{\text{red}}}((\tilde{X}, \mathbb{Z}_\ell(1))$ is torsion free. So we extract from (4.1) for any surface singularity the relations

$$H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1)) = \text{Pic}(D_{\text{red}})[\ell] = \text{Pic}^0(D_{\text{red}})[\ell]$$

and an exact sequence

$$0 \to \mathcal{T} \to H^2(U, \mathbb{Z}_\ell(1)) \to H^3_{D_{\text{red}}}((\tilde{X}, \mathbb{Z}_\ell(1)) \to 0$$

with finite $\mathcal{T}$ and torsion free $H^3_{D_{\text{red}}}((\tilde{X}, \mathbb{Z}_\ell(1))$. As Pic$^0(D_{\text{red}})$ is a semiabelian variety, we see that (4.2) implies that 1), 2) and 7) are equivalent conditions.

From the exact sequence

$$1 \to \mathcal{O}_D^\times \to \bigoplus_i \mathcal{O}^\times_{D_i} \to \bigoplus_{i<j} k^\times \to 1$$

one has that 6) and 7) are equivalent. Furthermore, from the structure of Pic($\tilde{X}$) explained in section 3, one has that 5) is equivalent to 7).

We show that 2) is equivalent to 3). The condition 2) implies that $H^1(U, \mu_{\ell^n}) \subset \mathcal{T}$ for all $n \geq 0$, thus there are finitely many $\mu_{\ell^n}$ torsors on $U$. This shows 2) implies 3). On the other hand, if Pic$^0(D_{\text{red}})$ is not trivial, then Pic$^0(D_{\text{red}})[\ell]$ contains $\mathbb{Z}_\ell$. Thus $H^1(U, \mathbb{Z}_\ell(1))$ contains $\mathbb{Z}_\ell$ as well by (4.2). Thus 3) implies 2).

Since obviously 4) implies 3), it remains to see that 3) implies 4). We assume 3). For any commutative finite $k$-group-scheme $G$, with Cartier dual $G' = \text{Hom}(G, \mathbb{G}_m)$, one has the exact sequence

$$0 \to H^1(X, G') \to H^1(U, G') \to \text{Hom}(G, \text{Pic}(U)) \to 0.$$  

(See [5, III, Théorème 4.1] and [5, III, Corollaire 4.9] for the 0 on the right, which we will use only on the proof of Theorem 4.2, as $k = \bar{k}$). We apply it for $G = \mathbb{Z}/\ell^n$ for some $n \in \mathbb{N}\setminus\{0, 1\}$. Since Pic(U) is an extension of a discrete (étale) group by Pic$^0(U)$ which is a product of $\mathbb{G}_a$s by 5), one has Hom($\mu_{\ell^n}$, Pic(U)) = 0. On the other hand, $A \xrightarrow{x \mapsto (x^{p^n} - x)} A$ is surjective, as $A$ is complete. Thus $H^1(U, \mathbb{Z}/\ell^n) = H^1(X, \mathbb{Z}/\ell^n) = 0$. This shows that 3) implies 4) and finishes the proof of the theorem.

\[\square\]

**Theorem 4.2.** Let $(X, x)$ be a surface singularity over an algebraically closed field $k$.

1) If $\pi^N_{\text{loc}}(U, X, x)$ is a finite group-scheme, $(X, x)$ is a rational singularity, in particular the dualizing sheaf $\omega_U$ has finite order.

2) If in addition, the order of $\omega_U$ is prime to $p$, then there is $(h : V \to U, \pi^N(U, x, y) \in \mathcal{C}_{\text{loc}}(U, x)$ such that the surface singularity $(Y, y)$ of the integral closure $\tilde{h} : Y \to X$ is a rational double point.

3) If $\pi^N_{\text{loc}}(U, X, x) = 0$, then $(X, x)$ is a rational double point.
Proof. We show 1). If $\pi_{\text{loc}}^N(U, X, x)$ is a finite group-scheme, then, by Lemma 2.5, the condition 3) of Theorem 4.1 is fulfilled, thus $\text{Pic}^0(\tilde{X}) = \text{Pic}^0(U)$ is a product of $G\alpha$s. We apply (4.5) to $G = \mathbb{Z}/p^n$. If $\text{Pic}^0(U)$ is not trivial, then $\text{Hom}(\mathbb{Z}/p^n, \text{Pic}(U)) \neq 0$ for all $n \geq 0$. Thus $U$ admits nontrivial $\mu_{p^n}$-torsors for all $n \geq 1$, which do not come from $X$. This contradicts the finiteness of $\pi_{\text{loc}}^N(U, X, x)$. Thus $\text{Pic}^0(U) = \text{Pic}^0(\tilde{X}) = 0$. We apply Lemma 3.1 to finish conclude that $(X, x)$ is a rational singularity. Again by Lemma 3.1, all line bundles on $U$, in particular the dualizing sheaf $\omega_U$ of $U$, is torsion. This proves 1).

We show 2). So there is a $M \in N \setminus \{0\}$ such that $\omega_U^M \cong \mathcal{O}_U$. Choosing such a trivialization yields an $\mathcal{O}_U$-algebra structure on $A = \bigoplus_{i=0}^{M-1} \omega_U^i$ and thus a flat nontrivial $\mu_M$-torsor $h : V = \text{Spec} \mathcal{O}_U A \rightarrow U$. Since $(M, p) = 1$, $h$ is étale, thus $(Y, y)$ is normal. In fact one has $Y = \text{Spec} \mathcal{O}_X B$ where $B$ is the $\mathcal{O}_X$-algebra $j_* A, j : U \subset X$. By duality theory, $h_* \omega_Y = \text{Hom}_{\mathcal{O}_X}(h_* \mathcal{O}_Y, \omega_X) \cong \mathcal{O}_X h_* \mathcal{O}_Y$. Let $y \in Y$ be the closed point of $Y$. Thus $(Y, y)$ is a Gorenstein normal surface singularity. On the other hand, since $h$ is a $\mu_M$-torsor, one has $\pi^N(V, y) \subset \pi^N(U, x)$, thus $\pi_{\text{loc}}^N(V, Y, y) \subset \pi_{\text{loc}}^N(U, X, x)$, and therefore is a finite $k$-group-scheme. Thus by 1) it is a rational singularity. Thus $(Y, y)$ is a Gorenstein rational singularity, thus is a rational double point ([6]).

Now 3) follows directly from 2) as $\omega_U$ has then order 1.

We now refer to [3, Section 3] for the notation, and we go to Artin’s list [3, Section 4/5] to conclude using Theorem 4.2 3):

Corollary 4.3. If $\pi_{\text{loc}}^N(U, X, x) = 0$, then $X$ admits a finite morphism $f : \mathbb{A}^2 \rightarrow X$. The morphism $f$ is the identity (i.e. $(X, x)$ is smooth) except possibly in the cases:

1) $\text{char}(k) = 2, E_8^1, E_8^3$
2) $\text{char}(k) = 3, E_8^1$

References


[16] SGAII: *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*.

Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
E-mail address: essnaul@uni-due.de

Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
E-mail address: viehweg@uni-due.de