A note on Cayley-Bacharach property for vector bundles

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Abstract. We study the Cayley-Bacherach property on smooth complex projective varieties for zero-dimensional subschemes, defined as the zero set of a global section of a rank n vector bundle, and for codimension 2 subschemes, defined by global sections of rank 2 vector bundles.

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The main purpose of this note is to present and to generalize results from [17] and to use them to study properties and the construction of vector bundles on smooth complex projective varieties X of dimension $n \geq 2$.

In [17], the first author proved that the Cayley-Bacharach property of a zero-dimensional complete intersection in X is equivalent to the k-very ampleness of some adjoint linear systems. In this paper, we show that the result remains true for the zero-dimensional subscheme defined by the zero set of a global section of a rank n vector bundle (Theorem 7), generalizing a theorem of Griffiths and Harris [8], p.677. Due to the Bogomolov inequality for rank 2 semistable vector bundles [4] [12], we can establish the Cayley-Bacharach theorem for codimension 2 subschemes defined by global sections of rank 2 vector bundles (Theorem 8 and Corollary 9). This result can be used to reprove Paoletti's theorem [14] [13], a generalization of the classical theorem of Halphen. As an application, we give an explicit construction of rank 2 vector bundles from codimension 2 subschemes (Theorem 10).

Throughout this paper we use the notion "k points" for any zero-dimensional subscheme of length k, not requiring the points to be distinct. The degree of an object is defined with respect to an fixed ample divisor A on X, hence the degree of a codimension r subscheme Y of X is defined by $\deg Y = A^{n-r}Y$, although A is not mentioned in the statements.

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1. An Exact Sequence

Let X be a smooth projective variety over $\mathbb{C}$ of dimension $n \geq 2$, and let Z be a subscheme of X of pure codimension $r \geq 2$.

Given a subscheme $Z' \subset Z$, the "complement" Z'' of Z' in Z is the canonical closed subscheme $Z'' \subset Z$ with sheaf of ideals $\mathcal{I}_{Z''} = [\mathcal{I}_Z : \mathcal{I}_{Z'}]$, i.e., for any open set $U \subset X$, we define

$$\mathcal{I}_{Z''}(U) := \{g \in \mathcal{O}_X(U) \,|\, g\mathcal{I}_{Z'}(U) \subset \mathcal{I}_Z(U)\},$$

or equivalently,

$$\mathcal{I}_{Z''}/\mathcal{I}_Z = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{Z'}, \mathcal{O}_Z).$$

The second description implies that $Z'' = Z$ if the support of Z' does not contain some of the irreducible components of Z. Moreover, if Z is reduced, then Z'' is the closure of $Z - Z'$. We call Z'' the *residual subscheme* of Z' in Z and denote it by

$$Z'' = Z - Z'.$$

Let E be a vector bundle on X of rank $r \geq 2$, let s be a global section of E and let $Z = Z(s) \subset X$ be its zero scheme. As above we will assume that Z is of pure codimension r, hence it is a local complete intersection. For a divisor L and a subscheme $\Delta \subset Z(s)$, we want to study hypersurfaces F in X satisfying the equations

$$\begin{cases} \Delta = Z(s) - Z(s)F, \\ L \equiv \det E - F. \end{cases} \tag{$*$}$$

Given Δ and L we will call (E, s, F) a solution of $(*)$ if $Z(s)$ is of pure codimension $r = \operatorname{rank}(E)$ and if the equation $(*)$ holds true.

Here and throughout this note $Z(s)F$ denotes the intersection subscheme of $Z(s)$ and a hypersurface (or effective divisor) F in X. If a hypersurface F satisfies the first equation in $(*)$ we will say that F does not pass through Δ. In a similar way, if Z' is a subscheme of F, we will say that F passes through Z'.

If $Z(s)$ is a reduced subscheme of X then F satisfies the first equation in $(*)$, if Δ is the union of all irreducible components of $Z(s)$ which are not contained in F.

Theorem 1. *Let E be a vector bundle on X of rank $r \geq 2$, and let s be a section whose zero subscheme $Z = Z(s)$ is of pure codimension r. Let $Z'' \subset Z'$ are two codimension r subschemes of Z and let L be a divisor. Then there exists a complex of vector spaces*

$$0 \longrightarrow H^0(\mathcal{I}_{Z-Z''}(\det E - L)) \xrightarrow{\alpha} H^0(\mathcal{I}_{Z-Z'}(\det E - L)) \xrightarrow{\mu}$$
$$H^{n-r+1}(\mathcal{I}_{Z'}(K_X + L)) \xrightarrow{\beta} H^{n-r+1}(\mathcal{I}_{Z''}(K_X + L)) \longrightarrow 0,$$

exact except at $H^{n-r+1}(\mathcal{I}_{Z'}(K_X+L))$*. If* E *is sufficiently ample, then the complex is exact everywhere.*

Remark. The condition "E is sufficiently ample" we used in the theorem stands for the following vanishing conditions:

$$H^j(X, \wedge^i E^\vee(\det E - L)) = 0, \quad \text{for } \; i, \; j = 1, \cdots, r-1. \tag{1}$$

If $X = \mathbb{P}^n$ and E splits (the hypersurface case), then (1) is always true. In general (1) can be enforced by replacing E by $E \otimes H$, for a sufficiently ample line bundle H (cf. Lemma 4 and the end of the proof of Theorem 1).

The connecting map μ is not "natural", but there is natural map to the dual of $\ker \beta$.

Throughout the proof of Theorem 1, F will denote an effective divisor on X with $F \equiv \det E - L$. We consider the Koszul complex of (E, s):

$$0 \longrightarrow \mathcal{E}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \xrightarrow{s} \mathcal{I}_Z \longrightarrow 0,$$

where $\mathcal{E}_0 = E^\vee$, $\mathcal{E}_i = \wedge^{i+1}\mathcal{E}_0$, and where s is the dual map of $\mathcal{O} \to \mathcal{E}_0^\vee$ given by the global section s of $\mathcal{E}_0^\vee$. Because $Z = Z(s)$ is a local complete intersection, the Koszul complex is exact (see [9], p.245). We split the Koszul complex as follows:

$$\begin{array}{l} 0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E}_0 \xrightarrow{s} \mathcal{I}_Z \longrightarrow 0, \\ 0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{F}_1 \longrightarrow 0, \\ \qquad \vdots \qquad\qquad\quad \vdots \\ 0 \longrightarrow \mathcal{F}_{r-1} \xrightarrow{s} \mathcal{E}_{r-2} \longrightarrow \mathcal{F}_{r-2} \longrightarrow 0, \end{array} \tag{2}$$

where $\mathcal{F}_{r-1} \cong \mathcal{E}_{r-1} \cong \det E^\vee$.

Lemma 2. *Assume that* Z' *is a subscheme of* Z*. If* (1) *holds, then*

$$H^{n-r+1}(\mathcal{I}_{Z'}(K_X+L))^\vee \cong \mathrm{Ext}^1(\mathcal{I}_{Z'}, \mathcal{F}_1(F)).$$

Proof. By Serre duality ([9], Theorem 7.6) one has an isomorphism

$$H^{n-r+1}(\mathcal{I}_{Z'}(K_X+L))^\vee \cong \mathrm{Ext}^{r-1}(\mathcal{I}_{Z'}, \mathcal{O}(-L)).$$

On the other hand, (1) implies that

$$\mathrm{Ext}^j(\mathcal{O}_X, \mathcal{E}_i(F)) \cong H^j(\mathcal{E}_i(F)) = 0, \quad \text{for } \; i \le r-2, \; 1 \le j \le r-1. \tag{3}$$

From (3) and from the exact sequence

$$0 \longrightarrow \mathcal{I}_{Z'} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{Z'} \longrightarrow 0, \tag{4}$$

we obtain easily that

$$\begin{aligned} \mathrm{Ext}^j(\mathcal{I}_{Z'}, \mathcal{E}_i(F)) &\cong \mathrm{Ext}^{j+1}(\mathcal{O}_{Z'}, \mathcal{E}_i(F)) \\ &\cong H^{n-j-1}(\mathcal{O}_{Z'}(\mathcal{E}_i^\vee(-F+K_X)))^\vee \\ &= 0, \end{aligned}$$

for $i \le r-2,\ 1 \le j \le r-2$. Considering the long exact sequences obtained from the short exact sequences in (2), we thereby have isomorphisms

$$\mathrm{Ext}^1(\mathcal{I}_{Z'}, \mathcal{F}_1(F)) \cong \mathrm{Ext}^2(\mathcal{I}_{Z'}, \mathcal{F}_2(F)) \cong \cdots \cong \mathrm{Ext}^{r-2}(\mathcal{I}_{Z'}, \mathcal{F}_{r-2}(F))$$

and an exact sequence

$$0 \longrightarrow \mathrm{Ext}^{r-2}(\mathcal{I}_{Z'}, \mathcal{F}_{r-2}(F)) \longrightarrow \mathrm{Ext}^{r-1}(\mathcal{I}_{Z'}, \mathcal{F}_{r-1}(F)) \xrightarrow{\tau} \mathrm{Ext}^{r-1}(\mathcal{I}_{Z'}, \mathcal{E}_{r-2}(F)).$$

Since $\mathcal{F}_{r-1}(F) \cong \mathcal{O}(-L)$, it remains to prove that the morphism τ is zero. Indeed, $\mathcal{E}_{r-2}^\vee \cong \mathcal{E}_0 \otimes \det E$ and by Serre duality τ is the dual morphism of

$$H^{n-r+1}(\mathcal{I}_{Z'} \otimes \mathcal{E}_0(K_X + L)) \xrightarrow{s} H^{n-r+1}(\mathcal{I}_{Z'}(K_X + L)).$$

On the other hand, from (4), we obtain a commutative diagram

$$\begin{array}{ccccc} H^{n-r}(\mathcal{O}_{Z'} \otimes \mathcal{E}_0(K_X + L)) & \longrightarrow & H^{n-r+1}(\mathcal{I}_{Z'} \otimes\ E_0(K_X + L)) & \longrightarrow & 0 \\ \big\downarrow s|_{Z'} & & \big\downarrow s & & \\ H^{n-r}(\mathcal{O}_{Z'}(K_X + L)) & \longrightarrow & H^{n-r+1}(\mathcal{I}_{Z'}(K_X + L)) & & \end{array}$$

Since s is vanishing on Z', we find $s|_{Z'}$ to be zero, which implies that the morphism s is zero as well. □

Lemma 3. *Under the assumptions made in Lemma 2, there is an exact sequence*

$$H^0(\mathcal{E}_0(F)) = \mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{E}_0(F)) \xrightarrow{s} H^0(\mathcal{I}_{Z-Z'}(F)) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_{Z'}, \mathcal{F}_1(F)) \longrightarrow 0.$$

Proof. Applying the functor $\mathrm{Hom}(\mathcal{I}_{Z'}, \cdot)$ to

$$0 \longrightarrow \mathcal{F}_1(F) \longrightarrow \mathcal{E}_0(F) \xrightarrow{s} \mathcal{I}_Z(F) \longrightarrow 0,$$

we obtain the exact sequence

$$\mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{E}_0(F)) \xrightarrow{s} \mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{I}_Z(F)) \longrightarrow \mathrm{Ext}^1(\mathcal{I}_{Z'}, \mathcal{F}_1(F)) \longrightarrow 0.$$

Note that the 0 term on the right hand side comes from (3) if $r \ge 3$, and for $r = 2$ from the morphism

$$\tau : \mathrm{Ext}^{r-1}(\mathcal{I}_{Z'}, \mathcal{F}_{r-1}(F)) \longrightarrow \mathrm{Ext}^{r-1}(\mathcal{I}_{Z'}, \mathcal{E}_{r-2}(F))$$

which is zero as we have seen in the proof of Lemma 2.

Because $Z - Z'$ is the residual subscheme of Z' in Z, we have (cf. [17])

$$\mathrm{Hom}(\mathcal{I}_{Z'}, \mathcal{I}_Z(F)) \cong H^0(\mathcal{I}_{Z-Z'}(F)),$$

completing the proof of Lemma 3. □

Proof of Theorem 1 for E sufficiently ample. By Lemma 2 and Lemma 3 for Z' and Z'', we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} s & \longrightarrow & H^0(\mathcal{I}_{Z-Z'}(F)) & \xrightarrow{\mu_{Z'}} & H^{n-r+1}(\mathcal{I}_{Z'}(K_X+L))^\vee & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \operatorname{Im} s & \longrightarrow & H^0(\mathcal{I}_{Z-Z''}(F)) & \xrightarrow[\mu_{Z''}]{} & H^{n-r+1}(\mathcal{I}_{Z''}(K_X+L))^\vee & \longrightarrow & 0 \end{array} \tag{5}$$

Note that the middle and right vertical morphisms are injective and by the Five Lemma we can see that they have the same cokernel Q, hence

$$0 \longrightarrow H^0(\mathcal{I}_{Z-Z''}(F) \longrightarrow H^0(\mathcal{I}_{Z-Z'}(F)) \longrightarrow Q \longrightarrow 0,$$

and

$$0 \longrightarrow Q^\vee \longrightarrow H^{n-r+1}(\mathcal{I}_{Z'}(K_X+L)) \longrightarrow H^{n-r+1}(\mathcal{I}_{Z''}(K_X+L)) \longrightarrow 0.$$

Choosing any isomorphism $Q \cong Q^\vee$ one obtains Theorem 1 from the two exact sequences above. □

For the general case we will replace the vector bundle E by $E \otimes H$, for some sufficiently ample line bundle H.

Lemma 4. *Assume that (E, s, F) is a solution of $(*)$ for fixed L and Δ. Let H be a sufficiently ample line bundle and $M \in H^0(E \otimes E^\vee \otimes H)$ a sufficiently general section, viewed as a morphism $M : E \to E \otimes H$. Let*

$$\widetilde{E} = E \otimes H, \quad \widetilde{s} = sM, \quad \widetilde{F} = F + Z(\det M).$$

Then $(\widetilde{E}, \widetilde{s}, \widetilde{F})$ is also a solution of $(*)$ for L and Δ.

Proof. We can assume that the divisor of $\det M$ does not contain any component of $Z(s)$. Let

$$\widetilde{\Delta} = Z(\widetilde{s}) - Z(\widetilde{s})\widetilde{F}$$

be the new residual subscheme. We only need to prove that $\widetilde{\Delta} = \Delta$, i.e., $\mathcal{I}_{\widetilde{\Delta}} = \mathcal{I}_\Delta$.

Indeed, by definition, it is clear that $\mathcal{I}_\Delta \subset \mathcal{I}_{\widetilde{\Delta}}$. Conversely, $\mathcal{I}_{\widetilde{\Delta}}$ consists of the local sections $\widetilde{g}$ such that $\widetilde{g} f \det M$ vanishes on $Z(\widetilde{s})$, where f is the local defining equation of F. Hence it also vanishes on $Z(s)$. Because $\det M$ does not vanish on any component of $Z(s)$, this implies that $\widetilde{g} f$ vanishes on $Z(s)$. Now we know that $\widetilde{g}$ is contained in $\mathcal{I}_\Delta$. So $\mathcal{I}_{\widetilde{\Delta}} \subset \mathcal{I}_\Delta$. □

Proof of Theorem 1 for arbitrary E. Keeping the notations and assumptions of Lemma 4 we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathcal{I}_{\widetilde{Z}-Z''}(\widetilde{F})) & \longrightarrow & H^0(\mathcal{I}_{\widetilde{Z}-Z'}(\widetilde{F})) & \longrightarrow & \widetilde{Q} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \phi_1} & & \uparrow{\scriptstyle \phi_2} & & \uparrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & H^0(\mathcal{I}_{Z-Z''}(F)) & \longrightarrow & H^0(\mathcal{I}_{Z-Z'}(F)) & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

where $Q = \operatorname{coker}\alpha$, and where ϕ_1 and ϕ_2 are defined as the multiplication by $\det M$. In particular ϕ_1 and ϕ_2 are injective. For H sufficiently ample, Theorem 1 holds true for $\widetilde{E}$, and $\widetilde{Q} = \ker\beta$. Thus we only need to prove that ψ is injective. By the Five Lemma, it is enough to prove that the induced natural map

$$\operatorname{coker}\phi_1 \to \operatorname{coker}\phi_2$$

is injective.

Indeed, let $G \equiv \widetilde{F}$ represent an element of $\operatorname{coker}\phi_1$, then G passes through $\widetilde{Z} - Z''$. If its image in $\operatorname{coker}\phi_2$ is zero, i.e., if $G = G' + \det M$ and G' passes through $Z - Z'$, we need to prove that G is also zero in $\operatorname{coker}\phi_1$, i.e., G' passes through $Z - Z''$. This is obvious because $\det M$ does not pass through $Z - Z''$, but G does. □

2. Solutions of the Equation $(*)$

Theorem 5. *Let Δ be a subscheme of X of pure codimension r and let L be a divisor on X. Then the following conditions are equivalent.*

1) $(*)$ *has a solution (E, s, F) for Δ and L, i.e., there are a hypersurface F, a rank r vector bundle E and a nonzero global section s of E whose zero set $Z = Z(s)$ is an $n - r$ dimensional subscheme such that $(*)$ holds, so Δ is the residual subscheme of $Z(s)F$ in $Z(s)$.*

2) *There is an element η in $H^{n-r+1}(\mathcal{I}_\Delta(K_X + L))^\vee$ such that for any proper codimension r closed subscheme $\Delta' \subsetneq \Delta$, η is not in the image of the following natural inclusion map:*

$$H^{n-r+1}(\mathcal{I}_{\Delta'}(K_X + L))^\vee \longrightarrow H^{n-r+1}(\mathcal{I}_\Delta(K_X + L))^\vee.$$

Proof. If $(*)$ has a solution, then by Lemma 4, $(*)$ has a solution with E sufficiently ample. Hence we are allowed to use the diagram (5) from the previous section for $\Delta' \subset \Delta \subset Z$ instead of $Z'' \subset Z' \subset Z$. Recall that the vertical maps in (5) are natural inclusions.

We claim that $\eta = \mu_\Delta(f)$ is the desired element, where $f \in H^0(\mathcal{I}_{Z-\Delta}(F))$ denotes the section with $F = Z(f)$. Indeed, if for some $\Delta' \subsetneq \Delta$ one has $\eta = \mu_{\Delta'}(f')$ with $f' \in H^0(\mathcal{I}_{Z-\Delta'}(F))$, then $\mu_\Delta(f) = \mu_{\Delta'}(f') = \mu_\Delta(f')$, thus $\mu_\Delta(f - f') = 0$. This implies that $f - f'$ as an element of the image of

$$s : \operatorname{Hom}(\mathcal{I}_\Delta, \mathcal{E}_0(F)) \to H^0(\mathcal{I}_{Z-\Delta}(F))$$

vanishes on Z and hence f vanishes on $Z - \Delta'$. We find $\Delta = Z - ZF \subset \Delta'$, contradicting the assumptions made.

Conversely, from a class η as in 2), we have to construct a solution (E, s, F). Let $F_1, \cdots, F_r$ be sufficiently ample hypersurfaces containing Δ. Assume that $Z = F_1 \cdots F_r$ is a complete intersection. From Theorem 1, we can find an $f \in H^0(\mathcal{I}_{Z-\Delta}(F))$ such that $\eta = \mu_\Delta(f)$, let $F = Z(f)$. Then we have the above commutative diagram for $Z' = \Delta$ and $Z'' = Z - ZF \subset \Delta$. If $Z'' \neq \Delta$, then $\eta = \mu_{Z''}(f)$, which contradicts our assumption. So $\Delta = Z - ZF$, and $(*)$ has a solution with $E = \oplus_{i=1}^r \mathcal{O}_X(F_i)$. □

Remark. From the proof of this theorem, we see that if (E, s, F) is a solution of $(*)$, then $(*)$ has a solution with splitting E, i.e., we can find hypersurfaces $F_1, \cdots, F_{r+1}$ such that $F_1 \cdots F_r$ is a complete intersection and

$$\begin{cases} \Delta = F_1 \cdots F_r - F_1 \cdots F_{r+1}, \\ L \equiv F_1 + \cdots + F_r - F_{r+1}. \end{cases} \qquad (**)$$

Corollary 6. *Let Δ be a codimension r subscheme of X and L a divisor on X. Then the following conditions are equivalent.*

1) $$h^{n-r+1}(\mathcal{I}_\Delta(K_X + L)) > h^{n-r+1}(K_X + L)$$

but for any subscheme $\Delta' \subsetneq \Delta$,

$$h^{n-r+1}(\mathcal{I}_{\Delta'}(K_X + L)) = h^{n-r+1}(K_X + L).$$

2) $(*)$ *has a solution (E, s, F) for Δ and L, but for any subscheme $\Delta' \subsetneq \Delta$,* $(*)$ *has no solution for Δ' and L.*

3. A Generalization of Griffiths-Harris Theorem

Let us first recall the definition of *k-very ampleness* (cf. [2] [3] or [11]). A linear system $|D|$ on X is called k-very ample if for any zero-dimensional subscheme Y of degree $k+1$, the restriction map

$$\rho_Y : H^0(\mathcal{O}(D)) \longrightarrow H^0(\mathcal{O}_Y(D))$$

is surjective, which is equivalent to the injectivity of

$$\beta_Y : H^1(\mathcal{I}_Y(D)) \longrightarrow H^1(\mathcal{O}(D)) \longrightarrow 0.$$

Note that "0-very ample" is equivalent to "base point free", and "1-very ample" is equivalent to "very ample".

Theorem 7. *For a fixed divisor L on X and a positive integer k, the following conditions are equivalent.*

1) *Let E be a rank n vector bundle with a nonzero global section s such that $Z = Z(s)$ is a zero-dimensional subscheme, and let $F \in |\det E - L|$. If F passes through a subscheme Z' of Z whose degree $\geq \deg Z - k$, then F passes through Z.*

2) $|K_X + L|$ *is $(k-1)$-very ample.*

Proof. The first condition means that for any F in the linear system, the degree of $\Delta := Z(s) - Z(s)F$ is zero or bigger than k. From Corollary 6, this is equivalent to the condition that for any zero dimensional subscheme Y of degree $\leq k$,

$$\beta_Y : H^1(\mathcal{I}_Y(K_X + L)) \longrightarrow H^1(\mathcal{O}_X(K_X + L))$$

is injective. Now by definition, this is just saying that $|K_X + L|$ is $(k-1)$-very ample. $\square$

If $L = -K_X$, then obviously $|K_X + L|$ is base point free. Hence the first part of Theorem 7 is true for $k = 1$. This is the Cayley-Bacharach Theorem due to Griffiths and Harris without the assumption that $Z(s)$ is reduced (cf. [8], p.677).

Since $\mathcal{O}_{\mathbb{P}^n}(k)$ is k-very ample, one obtains a Cayley-Bacharach theorem on $\mathbb{P}^n$ [17]. In fact, this theorem is sharp.

4. Rank 2 Vector Bundle Case

A divisor L is called numerically effective (nef) if the intersection number $L.C$ is non negative, for all irreducible curves C on X.

Theorem 8. *Let L be a nef divisor, let E be a rank 2 vector bundle on X, and let s be a global section such that $Z = Z(s)$ is of pure codimension 2. For some $F \in |\det E - L|$ let*

$$\Delta = Z(s) - Z(s)F$$

be the residual subscheme of $Z(s)F$ in $Z(s)$. If $\deg \Delta < \deg L^2/4$, then either Δ is empty or there exists an effective divisor D passing through Δ such that

$$\deg DL - \deg \Delta \leq \deg D^2 < \frac{1}{2} \deg DL \leq \frac{1}{4}\left(\deg L^2 - \sqrt{\deg L^2}\sqrt{\deg L^2 - 4\deg \Delta}\right) < \deg \Delta.$$

Proof. By Lemma 4, we can assume that E is sufficiently ample. So we have a class $\eta = \mu_\Delta(f)$ satisfying the condition 2) of Theorem 5. As in the proof of Theorem 5, we can find a new solution (E', s', F_3) of $(*)$ with $E' = \mathcal{O}(F_1) \oplus \mathcal{O}(F_2)$, $s' = (f_1, f_2)$, i.e.,

$$\begin{cases} \Delta = F_1F_2 - F_1F_2F_3, \\ L \equiv F_1 + F_2 - F_3. \end{cases}$$

Now we use [17], Corollary 2.2, to complete the proof. $\square$

If we take $\deg \Delta = 1$ or 2, then we can find a theorem of Reider's type (cf. [16]).

The following Corollary is a generalization of the classical Cayley-Bacharach Theorem [1] [6].

Corollary 9. *Keeping the notations introduced above, let H be an ample divisor on X, let ℓ be a positive integral, and let $F \in |\det E - \ell H|$. If F passes through an $(n-2)$-dimensional subscheme of $Z(s)$ whose degree is larger than or equal to $\deg Z(s) - \ell + 2$, then F passes through $Z(s)$.*

Remark. Theorem 8 can be used to study the codimension 2 subvarieties in projective space. We assume that Y is a codimension 2 projective subscheme of $\mathbb{P}^n$. We are interested in the following invariants.

$$\begin{aligned} d &= \deg Y, \\ s &= \min\{\, m \mid H^0(\mathcal{I}_Y(m)) \neq 0 \,\}, \\ e &= \max\{\, m \mid H^{n-1}(\mathcal{I}_Y(m)) \neq 0 \,\}. \end{aligned}$$

Note that $H^{n-1}(\mathcal{I}_Y(m)) = H^{n-2}(\mathcal{O}_Y(m))$.

Let $\ell = e + n + 1$. For a reduced and irreducible subscheme Y Theorem 5 implies that $(**)$ has a solution. Hence there are 3 hypersurfaces F_1, F_2, F_3 of degree d_1, d_2 and d_3, respectively, such that

$$\begin{cases} Y = F_1F_2 - F_1F_2F_3, \\ \ell = d_1 + d_2 - d_3. \end{cases}$$

In fact, we can choose F_1 such that $\deg F_1 = s$. If $\ell \geq 2\sqrt{d}$, then by Theorem 8,

$$s \leq \frac{1}{2}\ell - \frac{1}{2}\sqrt{\ell^2 - 4d}.$$

This reproves a theorem of Paoletti in [14].

5. An Explicit Construction of Rank 2 Vector Bundles

As wellknown, a codimension 2 subscheme Δ of X is the zero subscheme of a global section of a rank 2 vector bundle, provided it satisfies certain cohomological conditions. In this section, we are going to give an explicit construction of the corresponding vector bundle using the methods developed above.

We will prove that under those cohomological conditions on Δ, we can find three hypersurfaces F_1, F_2 and F_3 such that F_1 and F_2 have no common components,

$$\Delta = F_1F_2 - F_1F_2F_3,$$

and $F_1F_2F_3$ is pure codimension 2 and Cohen-Macaulay. We denote by $\mathcal{F}$ the syzygy sheaf of F_1, F_2, F_3, i.e.,

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_X(-F_i) \longrightarrow \mathcal{I}_{F_1F_2F_3} \longrightarrow 0.$$

Then we will construct a global section of the rank 2 vector bundle $\mathcal{E} = \mathcal{F}(F_1+F_2)$ whose zero subscheme is Δ.

Theorem 10. *Let $\Delta \subset X$ be a subscheme of pure codimension 2. Then the following are equivalent:*

1) *Δ is the zero subscheme of a section of a rank 2 vector bundle $\mathcal{E}$.*

2) *Δ is a local complete intersection, ω_Δ can be extended to an invertible sheaf $\mathcal{W}$ on X, and there is an element $\eta \in H^{n-1}(\mathcal{I}_\Delta(\mathcal{W}))^\vee$ such that for any codimension 2 subscheme $\Delta' \subset \Delta$ with $\deg \Delta' < \deg \Delta$, η is not contained in the image of the following inclusion map*

$$H^{n-1}(\mathcal{I}_{\Delta'}(\mathcal{W}))^\vee \longrightarrow H^{n-1}(\mathcal{I}_\Delta(\mathcal{W}))^\vee.$$

3) *There are three hypersurfaces F_1, F_2 and F_3 such that F_1 and F_2 have no common components, $\Delta = F_1F_2 - F_1F_2F_3$, and such that $F_1F_2F_3$ is of pure codimension 2 and Cohen-Macaulay.*

Furthermore, if 1), 2) and 3) hold true, then

$$c_1(\mathcal{E}) \equiv \mathcal{W} - K_X \equiv F_1 + F_2 - F_3.$$

Proof. 1) $\Longrightarrow$ 2): It is wellknown that Δ is a local complete intersection and

$$\omega_\Delta = (\det \mathcal{E} + K_X)|_\Delta,$$

so $\mathcal{W} = \det \mathcal{E} + K_X$. For $Z = \Delta$, $F = 0$ and for $L = \det \mathcal{E}$ the equation $(*)$ is satisfied. By Theorem 5, we obtain an element η satisfying the desired conditions.

2) $\Longrightarrow$ 3): Let F_1 and F_2 be two sufficiently ample hypersurfaces containing Δ. Assume that they have no common components. Due to Theorem 5, there is an $F_3 \equiv F_1 + F_2 - L$ (here $L = \mathcal{W} - K_X$) such that

$$\Delta = F_1F_2 - F_1F_2F_3.$$

Now we only need to prove that $F_1F_2F_3$ has pure codimension 2. In fact, if S is the pure codimension 2 part of $F_1F_2F_3$, then Δ and S are linked, so S is also Cohen-Macaulay [15]. Because Δ is a local complete intersection, we can assume that Δ and S have no common components. On the other hand, we claim that for $Z = F_1F_2$ the sequence

$$0 \longrightarrow \omega_\Delta \longrightarrow \omega_Z \longrightarrow \mathcal{O}_S \otimes \omega_Z \longrightarrow 0 \tag{6}$$

is exact. Indeed, let Z be embedded in some $\mathbb{P}^N$, and let s be its codimension. Because Z is Cohen-Macaulay, $\Delta \cap S$ is a divisor in Δ (see [7], p. 454). So $\Delta \cap S$ has codimension $s+1$, thus we have

$$\mathcal{E}xt^s_{\mathbb{P}^n}(\mathcal{O}_{\Delta\cap S}, \omega_{\mathbb{P}^N}) = 0.$$

Since Δ and S are Cohen-Macaulay, we have

$$\mathcal{E}xt^{s+1}_{\mathbb{P}^n}(\mathcal{O}_\Delta, \omega_{\mathbb{P}^N}) = 0, \quad \mathcal{E}xt^{s+1}_{\mathbb{P}^n}(\mathcal{O}_S, \omega_{\mathbb{P}^N}) = 0.$$

Applying $\mathcal{E}xt(., \omega_{\mathbb{P}^N})$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_\Delta \oplus \mathcal{O}_S \longrightarrow \mathcal{O}_{\Delta\cap S} \longrightarrow 0,$$

one obtaines an exact sequence,

$$0 \longrightarrow \omega_\Delta \oplus \omega_S \longrightarrow \omega_Z \longrightarrow \omega_{\Delta\cap S} \longrightarrow 0. \tag{7}$$

Thus the sequences

$$0 \longrightarrow \omega_S \longrightarrow \omega_Z/\omega_\Delta \longrightarrow \omega_{\Delta\cap S} \longrightarrow 0 \tag{8}$$

and
$$0 \longrightarrow \omega_\Delta|_S \oplus \omega_S \longrightarrow \omega_Z|_S \longrightarrow \omega_{\Delta\cap S} \longrightarrow 0$$

are exact. Since $\omega_Z|_S$ is invertible and since $\omega_\Delta|_S$ is a torsion sheaf, $\omega_\Delta|_S = 0$. One obtains an exact sequence

$$0 \longrightarrow \omega_S \longrightarrow \omega_Z|_S \longrightarrow \omega_{\Delta\cap S} \longrightarrow 0. \tag{9}$$

Note that there is a natural surjective morphism $\phi : \omega_Z/\omega_S \to \omega_Z|_S$. Comparing (8) and (9), we find ϕ to be an isomorphism. This proves (6).

From $\omega_\Delta = \mathcal{W}|_\Delta$ and $\omega_Z = (F_1 + F_2 + K_X)|_Z$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}(-F_3)|_\Delta \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

On the other hand,

$$0 \longrightarrow \mathcal{O}_\Delta(-S\cap\Delta) \longrightarrow \mathcal{O}_{F_1F_2} \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

so $F_3 \cap \Delta = S \cap \Delta$. This implies that $F_1F_2F_3$ has pure codimension 2.

3) $\Longrightarrow$ 1): Let $\mathcal{F}$ be the syzygy sheaf of F_1, F_2, F_3. Since $F_1F_2F_3$ is of pure codimension 2 and Cohen-Macaulay, we know that $\mathcal{F}$ is locally free (cf. [5], [10] or [18]). Considering the composition

$$\phi : \mathcal{F} \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_X(-F_i) \longrightarrow \mathcal{O}(-F_3),$$

we can see that the image of ϕ in $\mathcal{O}_X(-F_3)$ is $\mathcal{I}_\Delta(-F_3)$ (cf. the definition of $\mathcal{I}_\Delta$). Thus $\ker\phi$ is an invertible sheaf. By comparing the first Chern classes, we obtain

$$0 \longrightarrow \mathcal{O}_X(-F_1 - F_2) \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_\Delta(-F_3) \longrightarrow 0,$$

i.e.,
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F}(F_1 + F_2) \longrightarrow \mathcal{I}_\Delta(F_1 + F_2 - F_3) \longrightarrow 0.$$

Thus the rank 2 vector bundle $\mathcal{E} = \mathcal{F}(F_1+F_2)$ has a section s whose zero subscheme is Δ, and $\det\mathcal{E} = F_1 + F_2 - F_3$. □

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