POSITIVITY OF DIRECT IMAGE SHEAVES AND APPLICATIONS TO FAMILIES OF HIGHER DIMENSIONAL MANIFOLDS

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Let Y be a projective algebraic curve over $\mathbb{C}$, and let $S \subseteq Y$ be a finite set of points. I. R. Shafarevich's conjecture for families of curves, proved by A. Parshin and A. Arakelov (see [25] and [2]), states that

(I) There are only finitely many isomorphism classes of smooth non-isotrivial families of curves over $Y - S$.

(II) If $2g(Y) - 2 + \#S \leq 0$, then there are no such families.

L. Caporaso [4] has shown recently, that the number of non-isotrivial families in (I) is bounded by a constant depending only on the genus q of the fibre, on the genus g of Y and on $s = \#S$.

Arakelov's theorem follows from the observation, that for a non-isotrivial family of curves $f : X \to Y$ the relative dualizing sheaf $\omega_{X/Y}$ is nef and big, that $\det f_*\omega^\nu_{X/Y}$ is ample on Y, and that the degree of $\det f_*\omega^\nu_{X/Y}$ is bounded by a constant, depending on q, g and s.

Arakelov's methods were taken up in a series of papers, concerning Iitaka's conjecture on the subadditivity of the Kodaira dimension, starting with T. Fujita [7]. In the higher dimensional case, an analogue of the first two properties, the bigness of $\omega_{X/Y}$ and of $\lambda_\nu = \det f_*\omega^\nu_{X/Y}$ has been obtained by Y. Kawamata, J. Kollár and by the author for all morphisms $f : X \to Y$ whose general fibre F is either of general type, or allows a good minimal model (see [12], [14], [30], [31] or the excellent survey article [22]). Over a higher dimensional base, the non-isotriviality has to be replaced by the condition, that the fibres of f are varying in all directions, i.e. that Y is not covered by curves over which the restriction of f is isotrivial. As an intermediate step, one studies positivity properties of direct images of powers of dualizing sheaves, properties which reappear in [32] as a tool for the construction of quasi-projective moduli schemes for polarized manifolds with numerically effective canonical sheaves.

The boundedness of the degree of $\det f_*\omega^\nu_{X/Y}$, as well as analogues of (II) for families of higher dimensional varieties were only obtained recently.

For families of surfaces of general type over a curve, or more generally for families of canonically polarized manifolds, the analogue of (II) was verified by L. Migliorini [21], and S. Kovács [17], [19]. In [24] K. Oguiso and the author extended their result to all surfaces of non-negative Kodaira dimension. Recently, S. Kovács [20] proved (II) for all families with F a minimal model of general type. More or less at the same time K. Zuo and the author [33]

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obtained a slightly more general result, by verifying (II) for all families with $\deg(\lambda_\nu) > 0$.

In the higher dimensional case (I) is too much to hope for. For fixed Y and S there exist non-trivial deformations of families of abelian varieties over $Y - S$, and there exist smooth families of surfaces of general type over projective curves with non-trivial deformations. So (I) splits up in three questions: boundedness, finiteness of deformation types, and rigidity.

To be a bit more precise, let us fix some polynomial $h \in \mathbb{Q}[t]$. If $\deg(h) = 2$, we define M_h to be the moduli scheme of manifolds F with a numerically effective canonical sheaf, together with a polarization $\mathcal{L}$, with Hilbert polynomial $h(\mu) = \chi(\mathcal{L}^\mu)$ (see [32], for example).

(I) should be replaced by three sub-problems:

> (B) Find in terms of h, $g(Y)$ and $\#S$ some upper bound for $\deg(\lambda_\nu)$, for all non-isotrivial family $f : X \to Y$, which allow a polarization with Hilbert polynomial h.
>
> (F) Show that the non-trivial morphisms $Y - S \to M_h$, which are induced by smooth projective maps $g_0 : X_0 \to Y - S$, are parameterized by some scheme of finite type.
>
> (R) Under which additional conditions are the morphisms $Y - S \to M_h$ in (B) rigid.

For families of canonically polarized manifolds, and of surfaces of general type, (B) has been solved by E. Bedulev and the author in [3]. K. Zuo and the author obtained (B) in [33] for all families with $\deg(\lambda_\nu) > 0$, and independently S. Kovács [20] handled the case of families with F a minimal model of general type.

(B) implies (F), if one has a good compactification of the moduli scheme (see [3]). At present, such compactifications exist only for moduli of curves or surfaces of general type.

All the known results concerning (II) or (B), as well as the construction of quasi-projective moduli schemes in [32] are using positivity properties of certain direct image sheaves. In the first section, we formulate those results for families of curves over a curve, and we indicate why they imply (II) and the boundedness (B).

Next we consider families of curves over a higher dimensional base. We indicate how to obtain the positivity of the sheaf of logarithmic differential forms on moduli spaces of curves. Possible generalizations to moduli of higher dimensional manifolds are discussed in the last section.

Sections three and four contain the proof of some of the results used in the first two sections. The restriction to family of curves does not simplify the constructions, hence we consider arbitrary families of manifolds over curves. Whenever it is convenient, we assume that the canonical sheaf of the general fibre is semi-ample.

In section five, we indicate how the positivity properties are used in [33] to derive the boundedness and (II). We restrict ourselves to families of surfaces of general type, but the arguments easily generalize to families of higher dimensional canonically polarized manifolds.

These notes contain no new results, and most of the arguments are copied from earlier articles, in particular from [5], [3], [24] and [33]. Nevertheless, the proofs given are a bit sketchy. Trying to present the arguments in their most simple form, I might have added some inaccuracies.

Notations and Conventions

Throughout these notes we will use the standard notions of algebraic geometry, (see [8], for example). All schemes, varieties and manifolds are supposed to be defined over the field of complex numbers, and a scheme is supposed to be separated reduced and of finite type over $\mathbb{C}$. The definitions and results coming from the higher dimensional birational geometry are explained in [22].

An effective normal crossing divisor D on a manifold X is an effective divisor $D = \sum \nu_i D_i$ with non-singular components D_i intersecting each other transversely. A morphism $f : X \to Y$ from a manifold to a curve is called semistable, if all fibres are normal crossing divisors.

For a real number α the integral part is denoted by $[\alpha]$. Correspondingly we will write

$$[\Delta] = \sum_{i=1}^{\ell} [\alpha_i]\Delta_i$$

for the integral part of a $\mathbb{Q}$-divisor $\Delta = \sum_{i=1}^{\ell} \alpha_i \cdot \Delta_i$.

If $\mathcal{L}$ is an invertible sheaf and if D is a Cartier divisor on X we write sometimes $\mathcal{L}^N(D)$ instead of $\mathcal{L}^{\otimes N} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$. Hence $\mathcal{L}^N(D)^M$ stands for

$$\mathcal{L}^{\otimes N \cdot M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(M \cdot D).$$

The following properties of an invertible sheaf $\mathcal{L}$ on a scheme X will be used frequently:

- $\mathcal{L}$ is called semi-ample if for some $N \geq 0$ the sheaf $\mathcal{L}^N$ is generated by its global sections.
- $\mathcal{L}$ is called numerically effective or "nef" if for all projective curves C in X one has $\deg(\mathcal{L}|_C) = c_1(\mathcal{L}).C \geq 0$.

A family of varieties is a flat surjective morphism $f : X \to Y$. It is called (birationally) isotrivial, if all smooth fibres of f are birational, or equivalently, if there exists a finite cover $Y' \to Y$ and a birational map $X \times_Y Y' \xrightarrow{\sim} F \times Y'$.

If X is a reduced and normal variety or a Cohen-Macaulay scheme and if Y is a Gorenstein scheme we write $\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$, where ω_X and ω_Y denote the canonical sheaves. For a flat Cohen-Macaulay morphism $f : X \to Y$ $\omega_{X/Y}$ denotes the dualizing sheaf. In the latter case, the sheaf $\omega_{X/Y}$ is flat over Y and compatible with fibred products.

1. Families of Curves

Let $f : X \to Y$ be a family of curves over a curve Y. Hence X is a non-singular complex projective surface and Y a non-singular curve, and f a surjective morphism with connected general fibre F.

The starting point of "Positivity of direct image sheaves" is Fujita's theorem, saying that the locally free sheaf $f_*\omega_{X/Y}$ is numerically effective. In fact, as we will see in section three, the same holds true for families of higher dimensional varieties, and even to study families of curves, this will be needed several times.

Applying Fujita's theorem to certain cyclic coverings, one obtains that for all $\nu \geq 0$, the sheaf $f_*\omega^\nu_{X/Y}$ is nef. Moreover, $f_*\omega^\nu_{X/Y}$ has some weak stability property: If $\lambda_\nu = \det(f_*\omega^\nu_{X/Y})$ is ample, then $f_*\omega^\nu_{X/Y}$ is ample. The "weak stability" can be stated in a more precise form, but before doing so, we need some notations.

Definition 1.1. Let $\mathcal{E}$ be a locally free sheaf on a curve Y, and let λ be an invertible sheaf on Y.

a) $\mathcal{E}$ is numerically effective (nef) if for all finite covering $\pi : C \to Y$ and for all invertible quotients $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{N}$, one has $\deg \mathcal{N} \geq 0$.

b) For $a, b \in \mathbb{N}$, $b \neq 0$, we write
$$\mathcal{E} \succeq \frac{a}{b} \cdot \lambda$$
if $S^b(\mathcal{E}) \otimes \lambda^{-a}$ is nef.

If λ is ample and $\mathcal{E} \succeq \frac{a}{b} \cdot \lambda$, then $\mathcal{E}$ is ample. It is an easy exercise to verify, that $\mathcal{E}$ is nef, if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef.

Nef locally free sheaves on curves, have been used in [5] to study the height of points of curves over function fields. In [32], §2, and in the higher dimensional birational classification theory, one needs positive coherent torsionfree sheaves over higher dimensional manifolds, and "nef" is replaced by "weakly positive" (see 2.3, c). In the one-dimensional case, both notions coincide, and all the properties of weakly positive sheaves, listed in [22], [30] or [32] carry over to nef sheaves on curves. Let us recall one property:

Lemma 1.2. *Given $d \in \mathbb{N}$, assume that for all $\delta \in \mathbb{N} - \{0\}$, there exists a covering $\tau : Y' \to Y$ of degree δ such that $\tau^*\mathcal{E} \otimes \mathcal{H}$ is nef, for one, hence for all invertible sheaves of degree d. Then $\mathcal{E}$ is nef.*

Proof. Let $\pi : C \to Y$ and $\mathcal{N}$ be as in 1.1, a), and let C' be a component of the normalization of $C \times_Y Y'$. If
$$\begin{array}{ccc} C' & \xrightarrow{\tau'} & C \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\tau} & Y \end{array}$$
are the induced morphisms, then
$$0 \leq \deg(\tau'^*\mathcal{N} \otimes \pi'^*\mathcal{H}) = \deg(\tau') \cdot \deg(\mathcal{N}) + \deg(\pi') \cdot d$$
$$\leq \delta \cdot \deg(\mathcal{N}) + \deg(\pi) \cdot d.$$
This, for all $\delta \in \mathbb{N} - \{0\}$, implies that $\deg(\mathcal{N}) \geq 0$. □

The positivity results for families of curves over a curve are gathered in the following theorem.

Theorem 1.3. *Let $f : X \to Y$ be a family of curves over a curve. Let $g = g(Y)$ and $q = g(F)$ denote the genus of the base and of the general fibre, respectively. Then*

a) $f_*\omega^\nu_{X/Y}$ *is nef, for all* $\nu \geq 0$.
b) $f_*\omega^\nu_{X/Y} \succeq \frac{1}{e_\nu \cdot \text{rank}(f_*\omega^\nu_{X/Y})} \cdot \lambda_\nu$, *for* $e_\nu = \nu \cdot (2q-2)+1$, *and for all* $\nu > 1$.
c) *If* $f : X \to Y$ *is non-isotrivial, then* λ_ν *is ample.*

The converse of c) does not hold true. For example one can construct elliptic fibrations with constant moduli over $\mathbb{P}^1$, whose total space is of non-negative Kodaira dimension. For semistable families of curves (see 4.6 for generalizations), the two conditions in c) are equivalent. The property b), which we usually call "weak stability" will be essential for the constructions in these notes. The weak stability will lead to explicit upper bounds for the degree of λ_ν, as soon as we are able to construct a non-trivial morphism from $f_*\omega^\nu_{X/Y}$ to an invertible sheaf of low degree. Such morphisms are constructed below using the Kodaira Spencer map. As we will see in the proof of 1.3, similar results hold true for families of higherdimensional varieties. Before stating them, and sketching their proof in section three and four, let us discuss the Kodaira Spencer map, and some applications of 1.3. To this aim, let $S \subset Y$ be the discriminant divisor, or more general any reduced divisor in Y, such that for $\Delta = f^*S$ the restriction f_0 of f to

$$X_0 = X - \Delta \longrightarrow Y_0 = Y - S$$

is smooth. We write $s = \#S = \deg S$ and δ for the number of fibres of f, which are not reduced normal crossing divisors.

Corollary 1.4. *Let* $f : X \to Y$ *be a non-isotrivial family of curves of genus* $q \geq 1$ *over a curve* Y. *Then, using the notations introduced above,*
a) $2g - 2 + s \geq 1$
b) $\deg \lambda_2 \leq r_2 \cdot e_2 \cdot (2g-2+s+\delta)$, *for* $r_2 = \text{rank}(f_*\omega^2_{X/Y}) = 3(g-1)$ *and for* $e_2 = 2 \cdot (2q-2)+1$.

These inequalities, which are far from being optimal (see [29]), follow from the results in [5], for example. Let us recall, how 1.4 is obtained using 1.3.

Proof of 1.4. Assume first, that $2g-2+s \leq 0$. Hence either Y is an elliptic curve, and $s = 0$, or $Y \simeq \mathbb{P}^1$ and $S \subset \{0, \infty\}$. In the second case we are allowed to enlarge S, and to assume thereby, that $S = \{0, \infty\}$.

Replacing Y by $Y' \to Y = \mathbb{P}^1$, a finite covering ramified in $\{0, \infty\}$, and X by a desingularization of $X \times_Y Y'$, we may assume that all fibres of f are reduced normal crossing divisors, hence that f is semistable, and $\delta = 0$.

In both cases, the inequality b) implies that $\deg \lambda_\nu \leq 0$, and 1.3 c) shows that f must be isotrivial.

The proof of b) uses the Kodaira-Spencer map. Recall that there is an exact sequence of sheaves of logarithmic differential forms (see [5] and [6], for example)

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0. \tag{1.4.1}$$

Its dual is the tautological sequence of tangent sheaves

$$0 \to T_{X/Y}(-\log \Delta) \to T_X(-\log \Delta) \to f^*(-\log \Delta) \to 0.$$

If f is non-isotrivial, then the edge morphism

$$\gamma : T_Y(-\log \Delta) = f_*f^*T_Y(-\log \Delta) \longrightarrow R^1 f_* T_{X/Y}(-\log \Delta) \tag{1.4.2}$$

is non-zero. The non-trivial deformation of a fibre $F = f^{-1}(x)$ in general position, given by f, induces a non-zero element $\epsilon \in H^1(F, T_F)$. Restricting γ to the residue field $\mathbb{C}(x)$, one obtains a morphism

$$\mathbb{C}(x) \cong T_Y(-\log \Delta) \otimes \mathbb{C} \longrightarrow H^1(F, T_F),$$

whose image is the subspace generated by ϵ.

By Serre-duality we obtain a non-trivial morphism

$$\gamma^\vee : f_* \omega_{X/Y} \otimes \Omega^1_{X/Y}(\log \Delta) \longrightarrow \Omega^1_Y(\log S) = \omega_Y(S).$$

Comparing the determinants of the three sheaves in (1.4.1) one finds

$$\Omega^1_{X/Y}(\log \Delta) = \omega_{X/Y}(\Delta_{\text{red}} - f^* S).$$

In particular, if $D \leq S$ denotes the divisor given by those $y \in Y$ with $f^{-1}(y)$ non-reduced, one obtains non-trivial morphisms

$$f_* \omega^2_{X/Y} \longrightarrow \omega_Y(S + D),$$

and

$$S^{e_2 \cdot r_2}(f_* \omega^2_{X/Y}) \otimes \lambda_2^{-1} \longrightarrow \omega_Y(S + D)^{e_2 \cdot r_2} \otimes \lambda_2^{-1}.$$

By 1.3, b), the sheaf on the left hand side is nef, hence

$$\deg(\omega_Y(S + D)^{e_2 \cdot r_2} \otimes \lambda_2^{-1}) = e_2 \cdot r_2 \cdot (2g - 2 + s + \delta) - \deg \lambda_2 \geq 0.$$

□

2. Moduli of Curves

To understand the relation between 1.4 and the Parshin-Arakelov finiteness theorem (I), stated in the introduction, one has to consider moduli. Recall that a reduced projective curve C is called stable, if the singularities of C are normal crossings, and if ω_C is ample.

Theorem 2.1 ((Mumford [23])). *For $g \geq 2$, define*

$$\bar{\mathcal{M}}_g(\mathbb{C}) = \{ \text{ stable curves of genus } g, \text{ defined over } \mathbb{C}\} / \cong .$$

Then there exists a quasi-projective coarse moduli scheme $\bar{M}_g$ for $\bar{\mathcal{M}}_g$, of dimension $3g - 3$. i.e. a variety $\bar{M}_g$ and a natural bijection $\bar{\mathcal{M}}_g(\mathbb{C}) \cong \bar{M}_g(\mathbb{C})$ where $\bar{M}_g(\mathbb{C})$ denotes the $\mathbb{C}$-valued points of $\bar{M}_g$.

We will not recall the definition of a coarse moduli scheme. Let us just remark that "natural" means, that for each flat morphisms $g : \mathcal{X} \to Z$, whose fibers $g^{-1}(z)$ belong to $\bar{\mathcal{M}}_g(\mathbb{C})$, the induced map $Z(\mathbb{C}) \to \bar{M}_g(\mathbb{C})$ comes from a morphism of schemes $\phi : Z \to \bar{M}_g$. It follows from the construction of $\bar{M}_g$, that for all $\nu > 0$ and for some $p \gg \nu$ there exists an invertible sheaf $\lambda_\nu^{(p)}$, such that for all families $g : \mathcal{X} \to Z$,

$$\det(g_* \omega^\nu_{\mathcal{X}/Z})^p = \phi^*(\lambda_\nu^{(p)}).$$

Addendum 2.2 ((Mumford [23])). *For ν, μ and p sufficiently large and divisible, for*

$$\alpha = (2g - 2) \cdot \nu - (g - 1) \quad \text{and} \quad \beta = (2g - 2) \cdot \nu \cdot \mu - (g - 1)$$

the sheaf $\lambda_{\nu \cdot \mu}^{(p)^\alpha} \otimes \lambda_\nu^{(p)^{-\beta \cdot \mu}}$ is ample.

The moduli scheme $\bar{M}_g$ is normal, connected and reduced. The subscheme M_g, corresponding to non-singular curves of genus g, is open in $\bar{M}_g$.

For a flat family of stable curves, the arguments used in the next section for Y a curve, carry over to show some positivity properties for families over higher dimensional bases.

Definition 2.3. Let Z be a quasi-projective variety and $\mathcal{E}$ a locally free sheaf on Z. Let $U \subset Z$ be an open dense subset, and let $\mathcal{H}$ be an ample invertible sheaf on Z.

a) $\mathcal{E}$ is called nef, if for all morphisms $\pi : C \to Z$, from a curve C to Z, and for all invertible quotients $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{N}$ one has deg $\mathcal{N} \geq 0$.
b) $\mathcal{E}$ is called ample with respect to U, if for some $\eta > 0$ there exists a morphism
$$\bigoplus \mathcal{H} \longrightarrow S^\eta(\mathcal{E}),$$
surjective over U.
c) $\mathcal{E}$ is weakly positive over U, if for all $\alpha > 0$ the sheaf $S^\alpha(\mathcal{E}) \otimes \mathcal{H}$ is ample with respect to U.

If is quite easy to see that on a projective variety Z the sheaf $\mathcal{E}$ is nef, if and only if $\mathcal{E}$ is weakly positive with respect to Z.

Sometimes it is of help, to extend the definition of "weak positivity" to torsion free coherent sheaves $\mathcal{F}$. To this aim, let $\iota : V \to Z$ be the largest open subscheme with $\mathcal{F}|_V$ locally free. For an open subscheme $U \subset V$ one calls $\mathcal{F}$ weakly positive with respect to U if $\mathcal{F}|_V$ has this property.

Part of theorem 1.3 generalizes to families of stable curves over a higher dimensional base. For example, the arguments explained in the next two sections, can also be used to prove:

Proposition 2.4. *Let $g : \mathcal{X} \to Z$ be a flat morphism, all of whose fibres are stable curves and whose general fibre is non-singular. Then*

a) *$g_*\omega^\nu_{\mathcal{X}/Z}$ is locally free and nef, for all $\nu > 0$.*
b) *If $\lambda_\eta = \det(g_*\omega^\eta_{\mathcal{X}/Z})$ is ample with respect to $U \subset Z$, for some $\eta > 0$, then $g_*\omega^\nu_{\mathcal{X}/Z}$ is ample with respect to U, for all $\nu > 1$.*

The moduli schemes $\bar{M}_g$ have finite coverings $\phi : Z \to \bar{M}_g$ which carry a universal family $g : \mathcal{X} \to Z$. For example, one could use "level μ-structures", or one can apply the construction of Kollár-Seshadri ([15] or [32], §9). 2.4, a), implies in particular that $\lambda_\eta = \det g_*\omega^\eta_{\mathcal{X}/Z}$ is nef. By 2.2 one finds $\phi^*\lambda^{(p)}_\nu$ to be ample for all $\nu \gg 0$, and using 2.4, b), one obtains the same for all $\nu > 1$. So $\lambda^{(p)}_\nu$ are other ample sheaves on the moduli spaces. Arakelov obtained the same result in [2], by using Siegel modular forms. The ampleness of $\lambda^{(p)}_2$, together with the upper bound for the degree of the pullback of $\lambda^{(p)}_2$ in 1.4 imply:

Corollary 2.5. *Let Y be a curve and $S \subset Y$ a finite subset. The non-trivial morphisms $\varphi : Y \to \bar{M}_g$, with $\varphi^{-1}(\bar{M}_g) \subset S$ and which are induced by a flat family $f : X \to Y$ of stable curves, are parameterized by a scheme of finite type.*

The necessary arguments, needed to deduce 2.5 from 1.4 will be sketched in the last section, when we consider families of surfaces. A more precise description of the parameter scheme is given in [4].

The second part of 2.4 allows to generalize the "$2g-2+s>0$"-part of 1.4:

Proposition 2.6. *Let Z be a projective manifold, S a normal crossing divisor and $Z_0 = Z - S$. Let $\varphi_0 : Z_0 \to M_g$ be a morphism, étale over its image, and assume that φ_0 is induced by a smooth family of curves $f_0 : \mathcal{X}_0 \to Z_0$, which extends to a morphism, $f : \mathcal{X} \to Z$, semistable over the general points of the components of S. Then $\Omega^1_Z(\log S)$ is ample with respect to Z_0.*

Proof. In order to prove 2.6 we may replace Z by the complement of a closed subset of codimension two. Hence we may assume f to be flat with semistable fibres, and that $\Delta = f^*S$ is a normal crossing divisor.

As in the proof of 1.4, one has a tautological sequence

$$0 \to f^*\Omega^1_Z(\log S) \to \Omega^1_{\mathcal{X}}(\log \Delta) \to \Omega^1_{\mathcal{X}/Z}(\log \Delta) \to 0$$

and its dual

$$0 \to T_{\mathcal{X}/Z}(-\log \Delta) \to T_{\mathcal{X}}(-\log \Delta) \to f^*T_Z(\log S) \to 0.$$

The edge morphism

$$T_y(-\log S) \longrightarrow R^1 f_* T_{\mathcal{X}/Z}(-\log \Delta)$$

locally splits over Z_0, and by Serre duality one obtains a morphism

$$f_*\omega^2_{\mathcal{X}/Z} \longrightarrow \Omega^1_Z(\log S),$$

surjective over Z_0.

φ_0 extends to a morphism $\varphi : Z \to \bar{M}_g$, and $\lambda^p_\eta = \det(f_*\omega^\eta_{\mathcal{X}/Z})^p$ is ample, with respect to Z_0. By 2.4, $f_*\omega^2_{\mathcal{X}/Z}$ is ample with respect to Z_0, hence $\Omega^1_Z(\log S)$, as well. □

2.6 implies the "rigidity property" (R), stated in the introduction, for family of curves. In fact, if for some curve T_0 there exists a smooth family of curves

$$\mathcal{X}_0 \longrightarrow Z_0 = T_0 \times Y_0,$$

then the induced morphism $Z_0 \to M_g$ can not be generically finite. Otherwise, replacing Y_0 and T_0 by some covering, one would find for compactifications T and Y of T_0 and Y_0 the sheaf

$$\Omega^1_{T\times Y}(\log (T \times (Y - Y_0) + (T - T_0) \times Y)) = pr_1^*\Omega^1_T(\log (T - T_0)) \oplus pr_2^*\Omega^1_Y(\log (Y - Y_0))$$

to be ample over some dense open subset. Obviously this can not be true.

3. Positivity of Direct Images

We will prove 1.3, a) in this section. Since the arguments remain more or less the same, we will not assume that the fibres of f are curves, and we will consider arbitrary morphisms $f : X \to Y$ from a projective manifold to a curve Y. We will assume that f is smooth over $Y_0 = Y - S$, and that $f^*(S)$ is a normal crossing divisor.

As a start up, consider a finite covering $\tau : Y' \to Y$, with Y' non-singular. If τ is étale over a neighborhood of S, the fibre product $X' \times_Y Y'$ is nonsingular, and by flat base change $\tau^* f_* \omega^\nu_{X/Y} = f'_* \omega^\nu_{X'/Y'}$. Otherwise, consider the commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{\ \sigma\ } & Z & \xrightarrow{\ \gamma\ } & X \times_Y Y' & \xrightarrow{\ pr_1\ } & X \\ & \searrow^{f'} & \downarrow & & \downarrow & \swarrow^{f} & \\ & & Y' & \xrightarrow{\ \tau\ } & Y, & & \end{array}$$

where γ is the normalization and σ a desingularization. As a pullback of a flat Gorenstein morphism, $pr_2 : X \times_Y Y' \to Y'$ is Gorenstein, and by duality for the finite morphism γ one obtains a map $\gamma_* \omega_{Z/Y'} \to \omega_{X\times_Y Y'/Y'}$. Altogether one finds

$$\sigma_* \omega_{X'/Y'} \longrightarrow \omega_{Z/Y'} \quad \text{and} \tag{3.0.1}$$

$$\gamma_* \sigma_* \omega_{X'/Y'} \longrightarrow \gamma_* \omega_{Z/Y'} \longrightarrow \omega_{X\times_Y Y'/Y'} = pr_1^* \omega_{X/Y}. \tag{3.0.2}$$

Lemma 3.1. *Keeping the notations from above, for all* $\nu > 0$ *one has a natural inclusion* $f'_* \omega^\nu_{X'/Y'} \to \tau^* f_* \omega^\nu_{X/Y}$. *Moreover, if* f *is semistable, both sheaves are isomorphic.*

Proof. $\omega_{Z/Y'}$ is a reflexive sheaf. The natural map

$$\gamma^* \gamma_* \omega_{Z/Y'} /_{\text{torsion}} \longrightarrow \gamma^* \omega_{X\times_Y Y'/Y'}$$

factors through an inclusion $\omega_{Z/Y'} \to \gamma^* \omega_{X\times_Y Y'/Y'}$. One finds for the reflexive hull $\omega^{[\nu-1]}_{Z/Y'}$

$$\sigma_* \omega^\nu_{X'/Y'} \longrightarrow \omega^{[\nu-1]}_{Z/Y'} \otimes \omega_{Z/Y'} \longrightarrow \gamma^* \omega^{\nu-1}_{X\times_Y Y'/Y'} \otimes \omega_{Z/Y'}$$

and by (3.0.1)

$$\gamma_* \sigma_* \omega^\nu_{X'/Y'} \longrightarrow \omega^{\nu-1}_{X\times_Y Y'/Y'} \otimes \gamma_* \omega_{Z/Y'} \longrightarrow \omega^\nu_{X\times_Y Y'/Y'}.$$

The first part of 3.1 follows by flat base change. If f is semistable, $X \times_Y Y'$ is non-singular in codimension one, and Gorenstein, hence normal. Moreover the discriminant of pr_1 is a normal crossing divisor, and $Z \cong X \times_Y Y'$ has at most rational double points. So

$$\sigma_* \omega^\nu_{X'/Y'} = \omega^\nu_{Z/Y'} = \omega^\nu_{X\times_Y Y'/Y'}$$

and by flat base change $f'_* \omega^\nu_{X'/Y'} \cong \tau^* f_* \omega^\nu_{X/Y}$. □

In the special case, that f is a finite morphism, one can choose Y' to be the normalization of Y in the Galois hull of $\mathbb{C}(X)$ over $\mathbb{C}(Y)$. Then X' is the disjoint union of copies of Y' and one obtains the wellknown

Corollary 3.2. *If* $f : X \to Y$ *is a finite morphism, then* $f_* \omega^\nu_{X/Y}$ *is nef.*

The same argument shows, that for a morphism $f : X \to Y$ with non connected fibres and with Stein factorization $g : X \to Z$, the sheaves $f_* \omega^\nu_{X/Y}$ are nef, if the same holds true for $g_* \omega^\nu_{X/Z}$.

For $\nu = 1$, Fujita's proof in [7] of the analogue of 1.3 a), for families of higher dimensional manifolds, used variations of Hodge structures (see 5.6).

Kawamata in [11] used similar tools for his generalization to higher dimensional bases. Kollár realized, that both, Fujita's and Kawamata's result, are corollaries of his vanishing theorem [13]. We present here a variant of his proof.

Once one knows that $f_*\omega^\nu_{X/Y}$ is nef, for $\nu = 1$, one obtains the same for all $\nu > 0$ by considering cyclic coverings. In this section those coverings will not appear, but they are hidden behind the slight generalization of Kollárs vanishing, formulated in 3.3. Its proof can be found in [6], 5.12 a), for example.

Let $\mathcal{L}$ be an invertible sheaf on X. We consider in addition a normal crossing divisor $D = \sum_{i=1}^r \alpha_j D_j$ on X, and some $N \in \mathbb{N}-\{0\}$. Define $\mathcal{L}^{(1)} = \mathcal{L}\left(-\left[\frac{D}{N}\right]\right)$.

Theorem 3.3. *Assume that the sheaf $\mathcal{L}^N(-D)$ is semi-ample. Let B be an effective divisor with*

$$H^0(X, (\mathcal{L}^N(-D))^\nu \otimes \mathcal{O}_X(-B)) \neq 0,$$

for some $\nu > 0$. Then the adjunction maps

$$H^i(X, \mathcal{L}^{(1)} \otimes \omega_X(B)) \longrightarrow H^i(B, \mathcal{L}^{(1)} \otimes \omega_B)$$

are surjective, for all i.

Corollary 3.4. *Let $f : X \to Y$ be a surjective morphism to a non-singular curve. If $\mathcal{L}^N(-D)$ is semi-ample, then $f_*\omega_{X/Y} \otimes \mathcal{L}^{(1)}$ is nef.*

Proof. Replacing $\mathcal{L}$ by $\mathcal{L}\left(-\left[\frac{D}{N}\right]\right)$ and D by $D - N\cdot\left[\frac{D}{N}\right]$ we may assume that $\mathcal{L}^{(1)} = \mathcal{L}$. For a point $y \in Y$ write $F = f^{-1}(y)$ and $\mathcal{L}' = \mathcal{L}(F)$. The sheaf ${\mathcal{L}'}^N(-D)$ is again semi-ample and obviously for $B = F$, the assumptions made in 3.3 are satisfied. One obtains a surjection

$$H^0(X, \mathcal{L}' \otimes \omega_X(F)) = H^0(Y, f_*(\omega_{X/Y} \otimes \mathcal{L}) \otimes \omega_Y(2\cdot y)) \to H^0(F, \mathcal{L}(F) \otimes \omega_F)$$

and thereby

Claim 3.5. Under the assumptions made in 3.4 the sheaf

$$f_*(\omega_{X/Y} \otimes \mathcal{L}) \otimes \omega_Y(2\cdot y)$$

is generated by global section in y.

Consider the r-fold product

$$f^r : X^r = X \times_Y \ldots \times_Y X \longrightarrow Y,$$

the sheaf $\mathcal{M} = \bigotimes_{i=1}^r pr_i^*\mathcal{L}$ and the divisor $\Gamma = \sum_{i=1}^r pr_i^* D$. We choose a desingularization $\sigma : X^{(r)} \to X^r$ such that $\Gamma' = \sigma^*\Gamma$ is a normal crossing divisor. On $X^{(r)}$ the sheaf $\mathcal{M}' = \sigma^*\mathcal{M}$ and the divisor Γ' again satisfy the assumptions made, and 3.5 implies that

$$(f^r \circ \sigma)_*(\omega_{X^{(r)}/Y} \otimes {\mathcal{M}'}^{(1)}) \otimes \omega_Y(2\cdot y)$$

is generated by global sections in y. The morphism f^r is flat and Gorenstein and $\omega_{X^r/Y} = \bigotimes_{i=1}^r pr_i^*\omega_{X/Y}$. As in (3.0.1) one finds an inclusion $\sigma_*\omega_{X^{(r)}/Y} \subset \omega_{X^r/Y}$ and flat base change induces

$$(f^r \circ \sigma)_*(\omega_{X^{(r)}/Y} \otimes {\mathcal{M}'}^{(1)}) \subset (f^r \circ \sigma)_*(\omega_{X^{(r)}/Y} \otimes \mathcal{M}') \subset$$
$$f^r_*\omega_{X^r/Y} \otimes \mathcal{M} = \bigotimes^r f_*\omega_{X/Y} \otimes \mathcal{L}.$$

Since those sheaves are all isomorphic over some neighborhood of y,

$$(\bigotimes^r f_*\omega_{X/Y} \otimes \mathcal{L}) \otimes \omega_Y(2 \cdot y)$$

is again generated by global sections in a neighborhood of y.

If $\tau : Y' \to Y$ is a finite covering of degree d, and if $\mathcal{N}$ is an invertible quotient of $\tau^*(f_*\omega_{X/Y} \otimes \mathcal{L})$, one finds

$$r \cdot \deg(\mathcal{N}) + d \cdot 2 \cdot g(Y) \geq 0,$$

for all $r > 0$, which implies $\deg(\mathcal{N}) \geq 0$. □

Corollary 3.6. *Let $f : X \to Y$ be a surjective morphism from a projective manifold X to a curve Y. Then for all $\nu > 0$ the sheaf $f_*\omega^\nu_{X/Y}$ is nef.*

Proof. For some $y \in Y$ define

$$\mu = \text{Min } \{\eta > 0, f_*\omega^\nu_{X/Y} \otimes \mathcal{O}_Y((\eta \cdot \nu - 1) \cdot y) \ \text{ nef}\}$$

Then $f_*\omega^\nu_{X/Y} \otimes \mathcal{O}_Y(\mu \cdot \nu \cdot y)$ is ample. Blowing up X, the image of

$$f^* f_*\omega^\nu_{X/Y} \longrightarrow \omega^\nu_{X/Y}$$

is of the form $\omega^\nu_{X/Y}(-D)$, for a normal crossing divisor D. For

$$\mathcal{L} = \omega^{\nu-1}_{X/Y} \otimes f^*\mathcal{O}_Y(\mu \cdot (\nu - 1) \cdot y)$$

the sheaf $\mathcal{L}^\nu(-(\nu - 1) \cdot D)$ is semi-ample and 3.4 implies that

$$f_*\omega_{X/Y} \otimes \mathcal{L}^{(1)} = f_*\omega^\nu_{X/Y}\big(- [\frac{(\nu - 1) \cdot D}{\nu}]\big) \otimes \mathcal{O}_Y(\mu \cdot (\nu - 1) \cdot y) = f_*\omega^\nu_{X/Y} \otimes \mathcal{O}_Y(\mu \cdot (\nu - 1) \cdot y)$$

is nef. By the choice of μ this is only possible if $\mu \cdot (\nu - 1) > (\mu - 1) \cdot \nu - 1$ or, equivalently, if $\mu \leq \nu$. If $\tau : Y' \to Y$ is any finite covering of degree δ, whose ramification divisor lies outside of the discriminant of f, then $X' = X \times_Y Y'$ is again a manifold and, for $f' : X' \to Y'$ and $y' \in Y'$ one finds

$$f'_*\omega^\nu_{X'/Y'} \otimes \mathcal{O}_{Y'}((\nu^2 - 1) \cdot y') = \tau^*(f_*\omega^\nu_{X/Y}) \otimes \mathcal{O}_{Y'}((\nu^2 - 1) \cdot y')$$

to be nef. 3.6 follows from 1.2. □

Similar arguments apply to arbitrary morphisms $g : \mathcal{X} \to Z$. However, for $\dim(Z) > 0$ one can no longer assume the sheaves $g_*\omega^\nu_{\mathcal{X}/Z}$ to be locally free. Moreover, it is no longer true that a locally free sheaf which contains a nef subsheaf of the same rank, is again nef. For this reason, one only obtains that for some open dense subset $U \subset Z$, the sheaf $g_*\omega^\nu_{\mathcal{X}/Z}$ is weakly positive over U.

If $g : \mathcal{X} \to Z$ is flat, if all fibres of g are reduced normal crossing divisors (or stable curves) the product $\mathcal{X} \times_Z \ldots \times_Z \mathcal{X}$ is normal with at most rational singularities. This allows to prove analogues of 3.4 and 3.6 over Z.

Let us end this section with a simple example, indicating how 3.4 together with base change can be used to obtain stronger positivity results. The idea, which will be further exploited in the next section, is quite simple. If the divisor D in 3.4 contains a general fibre, we can replace Y by a finite covering, such that the multiplicity of the divisor becomes arbitrarily large. Hence this divisor will survive, if one is taking integral parts.

Corollary 3.7. *Let $f : X \to Y$ be a morphism between a manifold X and a curve Y, and let $\mathcal{L}$ be an invertible sheaf on X. If $\mathcal{L}$ is nef and big, then $f_*(\mathcal{L} \otimes \omega_{X/Y})$ is ample.*

Proof. Let $y \in Y$ be a point in general position and $F = f^{-1}(y)$. For $\mu \gg 1$ one finds an ample invertible subsheaf $\mathcal{H}$ of $\mathcal{L}^\mu$. Replacing N and $\mathcal{H}$ by some multiple, we may assume that $\mathcal{H}(-F)$ is ample, as well. Hence, for some divisor $\Gamma > 0$, and for all $N > \mu$, the sheaf $\mathcal{L}^N(-\Gamma - F)$ is ample, hence in particular semi-ample. The latter remains true, if one replaces X by a blowing up. So one is allowed to assume that $\Gamma + F$ is a normal crossing divisor, and that the multiplicities of the components of Γ are strictly smaller than N.

Let $\tau : Y' \to Y$ be a finite covering of degree N, étale over a neighborhood of S and totally ramified in y, and let $f' : X' \to Y'$ be the pullback family. Then, for $y' = \tau^{-1}(y)$, for $F' = f'^{-1}(y')$ and for the pullbacks $\mathcal{L}'$ of $\mathcal{L}$ and Γ' of Γ, the sheaf $\mathcal{L}^N(-\Gamma' - N \cdot F')$ is semi-ample. Applying 3.4, one obtains that $f'_*(\omega_{X'/Y'} \otimes \mathcal{L}'(-F'))$ is nef, hence $f'_*(\omega_{X'/Y'} \otimes \mathcal{L}')$ must be ample. 3.7 follows by flat base change. □

4. Effective Bounds for the Positivity of Direct Images

As for the positivity results in the last section, there is little additional effort to formulate and proof 1.3 b), directly for families of higher dimensional manifolds over curves. So we consider in this section again a surjective morphism $f : X \to Y$ from a projective manifold to a curve Y, smooth outside of a normal crossing divisor $\Delta = f^*(S)$. However, for simplicity we will assume that the canonical sheaf of the general fibre is semi-ample. The formulation and a proof without this assumption can be found in [33].

First recall the definition of the (algebraic) multiplier sheaves. We consider a surjective morphism $f : X \to Y$, with connected general fibre F, where X is an (n+1)-dimensional complex projective manifold, and Y a non-singular projective curve. If Γ is an effective divisor on X,

$$\omega_{X/Y}\big\{ - \frac{\Gamma}{N}\big\} = \tau_*\big(\omega_{X'/Y}\big(- \big[\frac{\Gamma'}{N}\big]\big)\big)$$

where $\tau : X' \to X$ is any blowing up with $\Gamma' = \tau^*\Gamma$ a normal crossing divisor. One easily finds this definition to be independent of τ. Moreover, for $i > 0$, the higher direct images

$$R^i\tau_*\big(\omega_{X'/Y}\big(- \big[\frac{\Gamma'}{N}\big]\big)\big) = 0 \tag{4.0.1}$$

(see for example [6], 7.4, or [32], section 5.3). The corollary 3.4 implies:

Lemma 4.1. *Let $\mathcal{N}$ be an invertible sheaf on X and let Γ be an effective divisor. Assume that for some $N > 0$ there exists a nef locally free sheaf $\mathcal{E}$ on Y and a surjection $f^*\mathcal{E} \to \mathcal{N}^N(-\Gamma)$. Then*

$$f_*\big(\mathcal{N} \otimes \omega_{X/Y}\big\{ - \frac{\Gamma}{N}\big\}\big)$$

is nef.

Proof. Let $p \in Y$ be a point. Then $\mathcal{E} \otimes \mathcal{O}_Y(N \cdot p)$ is ample, hence

$$\mathcal{N}^N(-\Gamma) \otimes f^*\mathcal{O}_Y(N \cdot p)$$

is semi-ample. 3.4 implies that the sheaf

$$f_*\big(\mathcal{N}\otimes\omega_{X/Y}\{-\frac{\Gamma}{N}\}\big)\otimes\mathcal{O}_Y(p)$$

is nef. Since the same holds true over all Y', finite over Y and unramified in S, one obtains 4.1 from 1.2 □

In 1.3 b), the constant e was given explicitly. For higher dimensional fibres recall a definition, given in [5], [6], § 7 and [32], section 5.3.

Definition 4.2. Let $\mathcal{L}$ be an invertible sheaf on a quasi-projective manifold Z with $H^0(Z,\mathcal{L})\neq 0$, and let Γ be an effective divisor. Then

$$e(\Gamma)=\text{Min}\left\{N\in\mathbb{N}-\{0\};\ \omega_Z\left\{-\frac{\Gamma}{N}\right\}=\omega_Z\right\}\qquad\text{and}$$

$$e(\mathcal{L})=\text{Sup}\left\{e(\Gamma);\ \Gamma\text{ the zero set of }\sigma\in H^0(Z,\mathcal{L})-\{0\}\right\}.$$

If Z is projective, then $e(\mathcal{L})<\infty$. If $\mathcal{L}$ is ample, and if $\mathcal{L}^\alpha$ is very ample, then

$$e(\mathcal{L})\leq\alpha^{\dim Z-1}\cdot c_1(\mathcal{L})^{\dim Z}+1.$$

We will use two properties of $e(\mathcal{L})$, shown in [5] (see also [6], § 7 or [32], section 5.4).

Lemma 4.3.

a) *$e(\mathcal{L})$ is upper semi-continuous. Moreover, if Γ is an effective divisor on X, and if $F=f^{-1}(y)$ is not contained in Γ, then there exists a neighborhood V of y with $e(\Gamma|_{f^{-1}(V)})\leq e(\Gamma|_F)$.*

b) *If F is projective, and $\mathcal{L}$ an invertible sheaf on F, then for $Z=F\times\ldots\times F$ and $\mathcal{M}=\bigotimes_{i=1}^r pr_i^*\mathcal{L}$ one has $e(\mathcal{M})=e(\mathcal{L})$.*

To illustrate the use of multiplier ideals, let us start with the following proposition, which will not be used in the sequel.

Proposition 4.4. *Let Γ be an effective divisor on X and let $y\in Y$ be a point in general position and $F=f^{-1}(y)$. Assume that ω_F is semi-ample, and that $\mathcal{O}_X(\Gamma)=\omega_{X/Y}^\eta\otimes f^*\mathcal{O}_Y(-N\cdot y)$, for some $N>0$. Then, for $\nu>1$ and $e=\text{Max}\{e((\nu-1)\cdot\Gamma),\eta\}$,*

$$f_*\omega_{X/Y}^\nu\succeq\frac{(\nu-1)\cdot N}{e}\cdot\mathcal{O}_Y(y).$$

Proof. In order to prove 4.4 we may replace Y by some finite covering, étale over a neighborhood of S, and totally ramified of order e in y, and $f:X\to Y$ by $pr_2:X\times_Y Y'$. Thereby we are allowed to assume that N is divisible by e. One has, for all $\mu>0$,

$$\omega_{X/Y}^{(\nu-1)\cdot e\cdot\mu}(-\mu\cdot(\nu-1)\cdot\Gamma-\mu\cdot(\nu-1)\cdot N\cdot F)=\omega_{X/Y}^{\mu\cdot(\nu-1)\cdot(e-\eta)}.$$

If $e=\eta$ we choose $\mu=1$, otherwise we choose μ such that $\omega_F^{\mu\cdot(\nu-1)\cdot(e-\eta)}$ is generated by global sections. In both cases 4.1 implies that for some divisor Γ' supported in fibres,

$$f_*\omega_{X/Y}^{\nu-1}\otimes\omega_{X/Y}\{-\frac{(\nu-1)\cdot\Gamma+(\nu-1)\cdot N\cdot F+\Gamma'}{e}\}$$

is weakly positive. The choice of e and 4.3, b), imply that

$$\omega_{X/Y}\{-\frac{(\nu-1)\cdot\Gamma+(\nu-1)\cdot N\cdot F}{e}\}|_{f^{-1}(V)}=\omega_{X/Y}|_{f^{-1}(V)},$$

for some neighborhood V of y. Hence

$$f_*\omega^{\nu}_{X/Y}\otimes\mathcal{O}_Y\big(-\frac{(\nu-1)\cdot N}{e}\cdot y\big)$$

contains a nef subsheaf of full rank, hence it is nef itself. □

In 4.4 we require $f_*\omega^{\eta}_{X/Y}$ to contain an ample invertible subsheaf. In general, since we have no control on the degree of such subsheaves, we want to use the determinant λ_η as a subsheaf of $\bigotimes^r f_*\omega^{\eta}_{X/Y}$. This only defines a divisor on the r-th fibre product, and the property b) in 4.3 is needed.

Proposition 4.5. *For $f:X\to Y$, choose $\eta>0$ with $f_*\omega^{\eta}_{X/Y}\neq 0$ and*

$e=\mathrm{Max}(\eta+1,e(\omega^{\eta}_F))$
$r=\mathrm{rank}(f_*\omega^{\eta}_{X/Y})$
$\lambda=\det(f_*\omega^{\eta}_{X/Y})$.

If λ is ample, then for all $\nu>1$ with $f_*\omega^{\nu}_{X/Y}\neq 0$ one finds $f_*\omega^{\nu}_{X/Y}\succeq\frac{1}{r\cdot e}\cdot\lambda$.

Proof. For some $\mu\gg 1$ there exists an effective divisor Σ_1, disjoint from S with $\lambda^{\mu}=\mathcal{O}_Y(\Sigma_1)$. By 1.2 and by flat base change, we are free to replace Y by any Y', finite over Y and unramified over a neighborhood of S. Hence we are allowed to assume that $\Sigma_1=e\cdot\mu\cdot(\nu-1)\cdot\Sigma$ or that

$$\lambda=\mathcal{O}_Y(e\cdot(\nu-1)\cdot\Sigma).$$

As in the proof of 3.5 consider the r-fold fibre product

$$f^r:X^r=X\times_Y X\ldots\times_Y X\longrightarrow Y,$$

and a desingularization $\sigma:X^{(r)}\to X^r$. Using flat base change, and the natural maps

$$\mathcal{O}_{X^r}\longrightarrow\sigma_*\mathcal{O}_{X^{(r)}}\quad\text{and}\quad\sigma_*\omega_{X^{(r)}}\longrightarrow\omega_{X^r},$$

one finds, for all $\mu>0$, morphisms

$$\bigotimes^r f_*\omega^{\mu}_{X/Y}\longrightarrow f^{(r)}_*\sigma^*(\omega^{\mu}_{X^r/Y})\quad\text{and}\tag{4.5.1}$$

$$f^{(r)}_*\sigma^*(\omega^{\nu-1}_{X^r/Y}\otimes\omega_{X^{(r)}/Y})\longrightarrow f^r_*(\omega^{\nu}_{X^r/Y})=\bigotimes^r f_*\omega^{\nu}_{X/Y},\tag{4.5.2}$$

and both are isomorphism over some open dense subset of Y. In particular, since $\lambda\subset\bigotimes^r f_*\omega^{\eta}_{X/Y}$, the sheaf $f^{(r)^*}\lambda$ is a subsheaf of $\sigma^*\omega^{\eta}_{X^r/Y}$. Let Γ be the zero divisor of the corresponding section of

$$f^{(r)*}\lambda^{-1}\otimes\sigma^*\omega^{\eta}_{X^r/Y}\quad\text{hence}\quad\mathcal{O}_{X^{(r)}}(-\Gamma)=f^{(r)*}\lambda\otimes\sigma^*\omega^{-\eta}_{X^r/Y}.$$

For the sheaf

$$\mathcal{M}=\sigma^*(\omega_{X^r/Y}\otimes\mathcal{O}_{X^r}(-f^{r*}\Sigma))$$

one finds

$$\mathcal{M}^{e\cdot(\nu-1)}(-\Gamma)=\sigma^*\omega^{e\cdot(\nu-1)}_{X^r/Y}\otimes f^{(r)*}\lambda^{-1}\otimes\mathcal{O}_{X^{(r)}}(-\Gamma)=\sigma^*\omega^{e\cdot(\nu-1)-\eta}_{X^r/Y}.$$

So for $\mu \gg 1$ the sheaf $\mathcal{M}^{\mu\cdot e\cdot(\nu-1)}(-\mu\cdot\Gamma)|_F$ will be generated by global sections and (4.5.1) allows to find an effective divisor Γ', supported in the fibres of $f^{(r)}$, and a surjection

$$\bigotimes^r f_*\omega_{X/Y}^{\mu\cdot(e\cdot(\nu-1)-\eta)} \longrightarrow \mathcal{M}^{\mu\cdot e\cdot(\nu-1)}(-\mu\cdot\Gamma-\Gamma').$$

By 4.1 we find

$$f_*^{(r)}\mathcal{M}^{\nu-1}\otimes\omega_{X^{(r)}/Y}\big\{-\frac{\mu\cdot\Gamma+\Gamma'}{\mu\cdot e}\big\}$$

to be nef. By the choice of e,

$$\omega_F\big\{-\frac{(\mu\cdot\Gamma+\Gamma')|_F}{\mu\cdot e}\big\}=\omega_F$$

and by (4.5.2) the sheaf $\mathcal{O}_Y(-(\nu-1)\cdot\Sigma)\otimes\bigotimes^r f_*\omega^\nu_{X/Y}$ contains a nef subsheaf of full rank, hence it is nef itself, as well as its e-th tensor power $\lambda^{-1}\otimes\bigotimes^{r\cdot e} f_*\omega^\nu_{X/Y}$. Since the symmetric product is a quotient of the tensor product, we obtain 4.5. □

4.5 implies that the ampleness of λ_ν is equivalent to the ampleness of $f_*\omega^\nu_{X/Y}$, for all $\nu>1$ with $f_*\omega^\nu_{X/Y}\neq 0$. Forgetting about the explicit bounds, one obtains:

Proposition 4.6. *For a surjective morphism $f: X\to Y$ between a manifold X and a curve Y, with connected general fibre F, the following conditions are equivalent:*

i) *For all $\nu>1$ the sheaf $f_*\omega^\nu_{X/Y}$ is ample, if non-trivial.*

ii) *There exists some $\eta>0$ such that $f_*\omega^\eta_{X/Y}$ contains an ample subsheaf.*

iii) *There exists some $\eta>0$ such that* $\det f_*\omega^\eta_{X/Y}$ *is ample.*

If f is semistable, then the conditions i), ii) and iii) imply that

iv) *f is not isotrivial.*

If F is either of general type, or if it is birational to a good minimal model, then the conditions i) - iv) are equivalent.

Proof. i) implies ii), and ii) together with 3.6 implies iii). Hence by 4.5 the first three conditions are equivalent. Since f is semistable, the sheaves $f_*\omega^\nu_{X/Y}$ are compatible with pullbacks. Hence if f is isotrivial, $\deg f_*\omega^\eta_{X/Y}=0$. On the other hand, as mentioned in the introduction, the assumptions in 4.6 are just the ones, which guaranty that for f non isotrivial and for some ν, sufficiently large and divisible, λ_ν is ample (see [12], [14], [30], [31] or [22]). □

The implication iv) ⇒ iii) was shown in [30] for families with ω_F semi-ample and big, by using local Torelli theorems for cyclic coverings. In [12] Kawamata extended this method to the case "ω_F semi-ample". Kollár found a proof in [14] for all families with F of general type. Finally, [31] reproves Kollárs theorem by using a simple "universal bundle construction", a method which also allowed the construction of moduli. As one sees again, positivity, moduli and thereby Torelli theorems are closely related.

The following corollary, although not needed in the sequel, is another illustration of the relation between positivity of direct image sheaves, and the isotriviality of morphisms.

Corollary 4.7. *Assume that the general fibre F of the surjective morphism $f : X \to Y$ is either a of general type, or birational to a good minimal model. If there exists a generically finite morphism $\tau : Z \to X$ with $f \circ \tau : Z \to Y$ isotrivial, then $f : X \to Y$ is isotrivial.*

Proof. We may assume both, f and $f\circ\tau$ to be semistable. The natural inclusion $\omega_{X/Y} \to \tau_*\omega_{Z/Y}$ induces an inclusion $f_*\omega^{\nu}_{X/Y} \to (f \circ \tau)_*\omega^{\nu}_{Z/Y}$, for all $\nu > 0$. Hence if f is not isotrivial, the condition ii) in 4.6 b) is satisfied for f'. □

5. Families of Higher Dimensional Manifolds over Curves

Let $f : X \to Y$ be a surjective morphism with connected general fibre F. We fix a reduced divisor S on Y which contains the discriminant divisor of f, i.e. a reduced divisor with

$$f_0 = f|_{X_0} : X_0 \longrightarrow Y_0$$

smooth, for $Y_0 = Y - S$ and $X_0 = f^{-1}(Y_0)$. Write $s = \deg(S)$, and $n = \dim(F)$. The genus of Y is again denoted by g, and δ is the number of singular fibres of f, which are not normal crossing divisors. The main result of [33] says:

Theorem 5.1. *Assume one of the following:*

i) *f is not isotrivial, and $\kappa(F) = \dim(F)$.*
ii) *f is not isotrivial, and F has a minimal model F' with $K_{F'}$ semi-ample.*
iii) *For all finite coverings $Y' \to Y$, étale over Y_0, for the family $f' : X' \to Y'$, obtained by desingularizing the pullback of X, and for some $\eta > 0$, the sheaf $\lambda_\eta = \det(f'_*\omega^{\eta}_{X'/Y'})$ is ample.*

Then

a) $2g - 2 + s > 0$.
b) *If $f_*\omega^{\nu}_{X/Y} \neq 0$, there exists a constant e, depending on ν and F, which is upper semi-continuous in families, with*

$$\deg(\lambda_\nu) \leq \text{rank}(f_*\omega^{\nu}_{X/Y}) \cdot \nu \cdot e \cdot (n \cdot (2g - 2 + s) + \delta).$$

In b) it is essential, that the constant e is upper semi-continuous. This will allow in the next section, to replace it by some constant, depending just on numerical invariants, whenever ω_F is semi-ample.

As in the proof of 1.4 it is easy to see that b) implies a). Moreover, by 4.6 the assumptions made in i) and ii) imply that iii) holds true. Before sketching the proof of b) for semistable families of surfaces of general type, let us give one application.

Corollary 5.2. *Let X be a manifold with $\kappa(X) \geq 0$, and let $f : X \to \mathbb{P}^1$ be a surjective morphism. Then f has at least three singular fibres.*

Proof. If not, we can assume that f is smooth over $\mathbb{P}^1 - \{0, \infty\}$. In particular, the Stein factorization of f is again a morphism $X \to \mathbb{P}^1$, with at most 2 singular fibres. Replacing $\mathbb{P}^1$ by some finite covering, we obtain a pullback family $f' : X' \to \mathbb{P}^1$, with two or less singular fibres, and with f' semistable. Since X' is a covering of X, its Kodaira dimension is again non-negative. One has

$$f'_*\omega^{\nu}_{X'/\mathbb{P}^1} = \bigoplus_{i=1}^{3(g-1)} \mathcal{O}_{\mathbb{P}^1}(\nu_i),$$

for some $\nu_1 \geq \ldots \geq \nu_r$. By 1.3 a), $\nu_r \geq 0$. Since $\kappa(X') \geq 0$, for some ν the sheaf $f'_*\omega^\nu_{X'}$ has a section, hence $\nu_1 \geq 2 \cdot \nu$. In particular $\deg \lambda_\nu > 0$, contradicting 5.1, b). □

Sketch of the proof of 5.1 b). We will restrict ourselves to the case of a semistable family of surfaces of general type, and we will assume that ω_F^ν is generated by global sections. As in the proof of 1.4 we can enlarge S and assume that $2g - 2 + s$ is non-negative.

If 5.1, b), does not hold, i.e. if for $r = \mathrm{rank}(f_*\omega^\nu_{X/Y})$ and for $e = \mathrm{Max}(\nu + 1, e(\omega^\nu_F))$

$$\deg(\lambda_\nu) > r \cdot \nu \cdot e \cdot 2 \cdot (2g - 2 + s), \tag{5.2.1}$$

we may replace Y by some finite cover, étale outside of S, and X by the pullback. Since we assumed f to be semistable, both sides of the inequality (5.2.1) are multiplied by the degree of the covering. We may assume thereby that the difference between both sides of (5.2.1) is strictly larger than $r \cdot \nu \cdot e \cdot 2$. By 4.5, one finds for an ample invertible sheaf $\mathcal{A}$ on Y:

$$\deg(\mathcal{A}) = 2 \cdot (2g - 2 + s + 1) \implies f_*\omega^\nu_{X/Y} \otimes \mathcal{A}^{-\nu} \text{ ample.} \tag{5.2.2}$$

For $\mu \gg 1$ the image of

$$f^*f_*\omega^{\mu\cdot\nu}_{X/Y} \longrightarrow \omega^{\mu\cdot\nu}_{X/Y}$$

is isomorphic to $\omega^{\mu\cdot\nu}_{X/Y} \otimes \mathcal{I}$ for some sheaf of ideals, with $\mathcal{O}_X/\mathcal{I}$ supported in the fibres. Let $\sigma : X' \to X$ be a blowing up with

$$\mathcal{O}_{X'}(-B_\mu) = \sigma^*\mathcal{I}/_{\text{torsion}}$$

for a normal crossing divisor B_μ. The semi-stability of f implies that

$$\Omega^2_{X/Y}(\log \Delta) = \omega_{X/Y}.$$

For some effective exceptional divisor E one obtains

$$\omega_{X'/Y}(-E) = \sigma^*\omega_{X/Y} = \sigma^*\Omega^2_{X/Y}(\log \Delta) \subset \Omega^2_{X'/Y}(\log \sigma^*(\Delta)).$$

For the divisor $B'_\mu = B_\mu + \mu \cdot E$ one finds

$$\omega^{(1)}_{X'/Y} := \omega_{X'/Y}\big(-\big[\frac{B'_\mu}{\mu}\big]\big) \subset \Omega^2_{X'/Y}(\log \sigma^*(\Delta)). \tag{5.2.3}$$

Writing $\omega^{(-1)}_{X'/Y}$ for the inverse of $\omega^{(1)}_{X'/Y}$, one finds

$$f'_*(\Omega^2_{X'/Y}(\log \sigma^*(\Delta)) \otimes \omega^{(-1)}_{X'/Y}) = \mathcal{O}_Y \quad \text{and} \tag{5.2.4}$$

$$f'^*f'_*\omega^{\mu\cdot\nu}_{X'/Y} \twoheadrightarrow \omega^{\mu\cdot\nu}_{X'/Y}(-B'_\mu). \tag{5.2.5}$$

Loosing the semi-stability, we will assume that (5.2.3) (5.2.4) and (5.2.5) hold true for $f : X \to Y$ itself, and for a normal crossing divisor B_μ on X.

Consider again the tautological sequence

$$0 \to f^*\Omega^1_Y(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/Y}(\log \Delta) \to 0, \tag{5.2.6}$$

and the sequence obtained by taking the second wedge product

$$0 \to f^*\Omega^1_Y(\log S) \otimes \Omega^1_{X/Y}(\log \Delta) \to \Omega^2_X(\log \Delta) \to \Omega^2_{X/Y}(\log \Delta) \to 0. \tag{5.2.7}$$

Hence one obtains edge morphisms

$$\tau_{1,1} : R^1 f_*(\Omega^1_{X/Y}(\log \Delta) \otimes \omega^{(-1)}_{X/Y}) \longrightarrow \Omega^1_Y(\log S) \otimes R^2 f_*(\omega^{(-1)}_{X/Y})$$
$$\tau_{2,0} : \mathcal{O}_Y \longrightarrow \Omega^1_Y(\log S) \otimes R^1 f_*(\Omega^1_{X/Y}(\log \Delta) \otimes \omega^{(-1)}_{X/Y})$$

for the cohomology of the exact sequences (5.2.6) and (5.2.7), tensorized with $\omega^{(-1)}_{X/Y}$.

Claim 5.3. The condition (5.2.2) implies that $\tau_{2,0}$ is injective. Moreover given an invertible subsheaf $\mathcal{N}$ of $\mathrm{Ker}(\tau_{1,1})$ or of $R^2 f_*(\omega^{(-1)}_{X/Y})$ its degree is

$$\deg(\mathcal{N}) < -2 \cdot (2g - 2 + s).$$

This obviously is a contradiction: since $\tau_{2,0}$ is injective, $R^1 f_*(\Omega^1_{X/Y} \otimes \omega^{(-1)}_{X/Y})$ contains a subsheaf $\mathcal{M}$ of degree larger than or equal to $-(2g-2+s)$. Hence $\tau_{1,1}(\mathcal{M}) \otimes (\Omega^1_Y(\log S))^{-1}$ is an invertible subsheaf of $R^2 f_*(\omega^{(-1)}_{X/Y})$, of degree larger than $-2 \cdot (2g-2+s)$. □

Short Break 5.4. For non-isotrivial families of curves, in the prove of 1.4 we only had to consider one edge morphism (1.4.2), and it is obvious, that this morphism is non-zero. In the surface case, one has to deal with the composite of two edge morphisms

$$\mathcal{O}_Y \xrightarrow{\tau_{2,0}} \Omega^1_Y(\log S) \otimes R^1 f_*(\Omega^1_{X/Y}(\log \Delta) \otimes \omega^{(-1)}_{X/Y}) \xrightarrow{\mathrm{id}_{\Omega^1_Y(\log S)} \otimes \tau_{2,0}} \Omega^1_Y(\log S)^2 \otimes R^2 f_*(\omega^{(-1)}_{X/Y}).$$

In [21], [17], [19], and in [3] one considers global cohomology instead of higher direct images, and one uses global vanishing theorems for twisted sheaves of differential forms, as the ones discussed in [6] to show that the corresponding composite of the edge-morphisms is non-zero.

Kovács [20] extends the vanishing theorems for logarithmic forms to singular varieties. This allows him to prove 5.1 a) and b) for F of general type, assuming that F has a minimal model.

For global vanishing theorem one needs a big sheaf, hence one has to assume that the general fibre is a surface of general type, or canonically polarized. For families of elliptic surfaces, in [24] we add logarithmic poles along some multi-section, in order to be able to use global vanishing theorems. The main difficulty there is the control of additional "bad" fibres introduced thereby.

The approach in [33], which we follow in these notes, replaces global vanishing theorems by the negativity of the kernel of the Kodaira Spencer map, as stated in 5.3.

Sketch of the proof of 5.3. In 5.2.2 we assumed that for some ample invertible sheaf $\mathcal{A}$, the sheaf $f_*\omega^{\nu}_{X/Y} \otimes \mathcal{A}^{-\nu}$ is ample. By (5.2.5), $\omega^{\mu\cdot\nu}_{X/Y}(-B_\mu) \otimes f^*\mathcal{A}^{-\mu\cdot\nu}$ is generated by global sections. The zero divisor H of a general section of this sheaf will be non-singular, however the fibres of $H \cap X_0 \to Y_0$ might be singular. Let us choose $S' \supset S$ such that H is smooth over $Y_1 = Y - S'$. Let $\sigma : X' \to X$ be a blowing up, $f' = f \circ \sigma$, such that $\Delta' = f'^{-1}(S')$ is a normal crossing divisor, as well as $\Delta' + H$. We write $\omega^{(-1)}_{X'/Y}$ for the pullback of $\omega^{(-1)}_{X/Y}$

to X'. One obtains morphisms

$$\sigma_{p,q} : R^q f_* \Omega^p_{X/Y}(\log \Delta) \otimes \omega^{(-1)}_{X/Y} \longrightarrow E'^{p,q} := R^q f'_* \Omega^p_{X'/Y}(\log \Delta' + H) \otimes \omega^{(-1)}_{X'/Y}.$$

By definition, $\sigma_{0,2}$ is an isomorphism, and $\sigma_{2,0}$ is injective. The same holds true for $\sigma_{1,1}$ since $\omega^{(-1)}_{X'/Y}|_H$ is anti-ample, and therefore $(f'|_H)_* \omega^{(-1)}_{X/Y}|_H = 0$.

The tautological exact sequences

$$\begin{aligned} (5.4.1) \qquad & 0 \to f'^* \Omega^1_Y(\log S') \to \Omega^1_{X'}(\log \Delta' + H) \\ & \qquad \to \Omega^1_{X'/Y}(\log \Delta' + H) \to 0 \quad \text{and} \\ (5.4.2) \qquad & 0 \to f'^* \Omega^1_Y(\log S') \otimes \Omega^1_{X'/Y}(\log \Delta' + H) \\ & \qquad \to \Omega^2_{X'}(\log \Delta' + H) \to \Omega^2_{X'/Y}(\log \Delta' + H) \to 0 \end{aligned}$$

are compatible with (5.2.6) and (5.2.7).

So $\tau_{p,q} \otimes \mathrm{id}_{\mathcal{A}^{-1}}$ commutes with the edge morphisms

$$\begin{aligned} \theta'_{1,1} : \mathcal{A}^{-1} \otimes R^1 f'_* (\Omega^1_{X'/Y}(\log \Delta' + H) \otimes \omega^{(-1)}_{X'/Y}) \\ \longrightarrow \mathcal{A}^{-1} \otimes \Omega^1_Y(\log S') \otimes R^2 f_* (\omega^{(-1)}_{X'/Y}) \\ \theta'_{2,0} : \mathcal{A}^{-1} \longrightarrow \mathcal{A}^{-1} \otimes \Omega^1_Y(\log S') \otimes R^1 f'_* (\Omega^1_{X'/Y} \otimes \omega^{(-1)}_{X'/Y}) \end{aligned}$$

for the cohomology of the exact sequences (5.2.6) and (5.2.7), tensorized with $\mathcal{A}^{-1} \otimes \omega^{(-1)}_{X'/Y}$. Hence 5.3 follows from

Claim 5.5. Let $\mathcal{N}$ be an invertible subsheaf of $\mathrm{Ker}(\theta'_{p,q})$. Then $\deg(\mathcal{N}) \leq 0$.

To prove 5.5 we have to recall some facts from the theory of variations of Hodge structures and the induced Higgs bundles. More details and references can be found in [33], section one.

A variation $\mathbb{V}_0$ of polarized Hodge structures of weight k on $Y_1 = Y - S'$ gives rise to

$$E_0 = \mathrm{gr}_F(\mathbb{V}_0 \otimes \mathcal{O}_{Y_1}) = \bigoplus_{p+q=k} E_0^{p,q},$$

together with a Higgs structure $\theta_0 = \oplus \theta_{p,q} : E_0 \to E_0 \otimes \Omega^1_{Y_1}$. Recall that the restricted Hodge metric on sub-bundles $\mathcal{N}$ of the kernel of θ_0 is negative semidefinite (see [9] and [26], for example).

Suppose the local monodromies of $\mathbb{V}_0$ around the components of S' are unipotent and let $\mathcal{V}$ be the Deligne extension of $\mathbb{V}_0 \otimes \mathcal{O}_{Y_1}$. The F-filtration extends to a filtration of $\mathcal{V}$ by subbundles, hence there exists a canonical extension E of E_0 to Y, and θ_0 extends to

$$\theta = \bigoplus_{p+q=k} \theta_{p,q} : E = \bigoplus_{p+q=k} E^{p,q} \longrightarrow E \otimes \Omega^1_Y(\log S') = \bigoplus_{p+q=k} E^{p,q} \otimes \Omega^1_Y(\log S').$$

Lemma 5.6. *If* $\mathcal{N} \subset E^{p,q}$ *is a sub-bundle with* $\theta_{p,q}(\mathcal{N}) = 0$, *then* $\deg(\mathcal{N}) \leq 0$.

The main point here is that the dual $\mathcal{N}^\vee$ of $\mathcal{N}$ is a quotient of a subbundle of a variation of Hodge structures, which allows to apply [14], 5.20. So the Chern forms of the induced Hodge metric on $(\mathcal{N}|_{Y_1})^\vee$ represent the corresponding Chern classes of $\mathcal{N}^\vee$. We get in particular $c_1(\mathcal{N}^\vee) \geq 0$, and hence $c_1(\mathcal{N}) \leq 0$.

Let $g : Z \to Y$ be a surjective morphism between a projective n-dimensional manifold Z and a non-singular curve Y, both defined over the complex numbers. Let $S' \subset Y$ be a divisor such that g is smooth outside of $\Pi = g^{-1}(S')$. We will assume Π to be a normal crossing divisor. The smooth projective morphism

$$g_1 : Z_1 = Z - \Pi \longrightarrow Y - S'$$

obtained by restricting g gives rise to a polarized variation of Hodge structures $\mathbb{V}_0 = R^k g_{1*}\mathbb{C}_{Z_1}$.

Using the notations introduced above we find

$$E^{p,q} = R^q g_* \Omega^p_{Z/Y}(\log \Pi).$$

The Kodaira Spencer maps $\theta_{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S')$ are again the edge morphisms for tautological exact sequences, similar to the ones considered in (5.2.6) and (5.2.7). The local monodromies for $\mathbb{V}_0 = R^k g_{1*}\mathbb{C}_{Z_1}$ are in general only quasi-unipotent, but following up, what happens under "unipotent reduction", one obtains:

Proposition 5.7. *Let $\mathcal{N}$ be an invertible subsheaf of $E^{p,q}$ with $\theta_{p,q}(\mathcal{N}) = 0$. Then* $\deg(\mathcal{N}) \leq 0$.

To prove 5.5 we choose Z to be the cyclic covering, obtained by taking the $\mu \cdot \nu$-th root out of the divisor H. Then the sheaves $E'^{p,q}$ are subsheaves of eigenvectors for the action of $\mathbb{Z}/\mu \cdot \nu$ on $E^{p,q}$, hence direct factors. Moreover, $\theta_{p,q}$ respects the group action, and its restriction to $E'^{p,q}$ coincides with $\theta'_{p,q}$. Hence 5.5 follows from 5.7. □

Remark 5.8. For semistable families of curves of genus $q \geq 2$ over $Y = \mathbb{P}^1$ the bound $s > 2$ in 5.1, a), is far from being optimal. Tan [29] has shown that f has at least five singular fibres. Using 3.7 and the methods from [3] or those employed in the proof of 1.4 one can reprove Tan's result under the additional assumption that X is a surface of general type.

Exercise 5.9. Let X be a surface, and let $f : X \to \mathbb{P}^1$ be a surjective semistable morphism with connected fibres.

a) Show that $\kappa(X) \geq 0$ implies that f is non isotrivial.
b) If X is a minimal surface of general type then show that f has at least 5 singular fibres.
c) Try to generalize b) to arbitrary surfaces X of general type.

Hints: For b) and c): If $\mathcal{L}$ is an invertible sheaf, nef and big, then $f_*(\omega_{X/Y} \otimes \mathcal{L})$ is ample (see 3.7).
For c): For a big invertible sheaf $\mathcal{L}$ and a normal crossing divisor D a generalization of the Bogomolov-Sommese vanishing theorem says that

$$H^0(X, \Omega^1_X(\log D) \otimes \mathcal{L}^{-1}) = 0.$$

Solution. a) If f is isotrivial, there exists some finite covering $\tau : Y' \to \mathbb{P}^1$, such that the pullback $X' = X \times_{\mathbb{P}^1} Y'$ is birational to $Y' \times F$. By 3.1 $\tau^* f_* \omega^\nu_{X/\mathbb{P}^1}$ is trivial, hence $f_* \omega^\nu_{X/\mathbb{P}^1}$, as well. So

$$H^0(X, \omega^\nu_X) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2 \cdot \nu) \otimes f_* \omega^\nu_{X/\mathbb{P}^1}) = 0,$$

for all $\nu > 0$.

For b) choose $\mathcal{L} = \omega_X$. By 3.7 $f_*(\omega_{X/\mathbb{P}^1} \otimes \mathcal{L}) = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes f_*\omega^2_{X/\mathbb{P}^1}$ is ample, hence

$$f_*\omega^2_{X/\mathbb{P}^1} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(\nu_i)$$

for $\nu_i > 2$. By part a) the dual of the Kodaira Spencer map

$$\gamma^\vee : f_*\omega^2_{X/\mathbb{P}^1} \longrightarrow \omega_{\mathbb{P}^1}(S)$$

considered in the proof of 1.4 is non-trivial, hence $-2 + s > 2$.

If $\sigma : X \to Z$ is the morphism to the minimal model, we choose in c)

$$\mathcal{L} = \sigma^*\omega_Z = \omega_X(-E).$$

As in b) one finds

$$f_*\omega^2_{X/\mathbb{P}^1}(-E) = \bigoplus \mathcal{O}_{\mathbb{P}^1}(\nu_i)$$

where $\nu_i > 2$. Again the dual of the Kodaira Spencer map $\gamma^\vee$ is non-trivial, but in order to prove c) one has to show that $f_*\omega^2_{X/\mathbb{P}^1}(-E)$ does not lie in the kernel of $\gamma^\vee$. Hence we consider a slightly different approach, closer to the one in [3]. Start again with the exact sequence

$$0 \to f^*\Omega^1_{\mathbb{P}^1}(\log S) \to \Omega^1_X(\log \Delta) \to \Omega^1_{X/\mathbb{P}^1}(\log \Delta) \to 0.$$

Tensorizing with $\mathcal{L}^{-1} \otimes f^*\omega_{\mathbb{P}^1}$ one obtains

$$H^0(X, \Omega^1_X(\log \Delta) \otimes \mathcal{L}^{-1} \otimes f^*\omega_{\mathbb{P}^1}) \to H^0(X, \mathcal{O}_X(E)) \to H^1(X, f^*\omega^2_{\mathbb{P}^1}(S) \otimes \mathcal{L}^{-1}).$$

Since $\mathcal{L} \otimes f^*\omega^{-1}_{\mathbb{P}^1}$ is big, the first group is zero. Hence $H^1(X, f^*\omega^2_{\mathbb{P}^1}(S) \otimes \mathcal{L}^{-1}) \neq 0$. So the dual of $f^*\omega^2_{\mathbb{P}^1}(S) \otimes \mathcal{L}^{-1}$ is not nef and big, which implies that the degree of $\omega^2_{\mathbb{P}^1}(S)$ must be strictly positive, or that $s > 4$. □

A similar argument, as the one used in c) seems to imply that for a semistable family of surfaces of general type, with $\kappa(X) = 3$, there are at least four singular fibres.

6. Moduli for Higher Dimensional Manifolds

As mentioned in the introduction, given a Hilbert polynomial h, one can construct a coarse quasi-projective moduli scheme M_h, parameterizing families of $n = \deg(h)$ dimensional canonically polarized manifolds (see [32]). The same holds true for manifolds F with ω_F semi-ample, if one considers pairs $(F, \mathcal{L})$, with $\mathcal{L}$ a polarization. This implies in particular, that the constant e in 5.1 only depends on the Hilbert polynomial h and on ν.

At present, there is no good compactification of M_h, i.e. no compactification as a moduli scheme of "stable varieties", except in the case of curves and surfaces of general type. Kollár and Shepherd-Barron [16] define stable surfaces, Alexeev [1] proves that the index of the singularities is bounded in terms of the coefficients of the Hilbert polynomial, which implies by [15] that M_h has a compactification $\bar{M}_h$, parameterizing families of stable surfaces (see also [32], section 9.6). Moreover, for ν sufficiently large and divisible and for some $p > 0$ there exists a very ample invertible sheaf $\lambda^{(p)}_\nu$ on $\bar{M}_h$ such that:

For $f : \mathcal{X} \to Z$ a flat morphism of stable surfaces with Hilbert polynomial h

and for the induced morphism $\varphi : Z \to \bar{M}_h$, the reflexive hull of ν-th power of $\omega_{\mathcal{X}/Z}$ is invertible and

$$\varphi^* \lambda_\nu^{(p)} = \det(f_* \omega_{\mathcal{X}/Z}^\nu)^p.$$

A different construction of $\bar{M}_h$ has been given by Karu in [10]. If one assumes that minimal models exist in dimension $n+1$, his construction extends to families of n-dimensional canonically polarized minimal models.

Let us return to families of surfaces of general type over curves Y, and smooth over Y_0. One obtains the generalization of 2.5:

Corollary 6.1. *$f_0 : X_0 \to Y_0$ be a smooth projective morphism, whose fibers are surfaces of general type with Hilbert polynomial h. Let $\Phi : Y \to \bar{M}_h$ be the induced morphism. Then* $\deg \Phi^* \lambda_\nu^{(p)}$ *is bounded above by a constant, depending on h, ν, g and s.*

Proof. Choose a non-singular projective compactification X of X_0 such that f_0 extends to $f : X \to Y$. By 4.5 the degree of $\det(f_* \omega_{X/Y}^\nu)$ is smaller than a constant depending on h, ν, g and s. There exists a finite covering $\tau : Y' \to Y$ and $f : X' \to Y' \in \bar{\mathcal{M}}_h(Y')$ which induces

$$Y' \xrightarrow{\tau} Y \xrightarrow{\Phi} \bar{M}_h.$$

We may assume, in addition, that f' has a semistable model $f'' : X'' \to Y'$, with X'' non-singular. By the definition of stable surfaces in [16],

$$f'_* \omega_{X'/Y'}^\nu = f''_* \omega_{X''/Y'}^\nu.$$

3.1 gives an injective map $f''_* \omega_{X''/Y'}^\nu \to \tau^* f_* \omega_{X/Y}^\nu$ which is an isomorphism over $\tau^{-1}(Y_0)$. Hence

$$\deg \Phi^* \lambda_\nu^{(p)} = \frac{\deg(f'_* \omega_{X'/Y'}^\nu)^p}{\deg \tau} \leq \deg(f_* \omega_{X/Y}^\nu)^p$$

is bounded, as well. □

Let us denote by $\mathbf{H} = \mathbf{Hom}((Y, Y_0), (\bar{M}_h, M_h))$ the scheme parameterizing morphism $\Phi : Y \to \bar{M}_h$ with $\Phi(Y_0) \subset M_h$. Since $\lambda_\nu^{(p)}$ is ample, 6.1 implies (see [3]):

Corollary 6.2. *Under the assumptions made in 6.1 there exists a subscheme $T \subset \mathbf{H}$, of finite type over k, which contains all points $[\Phi] \in \mathbf{H}$, induced by smooth morphisms $f_0 : X_0 \to Y_0$ whose fibres are surfaces of general type with Hilbert polynomial h.*

The way 2.6 was formulated, it does not refer to a compactification of M_h, and one may ask, in how far it generalizes to the higher dimensional case. In this situation, there is no hope for $\Omega_Y^1(\log S)$ to be ample with respect to some open set and the rigidity (R) formulated in the introduction does not hold true.

As well known, there exist projective curves Y in M_g and hence smooth non-isotrivial families $f : \mathcal{C} \to Y$. Choosing two such families, say $f_i : \mathcal{C}_i \to Y_i$, for $i = 1, 2$, and

$$g = (f_1 \times f_2) : \mathcal{X} = \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow Z = Y_1 \times Y_2$$

one obtains a non-trivial deformation of $\mathcal{C}_1 \times f_2^{-1}(p) \to Y_1 \times \{p\}$. Moreover,

$$\Omega^1_Z = pr_1^* \Omega^1_{Y_1} \oplus pr_2^* \Omega^1_{Y_2}$$

can not be ample over any non-empty open set.

Problem 6.3. Let M_h be the moduli scheme of canonically polarized manifolds with Hilbert polynomial h. Let Z be a projective manifold, S a normal crossing divisor and $Z_0 = Z - S$. Consider morphism $\varphi_0 : Z_0 \to M_h$, étale over its image, which is induced by a smooth family $g_0 : \mathcal{X}_0 \to Z_0$, which extends to $g : \mathcal{X} \to Z$, semistable over the general points of the components of S.

a) Is $\Omega^1_Z(\log S)$ weakly positive over Z_0?
b) Is $\det(\Omega^1_Z(\log S))$ ample with respect to Z_0 or with respect to some smaller non-empty subset?
c) Are there conditions on Ω^1_F, for a general fibre F of g, which imply that $\Omega^1_Z(\log S)$ is ample with respect to some dense open subset $U \subset Z_0$?

The assumption "étale over its image" is in fact not the right one. It is sufficient that the family $f : V \to U$ induces an étale map to the moduli stack, or in down to earth terms, that the induced Kodaira Spencer map

$$T_U \longrightarrow R^1 f_* T_{V/U}$$

is injective and locally split.

a) and b) have been verified by Zuo [34], under the additional assumption that the local Torelli theorem holds true for the general fibre F of g. It seems that a) implies b), at least for families of surfaces of general type. This would reprove the results obtained by Kovács in [18] by different methods.

What c) is concerned, one could hope that "Ω^1_F ample" or "Ω^1_F ample with respect to some dense open subset $V \subset F$" is the right type of condition. At present, there are no results in this direction, even for surfaces of general type.

References

[1] Alexeev, V.: Boundedness and K^2 for log surfaces. Int. Journ. Math. **5** (1994)
[2] Arakelov, A.: Families of algebraic curves with fixed deneracies. Math. U.S.S.R. Izv. **5** (1971) 1277 - 1302
[3] Bedulev, E., Viehweg, E.: On the Shafarevich conjecture for surfaces of general type over function fields. preprint (1999), to appear in Invent. Math.
[4] Caporaso, L,: On certain uniformity properties of curves over function fields. eprint AG9906156
[5] Esnault, H. and Viehweg, E.: Effective bounds for semi positive sheaves and the height of points on curves over complex function fields. Compositio math. **76** (1990) 69–85
[6] Esnault, H. and Viehweg, E.: Lectures on vanishing theorems. DMV Seminar **20** (1992) Birkhäuser, Basel Boston
[7] Fujita, T.: On Kähler fibre spaces over curves. J. Math. Soc. Japan **30** (1978) 779–794
[8] Hartshorne, R.: Algebraic geometry. Graduate Texts in Math. **52** (1977) Springer, New York Heidelberg Berlin
[9] Jost, J, Zuo, K.: Arakelov type inequalities for Hodge bundles over algebraic varieties, Part 1: Hodge bundles over algebraic curves. preprint (1999)
[10] Karu, K.: Minimal models and boundedness of stable varieties. J. Alg. Geom. to appear.
[11] Kawamata, Y.: Characterization of abelian varieties. Compositio math. **43** (1981) 253–276
[12] Kawamata, Y.: Minimal models and the Kodaira dimension of algebraic fibre spaces. Journ. Reine Angew. Math. **363** (1985) 1–46

[13] Kollár, J.: Higher direct images of dualizing sheaves. Ann. of Math. **123** (1986) 11–42
[14] Kollár, J.: Subadditivity of the Kodaira dimension: Fibres of general type. In: Algebraic geometry, Sendai 1985, Advanced Studies in Pure Math. **10** (1987) 361–398
[15] Kollár, J.: Projectivity of complete moduli. Journal Diff. Geom. **32** (1990) 235–268
[16] Kollár, J. and Shepherd-Barron, N. I.: Threefolds and deformations of surface singularities. Invent. math. **91** (1988) 299–338
[17] Kovács, S.: On the minimal number of singular fibres in a family of surfaces of general type. Journ. Reine Angew. Math. **487** (1997) 171 - 177
[18] Kovács, S.: Families over a base with a birationally nef tangent bundle. Math. Ann. **308** (1997) 347–359
[19] Kovács, S.: Algebraic hyperbolicity of fine moduli spaces. J. Alg. Geom. **9** (2000) 165 - 174
[20] Kovács, S.: Logarithmic Kodaira-Akizuki-Nakano Vanishing and Arakelov-Parshin Boundedness for singular varieties. eprint AG0003019
[21] Migliorini, L.: A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. J. Alg. Geom. **4** (1995) 353 - 361
[22] Mori, S.: Classification of higher dimensional varieties. In: Algebraic Geometry. Bowdoin 1985, Proc. Symp. Pure Math. **46** (1987) 269–331
[23] Mumford, D.: Geometric invariant theory. (1965)
Second enlarged edition: Mumford, D. and Fogarty, J.: Ergebnisse der Mathematik (3. Folge) **34** (1982) Springer, Berlin Heidelberg New York
[24] Oguiso, K., Viehweg, E.: On the isotriviality of families of elliptic surfaces. eprint AG9912100, J. Alg. Geom., to appear
[25] Parshin, A.: Algebraic curves over function fields I. Math. U.S.S.R. Isv. **2** (1968) 1145 - 1170
[26] Peters, C.: Arakelov-type inequalities for Hodge bundles. preprint (1999)
[27] Schmid, W.: Variation of Hodge structure: The singularities of the period mapping. Invent. math. **22** (1973) 211–319
[28] Seshadri, C. S.: Quotient spaces modulo reductive algebraic groups. Ann. of Math. **95** (1972) 511–556
[29] Tan, S. L.: The minimal number of singular fibers of a semistable curve over $\mathbb{P}^1$. Journ. Alg. Geometry **4** (1995) 591 - 596
[30] Viehweg, E.: Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces. In: Algebraic Varieties and Analytic Varieties, Advanced Studies in Pure Math. **1** (1983) 329–353
II. The local Torelli map. In: Classification of Algebraic and Analytic Manifolds, Progress in Math. **39** (1983) 567–589
[31] Viehweg, E.: Weak positivity and the stability of certain Hilbert points. Invent. math. **96** (1989) 639–667
II. Invent. math. **101** (1990) 191–223
III. Invent. math. **101** (1990) 521–543
[32] Viehweg, E.: Quasi-projective Moduli for Polarized Manifolds. Ergebnisse der Mathematik, 3. Folge **30** (1995), Springer Verlag, Berlin-Heidelberg-New York
[33] Viehweg, E., Zuo, K.: On the isotriviality of families of projective manifolds over curves. eprint AG0002203
[34] Zuo, K.: Negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. preprint (2000)

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