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EFFECTIVE IITAKA FIBRATIONS

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Abstract

For every n-dimensional projective manifold X of Kodaira dimension 2 we show that $\Phi_{|MK_X|}$ is birational to an Iitaka fibration for a *computable* positive integer $M = M(b, B_{n-2})$, where $b > 0$ is minimal with $|bK_F| \neq \emptyset$ for a general fibre F of an Iitaka fibration of X, and where B_{n-2} is the Betti number of a smooth model of the canonical $\mathbb{Z}/(b)$-cover of the $(n-2)$-fold F. In particular, M is a universal constant if the dimension $n \leq 4$.

Building upon the work of H. Tsuji, C. D. Hacon and J. McKernan in [HM] and independently S. Takayama in [Ta] have shown the existence of a constant r_n such that $\Phi_{|mK_X|}$ is a birational map for every $m \geq r_n$ and for every complex projective n-fold X of general type.

If the Kodaira dimension $\kappa = \kappa(X)$ is non-negative and $\kappa < n$, consider an Iitaka fibration $f : X \to Y$, i.e. a rational map onto a projective manifold Y of dimension κ with a connected general fibre F of Kodaira dimension zero. We define the *index* b of F to be

$$b = \min\{b' > 0 \mid |b'K_F| \neq \emptyset\},$$

and $B_{n-\kappa}$ to be the $(n-\kappa)$-th Betti number of a nonsingular model of the $\mathbb{Z}/(b)$-cover of F, obtained by taking the b-th root out of the unique member in $|bK_F|$, or as we will say, the *middle Betti number* of the *canonical covering* of F.

Question 0.1. Is there a constant $M := M(n, \kappa, b, B_{n-\kappa})$ such that $\Phi_{|MK_X|}$ is (birational to) an Iitaka fibration $f : X \to Y$ for all projective n-folds X of Kodaira dimension κ?

Assume that for all $s \leq n$ there exists an effective constant $a(s)$ such that for every projective s-fold V of non-negative Kodaira dimension, one has $|a(s)K_V| \neq \emptyset$ and such that the dimension of $|a(s)K_V|$ is at least one if

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$\kappa(V) > 0$. Then J. Kollár gives in [Ko86, Th. 4.6] a formula for the constant M in Question 0.1 in terms of $a(s)$ and n.

Question 0.1 has been answered in the affirmative by Fujino-Mori [FM] for $\kappa = 1$. In this note we show that the answer is also affirmative for $\kappa = 2$.

Theorem 0.2. *Let X be an n-dimensional projective manifold of Kodaira dimension 2 with Iitaka fibration $f : X \to Y$. Then there exists a computable positive integer M depending only on the index b of a general fibre F of f and on the middle Betti number B_{n-2} of the canonical covering of F, such that $\Phi_{|MK_X|}$ is birational to f.*

We now include a few words about the proof. In Section 1 we consider two $\mathbb{Q}$-divisors D_Y and L_Y on Y (with Y suitably chosen), such that the reflexive hull of $f_*\mathcal{O}_X(bNK_{X/Y})$ is isomorphic to $\mathcal{O}_Y(bN(D_Y + L_Y))$ for some constant N depending on $B_{n-\kappa}$. The divisor $bN(K_Y + D_Y + L_Y)$ is big, and in order to prove Theorem 0.2 it remains to bound a multiple M of bN for which $\Phi_{|MK_Y+MD_Y+ML_Y|}$ is birational.

The two divisors L_Y and D_Y are of different nature; D_Y is given by the multiplicities of fibres of f, and the pair (X, D_Y) is klt, whereas bL_Y is the semistable part of the direct image of $bK_{X/Y}$, hence nef. We apply the log minimal model program for surfaces in various ways, to reduce the problem to the case where on some birational model W the corresponding divisor $K_W + D_W + L_W$ is nef. The rest then will be easy if the sheaf $K_Y + D_Y$ is big, and will also not be too difficult if L_Y is big and $K_Y + D_Y$ is pseudoeffective. For the remaining cases, we consider in Section 3, the *pseudo-effective threshold*, i.e. the smallest real number e with $K_W + D_W + eL_W$ pseudo-effective, and we will show that e is bounded away from one.

Beginning with Section 2, our arguments and methods use $\dim(Y) = \kappa(X) = 2$, sometimes for convenience, and at other times used in an essential way. After finishing the proof of Theorem 0.2 we are a bit more precise (see Remarks 4.1 and 4.2).

If one assumes that the general fibre F of the Iitaka fibration has a good minimal model F', hence one with $bK_{F'} = 0$, then L_Y would be the pullback of a nef and big $\mathbb{Q}$-divisor on some compactification of a moduli scheme. As we will discuss in Remark 4.2, assuming the existence of good minimal models, the existence of nice compactifications of moduli schemes might lead to an affirmative answer to Question 0.1.

Conjecturally the index b and the Betti number $B_{\dim(F)}$ should be bounded by a constant depending only on the dimension of F. So one could hope for an affirmative answer to:

Question 0.3. Can one choose the constant M in Question 0.1 to be independent of b and $B_{n-\kappa}$?

For example, if F is an elliptic curve one has $b = 1$ and $B_1 = 2$. For surfaces F of Kodaira dimension zero, the index b divides 12, and 22 is an upper bound for the middle Betti number B_2 of the smooth minimal model of the canonical covering of F. Hence for $n \leq 4$, the constant M in Theorem 0.2 can be chosen to be *universal*, i.e. only depending on n. Since by [Mo, §10], [FM, Corollary 6.2], [CC, Th. 1.1], [HM] and [Ta] the same holds true for $\dim(X) = 3$ and $\kappa(X) = 0, 1$, or 3 we can state:

Corollary 0.4. *There is a computable universal constant M_3 such that $\Phi_{|M_3 K_X|}$ is an Iitaka fibration for every 3-dimensional projective manifold X.*

We remark that when $\dim X = 3$ and $\kappa(X) = 2$, Kollár [Ko94, (7.7)] has already shown that there exists a universal constant M' such that $H^0(X, mK_X) \neq 0$ for all $m \geq M'$, under the additional assumption that the Iitaka fibration is non-isotrivial. A direct proof of Corollary 0.4, using the existence of good minimal models, will be given at the end of Section 4. [1]

0.5. Conventions.

We adopt the conventions of Hartshorne's book, of [KMM] and of [KM]. However, if D is a $\mathbb{Q}$-divisor on X we will often write $H^0(X, D)$ or $H^0(X, \mathcal{O}_X(D))$ instead of $H^0(X, \mathcal{O}_X(\lfloor D \rfloor))$, and write $|D|$ instead of $|\lfloor D \rfloor|$. By abuse of notation we will not distinguish line bundles and linear equivalence classes of divisors.

1. Some auxiliary results

Set-up 1.1. Let X be a complex n-fold of Kodaira dimension κ. We will consider an Iitaka fibration $f : X \to Y$ of X with Y nonsingular, and F will denote a general fibre of f. Replacing X by some nonsingular blowup, as in [Vi83, §3] or [FM, §§2 and 4], one may assume that $f : X \to Y$ is a morphism, that the discriminant of f is contained in a normal crossing divisor of Y and that each effective divisor E in X, with $\mathrm{codim}_Y(f(E)) \geq 2$, is exceptional for some morphism $X \to X'$ with X' nonsingular. In particular, for all $i \geq 1$ and for all such divisors E one has

$$\begin{aligned} H^0(X, \mathcal{O}_X(ibK_X)) &= H^0(X, \mathcal{O}_X(ibK_X + E)) \\ &= H^0(Y, \mathcal{O}_Y(ibK_Y) \otimes f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}), \end{aligned}$$

where b again denotes the index of F and where $f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}$ is the invertible sheaf, obtained as the reflexive hull of $f_*\mathcal{O}_X(ibK_{X/Y})$. By

[1] After a first version of this article was submitted to the arXiv-server, we learned that Corollary 0.4 had been obtained independently by Adam T. Ringler in [Ri], using different arguments.

[FM, Corollary 2.5] one can define the semistable part of $f_*\mathcal{O}_X(ibK_{X/Y})$ as a $\mathbb{Q}$-Cartier divisor $iL^{ss}_{X/Y}$, compatible with base change, such that $\mathcal{O}_Y(iL^{ss}_{X/Y}) \subset f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}$ for i sufficiently divisible, and such that both sheaves coincide if $f: X \to Y$ is semistable in codimension one. In particular, $L^{ss}_{X/Y}$ is nef. We write

$$L_Y = \frac{1}{b} L^{ss}_{X/Y} \quad \text{and} \quad D_Y = \sum_P \frac{s_P}{b} P$$

for the $\mathbb{Q}$-divisors with $\mathcal{O}_Y(ib(L_Y + D_Y)) = f_*\mathcal{O}_X(ibK_{X/Y})^{\vee\vee}$. We remark that D_Y is supported on the discriminant locus of f and $b(L_Y + D_Y)$ is only a $\mathbb{Q}$-divisor (whose denominators may not be uniformly bounded); see [FM, Proposition 2.2].

Let $B_{n-\kappa}$ be the middle Betti number of the canonical covering of F, and

$$N = N(B_{n-\kappa}) = \operatorname{lcm}\{m \in \mathbb{Z}_{>0} \,|\, \varphi(m) \le B_{n-\kappa}\},$$

where φ denotes the Euler φ-function. By [FM, Theorem 3.1], $NbL_Y = NL^{ss}_{X/Y}$ is an integral Cartier divisor. By [FM, Proposition 2.8], if $s_P \neq 0$, there exists $u_P, v_P \in \mathbb{Z}_{>0}$ with $0 < v_P \le bN$ such that

$$0 < \frac{s_P}{b} = \frac{bNu_P - v_P}{bNu_P} < 1.$$

So all the non-zero coefficients of D_Y are contained in

$$\mathrm{A}(b, N) := \{\frac{bNu - v}{bNu} \,|\, u, v \in \mathbb{Z}_{>0};\ 0 < v \le bN\} \setminus \{0\}.$$

Lemma 1.2. *In Set-up* 1.1, *the following statements hold true.*

(1) The set $\mathrm{A}(b, N)$ is a DCC set in the sense of [AM, §2], and one has

$$\frac{1}{Nb} \le \operatorname{Inf} \mathrm{A}(b, N).$$

(2) (Y, D_Y) is klt.
(3) The $\mathbb{Q}$-divisor $K_Y + D_Y + L_Y$ is big.
(4) We have $H^0(X, mbK_X) \cong H^0(Y, mb(K_Y + D_Y + L_Y))$, for every $m \in \mathbb{Z}_{>0}$; further, the map $\Phi_{|mbK_X|}$ is birational to the Iitaka fibration f if and only if $|mb(K_Y + D_Y + L_Y)|$ gives rise to a birational map.
(5) NbL_Y is an integral Cartier divisor.
(6) If $s + 1 \in \mathbb{Z}_{>0}$ is divisible by Nb, then $(s+1)D_Y \ge \lceil sD_Y \rceil$.

Proof. Part (1) is obvious and (5) was mentioned already in Set-up 1.1. For (2), we remark that D_Y, as part of the discriminant locus, is a simple normal crossing divisor and that $s_P/b \in (0, 1)$. Parts (3) and (4) are obvious, since for all $i \ge 1$,

$$H^0(X,\, ibK_X) = H^0(Y,\, ib(K_Y + L_Y + D_Y)).$$

Finally (6) is a consequence of the description of the coefficients of D_Y as elements of $\mathrm{A}(b,N)$. In fact, since all $\beta = \frac{bNu-v}{bNu} \in \mathrm{A}(b,N)$ are larger than or equal to $1-\frac{1}{u}$ and since $(s+1)\beta \in \frac{1}{u}\cdot\mathbb{Z}$, one finds that $(s+1)\beta \geq \lceil s\beta \rceil$. $\square$

2. Log minimal models of surfaces and pseudo-effectivity

From now on we will restrict ourselves to the case $\kappa = 2$.

Remark 2.1. As we will see in proving Theorem 0.2, the constant $M(b, B_{n-2})$ (later written as $M(b,N)$) can be computed using the invariants $\beta(\mathrm{A})$ and $\epsilon(\mathrm{A})$ of the DCC set $\mathrm{A} = \mathrm{A}(b,N)$ (see [AM, Th. 4.12] and [Ko94, Complement 5.7.4], or [La, Th. 5.4]).

Lemma 2.2. *There is a birational morphism $\sigma : Y \to W$ such that the following hold true:*

(1) *$K_W + D_W + L_W$ is ample and $K_Y + D_Y + L_Y = \sigma^*(K_W + D_W + L_W) + E_\sigma$. Here $D_W := \sigma_* D_Y$, $L_W := \sigma_* L_Y$; $E_\sigma \geq 0$ is an effective σ-exceptional divisor.*
(2) *(W, D_W) is* klt.
(3) *Suppose that L_S is big for $S = Y$ or for $S = W$. Then $L_S \sim_{\mathbb{Q}} L'_S$ with $(S, D_S + L'_S)$* klt.

Before giving the proof, let us recall the log minimal model program (LMMP) for surfaces.

Construction 2.3. Starting with any two dimensional klt-pair (Y, Δ_Y), the LMMP provides us with a sequence $\gamma : Y \to Z$ of contractions of $(K_Y + \Delta_Y)$-negative extremal rays. If $K_Y + \Delta_Y$ is big or more generally pseudo-effective (which will be assumed here), γ will be birational. Writing $\Delta_Z = \gamma_* \Delta_Y$, the $\mathbb{Q}$-divisor $E := K_Y + \Delta_Y - \gamma^*(K_Z + \Delta_Z)$ is effective and γ-exceptional, the $\mathbb{Q}$-divisor $K_Z + \Delta_Z$ is nef and (Z, Δ_Z) is klt.

By the abundance theorem for klt log surfaces (see [Ko], for example), there is a morphism with connected fibres $\psi : Z \to W$ such that $K_Z + \Delta_Z$ is the pullback of an ample $\mathbb{Q}$-divisor H on W.

Proof of Lemma 2.2. As in the proof of [FM, Th. 5.2], the nefness of L_Y and the bigness of $K_Y + D_Y + L_Y$ allows us to find some $a \in \mathbb{Q}_{>0}$ and a klt-pair (Y, Δ_Y) such that

$$K_Y + L_Y + D_Y \sim_{\mathbb{Q}} a(K_Y + \Delta_Y).$$

If $K_Y + \Delta_Y$ is big, as we assumed in Lemma 2.2, the morphism $\sigma = \psi \circ \gamma : Y \to W$ in Construction 2.3 is birational and the ample $\mathbb{Q}$-divisor on W is $H = K_W + \Delta_W$. So $K_Z + \Delta_Z = \psi^*(K_W + \Delta_W)$ and hence the γ-exceptional divisor E is

$$K_Y + \Delta_Y - \sigma^*(K_W + \Delta_W).$$

In particular, (W, Δ_W) is again klt. We find

$$K_Y + L_Y + D_Y \sim_{\mathbb{Q}} a\sigma^*(K_W + \Delta_W) + aE = \\ \sigma^*\sigma_*(K_Y + L_Y + D_Y) + aE = \sigma^*(K_W + L_W + D_W) + aE.$$

(1) is true by Lemma 1.2(3). Part (2) follows from (1) and Lemma 1.2(2). For part (3), we refere to [KM, Proposition 2.61 and Corollary 3.5]. □

Remark and Assumption 2.4. The morphism $\sigma : Y \to W$, constructed in Lemma 2.2, factors through a minimal resolution $\widetilde{W}$ of W. So $\sigma = \pi \circ \xi$ for

$$Y \xrightarrow{\ \xi\ } \widetilde{W} \xrightarrow{\ \pi\ } W.$$

Applying the construction in Section 1 to $\widetilde{W}$ instead of Y one obtains divisors $L_{\widetilde{W}}$ and $D_{\widetilde{W}}$, which are just the direct images of the divisors L_Y and D_Y.

Using the calculation in Lemma 2.2, we see that Lemma 1.2 holds true even with Y replaced by the minimal resolution of W. By abuse of notation we will replace Y by $\widetilde{W}$ and assume in the sequel that the morphism $\sigma : Y \to W$ in Lemma 2.2 is a minimal desingularization.

The answer to Question 0.1 is quite easy when $K_Y + D_Y$ is big, and especially when $L_Y \equiv 0$.

Lemma 2.5. *Suppose that $K_Y + D_Y$ is big. Then there is a constant $M = M(b, N)$ such that for all $s \geq M(b, N)$ with $s + 1$ divisible by Nb, the linear system $|(s+1)(K_Y + D_Y + L_Y)|$ defines a birational map. In particular, $\Phi_{|MK_X|}$ is an Iitaka fibration for some $M = M(b, N)$ depending only on the set $\mathrm{A}(b, N)$, and hence only on b and $N = N(B_{n-2})$.*

Proof. Applying Construction 2.3 to (Y, D_Y), we get a birational morphism $\eta : Y \to Z$ such that $K_Z + D_Z = \eta_*(K_Y + D_Y)$ is ample and $E_\eta := K_Y + D_Y - \eta^*(K_Z + D_Z)$ is an η-exceptional effective $\mathbb{Q}$-divisor.

Note that the coefficients of $D_Z = \eta_* D_Y$ still belong to the same DCC set $\mathrm{A}(b, N)$. Remark (3) on page 60 of [La] allows us to apply [La, Th. 3.2]. As in [La, Th. 5.3] one finds a constant $M(b, N)$, depending only on the set $\mathrm{A}(b, N)$, such that the linear system

$$|K_Y + \lceil s\eta^*(K_Z + D_Z)\rceil|$$

gives rise to a birational map for every $s \geq M(b, N)$. The same [La, Th. 3.2] applies to

$$|K_Y + \lceil s(K_Y + D_Y) + (s + 1)L_Y\rceil|,$$

since $(s+1)L_Y$ is pseudo-effective and hence the boundary divisor of the above adjoint linear system has nef part larger than $s\eta^*(K_Z + D_Z)$.

Assume further that Nb divides $s+1$. Then by Lemma 1.2,

$$K_Y + \lceil sK_Y + sD_Y + (s+1)L_Y \rceil \leq (s+1)(K_Y + D_Y + L_Y).$$

This implies the first part of Lemma 2.5. Now the second part follows from the first part using Lemma 1.2(4). □

Lemma 2.6. *Suppose that L_Y is big and $K_Y + D_Y + eL_Y$ is pseudo-effective for some $e \in [0, 1)$. Then there is a constant $M = M(b, N, e)$ such that $\Phi_{|MK_X|}$ is an Iitaka fibration.*

Proof. Consider the Zariski decompositions

$$K_Y + D_Y + eL_Y = P_e + N_e \quad \text{and} \quad K_Y + D_Y + L_Y = P_Y + N_Y.$$

Then $P_Y \geq (1-e)L_Y + P_e$. Since NbL_Y is an integral divisor

$$P_Y^2 \geq (1-e)^2 L_Y^2 \geq \frac{(1-e)^2}{(Nb)^2}.$$

For a very general curve C_t, we have

$$P_Y.C_t \geq (1-e)L_Y.C_t \geq \frac{1-e}{Nb}.$$

Assume that $s(1-e) > 4Nb$. Applying [La, Th. 3.2] one finds that the adjoint linear system $|K_Y + \lceil s(K_Y + D_Y + L_Y) + L_Y \rceil|$ (whose boundary divisor has the nef part larger than sP_Y) gives rise to a birational map. Assume further that Nb divides $(s+1)$. The lemma follows from the observation that the latter system is included in the following (see Lemma 1.2):

$$|(s+1)(K_Y + D_Y + L_Y)|. \qquad \square$$

The most difficult part of the proof of Theorem 0.2 is the one where $K_Y + D_Y$ is not pseudo-effective, and hence where L_Y is not numerically trivial. As a first step, we will need the following construction, well known to experts as a consequence of [Ba] (see however [Ar]). This will be essential in the next section; for the completeness and for the need of the precise description of the end product (i.e., V), we give a proof.

Proposition 2.7. *Suppose that $K_Y + D_Y$ is not pseudo-effective. Then in addition to the birational morphism $\sigma : Y \to W$ constructed in Lemma 2.2 there are a birational morphism $\tau : W \to V$, some $e \in \mathbb{Q} \cap (0, 1)$ and effective divisors $E_{\tau\sigma}$, E_σ, E_{L_Y} on Y and E_τ on W satisfying:*

(a) *E_τ is τ-exceptional, E_σ and E_{L_Y} are σ-exceptional, and*

$$E_{\tau\sigma} = E_\sigma + (1-e)E_{L_Y} + \sigma^* E_\tau,$$

(b) *Writing* $D_W := \sigma_* D_Y, D_V := \tau_* D_W$ *etc. one has*

$$\begin{aligned} K_Y + D_Y + eL_Y &= \sigma^*(K_W + D_W + eL_W) + E_\sigma + (1-e)E_{L_Y} \\ &= \sigma^*\tau^*(K_V + D_V + eL_V) + E_{\tau\sigma}. \end{aligned}$$

(c) $\sigma^* L_W = L_Y + E_{L_Y}$*, and* L_W *is nef.*

(d) $e = \min\{e' \mid K_S + D_S + e'L_S$ *is pseudo-effective*$\}$*. Here* S *can be chosen to be equal to* Y*,* W*, or* V *and the resulting* e *is independent of this choice.*

(e) (V, D_V) *and hence* V *are* klt*.*

(f) *One of the following holds true:*

(1) $K_V + D_V + eL_V \equiv 0$*, the Picard number* $\rho(V) = 1$*, and* V *is a* klt *del Pezzo (rational) surface. In particular,* $-K_V$ *is an ample* $\mathbb{Q}$*-divisor and* V *has at most quotient singularities.*

(2) V *is the total space of a* $\mathbb{P}^1$*-fibration over a curve with general fibre* Γ*, the Picard number* $\rho(V) = 2$*, and* $K_V + D_V + eL_V \equiv \beta\Gamma$ *for some* $\beta \in \mathbb{Q}_{>0}$*.*

Proof. We start with the morphism $\sigma : Y \to W$ from Lemma 2.2. For $L_W := \sigma_* L_Y$ one has $\sigma^* L_W = L_Y + E_{L_Y}$ where E_{L_Y} is supported in the exceptional locus of σ. Since L_Y is nef, L_W is also nef, and E_{L_Y} is effective. By Lemma 2.2 one finds for all e',

$$K_Y + D_Y + e'L_Y = \sigma^*(K_W + D_W + e'L_W) + E_\sigma + (1-e')E_{L_Y}. \tag{2.1}$$

So the assertion (c) and the first equation in the assertion (b) holds true.

Starting from $W_0 = W$ we will construct for some $r \geq 0$ and for $i = 0, \ldots, r-1$ a chain of birational morphisms $\tau_i : W_i \to W_{i+1}$, such that W_r satisfies the conditions stated in Proposition 2.7(f), (1) or (2). We will show inductively that the following conditions (c1)–(c5) hold for $i = 1, \ldots, r$ and that (c6)–(c8) hold for $i = 1, \ldots, r-1$.

(c1) (W_i, D_i) is klt.

(c2) $K_i + D_i$ is not pseudo-effective.

(c3) $K_i + D_i + L_i$ is ample.

(c4) $e_i = \min\{e' \in (0,1) \mid K_i + D_i + e'L_i$ is nef$\}$ exists and is rational.

(c5) $1 > e_0 \geq e_1 \geq \cdots \geq e_r > 0$.

(c6) $\rho(W_{i+1}) = \rho(W_i) - 1$.

(c7) L_i is nef, and $\tau_i^* L_{i+1} = L_i + E_{L_i}$ for an effective τ_i-exceptional divisor E_{L_i}.

(c8) $K_i + D_i + e_i L_i = \tau_i^*(K_{i+1} + D_{i+1} + e_i L_{i+1})$.

Here $K_i = K_{W_i}$, and D_i or L_i denotes the pushdowns of D_Y or L_Y to W_i. We write $\rho(W_i)$ for the Picard number of W_i.

Claim 2.8. (c3) and (c7) are true for all $i \geq 0$, and (c1) and (c2) hold for $i = 0$.

Proof. Note that τ_i is birational. (c3) and (c7) are true for $i = 0$ and hence they are true for all $i \geq 0$ on surfaces; see Lemma 2.2 and the proof for the assertion (c) above. (c1) is also part of Lemma 2.2. For (c2), set $e' = 0$ in the equation (2.1) and use the non-pseudo-effectiveness of $K_Y + D_Y$. □

Claim 2.9.

(i) The conditions (c2) and (c3) for some i imply (c4) with $e_i \in (0, 1)$.

(ii) In particular, (c4) and (c5) hold for $i = 0$.

Proof. Knowing (c2) for some i, the condition (c3) allows us to deduce from [KMM, Th. 4-1-1] or [KM, Th. 3.5] that there exists a rational number

$$d_i = \max\{d \,|\, (K_i + D_i + L_i) + d(K_i + D_i) \text{ is nef}\}.$$

Since $K_i + D_i + L_i$ is ample, $d_i > 0$. Then $e_i = 1/(1 + d_i)$. □

Assume now that we have found the birational morphisms τ_i for $i < i_0$, that (c1)–(c5) hold for $i = 0, \ldots, i_0$ and that (c6)–(c8) hold for $i = 0, \ldots, i_0 - 1$.

By [KM, Complement 3.6], the condition (c2) implies the existence of a $K_{i_0} + D_{i_0}$-negative extremal ray R_{i_0}, perpendicular to $K_{i_0} + D_{i_0} + e_{i_0} L_{i_0}$. We choose $\tau_{i_0} : W_{i_0} \to W_{i_0+1}$ to be the contraction of R_{i_0} (i.e., of all the curves proportional to R_{i_0}). In particular, one finds

$$\tau_{i_0}^* \tau_{i_0 *}(K_{i_0} + D_{i_0} + e_{i_0} L_{i_0}) = K_{i_0} + D_{i_0} + e_{i_0} L_{i_0}. \tag{2.2}$$

Suppose that τ_{i_0} is birational. Then for $i = i_0$ the condition (c6) holds. (c8) follows from equation (2.2).

Knowing (c1)–(c8) for $i = i_0$, it is easy to verify (c1)–(c5) for $i = i_0 + 1$. We remark that (c7) and (c8) for i_0 imply that

$$K_{i_0} + D_{i_0} = \tau_{i_0}^*(K_{i_0+1} + D_{i_0+1}) + e_{i_0} E_{L_{i_0}},$$

so (c1) and (c2) for $i_0 + 1$ follow from the corresponding statements for i_0, and hence (c4) for $i_0 + 1$ follows from Claim 2.9.

By the choice of e_{i_0},

$$K_{i_0} + D_{i_0} + e_{i_0} L_{i_0} = \tau_{i_0}^*(K_{i_0+1} + D_{i_0+1} + e_{i_0} L_{i_0+1})$$

is nef. This is possible only if $K_{i_0+1} + D_{i_0+1} + e_{i_0} L_{i_0+1}$ is nef, and hence only if $e_{i_0} \geq e_{i_0+1}$, as claimed in (c5).

If τ_{i_0} is birational, we can continue this process. This way, one obtains birational morphisms $\tau_j : W_j \to W_{j+1}$ $(0 \leq j \leq r)$ satisfying the conditions (c1)–(c8). The condition (c6) implies that $r < \rho(W)$.

If τ_{i_0} is non-birational we set $V = W_{i_0}$ and $e = e_{i_0}$ in Proposition 2.7. The assertions (a) and the second half of (b) follow from (c5), (c7) and (c8), whereas (e) is the same as (c1). It remains to verify (d) and (f).

Case 1. If the image of τ_{i_0} is a point, we claim that we are in the first case in Proposition 2.7(f). By the construction, $\rho(V) = 1$.

Recall that the singularities of a klt surface are just quotient singularities. Since L_Y and hence $L_V = \tau_* \sigma_* L_Y$ cannot be numerically trivial, it must be a positive multiple of the generator of the Neron-Severi group of V. So the definition of e implies that $K_V + D_V + eL_V \equiv 0$. By [GZ, Lemma 1.3] a klt surface with $-K$ ample is rational.

Case 2. We claim that the second case in Proposition 2.7(f) occurs if τ_{i_0} has a curve W_{i_0+1} as its image. Let Γ denote a general fibre of τ_{i_0}.

For $V = W_{i_0}$ one finds $\rho(V) = 1 + \rho(W_{i_0+1}) = 2$. Our Γ generates the extremal ray R_{i_0} giving rise to the contraction τ_{i_0}. So every fibre of $V \to W_{i_0+1}$ is irreducible (also because $\rho(V) = 2$). Since the nef divisor $K_V + D_V + eL_V$ is perpendicular to R_{i_0} and hence to the nef divisor Γ, one finds that $K_V + D_V + eL_V \equiv \beta\Gamma$ for some $\beta > 0$.

Since $K_V + D_V + L_V \equiv (1-e)L_V + \beta\Gamma$ is ample, we have $\Gamma.L_V > 0$. Now $0 = \Gamma.\beta\Gamma = \Gamma.(K_V + D_V + eL_V) > \Gamma.K_V$ and hence $\Gamma \cong \mathbb{P}^1$.

We still have to characterize e as the pseudo-effective threshold as claimed in the assertion (d) of Proposition 2.7.

Clearly, when $S = V$, our $K_S + D_S + eL_S \equiv \beta\Gamma$ (setting $\beta = 0$ and Γ to be any ample divisor, in Case (1)) is pseudo-effective, so by the assertion (b) of Proposition 2.7 the same is true when $S = Y$ or $S = W$.

Conversely, suppose that $K_S + D_S + e'L_S$ is pseudo-effective for some e' and some $S \in \{Y, W, V\}$. Then the same holds for $S = V$ by considering the pushdown.

For $S = V$ we can write this divisor as $\beta\Gamma + (e' - e)L_V$. Thus $0 \leq \Gamma.(\beta\Gamma + (e'-e)L_V) = (e'-e)\Gamma.L_V$. Since $K_V + D_V + L_V \equiv \beta\Gamma + (1-e)L_V$ is ample, we have $\Gamma.L_V > 0$ in both Cases (1) and (2), and hence $e' \geq e$. $\square$

The next two lemmata give a universal upper bound for the threshold e in Proposition 2.7.

Lemma 2.10. *In the situation considered in Proposition* 2.7(f), Case (1), *there is a constant* $e(b, N) < 1$, *depending only on* b *and* N, *such that the threshold* $e \leq e(b, N)$.

Proof. Let $\pi : \widetilde{V} \to V$ be a minimal resolution. So one has a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\sigma} & W \\ \xi\Big\downarrow & & \tau\Big\downarrow \\ \widetilde{V} & \xrightarrow{\pi} & V. \end{array}$$

As usual, when there is a birational morphism $Y \to S$ we will write D_S and L_S for the direct images of D_Y and L_Y, respectively.

Write $\pi^* K_V = K_{\widetilde{V}} + J$ where J is an effective and π-exceptional $\mathbb{Q}$-divisor. Note that (V, D_V) and hence V and $(\widetilde{V}, J)$ are klt. Since $H_{Nb} := NbL_{\widetilde{V}}$ is a nef line bundle with $H_{Nb} - (K_{\widetilde{V}} + J)$ nef and big, [Ka, Th. 3.1] tells us that $|2H_{Nb}|$ is base point free.

So $L_{\widetilde{V}}$ is $\mathbb{Q}$-linearly equivalent to $L'_{\widetilde{V}} := H_{2Nb}/2Nb$ for a smooth divisor $H_{2Nb} \in |2H_{Nb}|$ intersecting $D_{\widetilde{V}}$ transversely and away from the fundamental point of the inverse of the birational morphism $\xi : Y \to \widetilde{V}$.

For $L'_Y := \xi^* L'_{\widetilde{V}}$, the pair $(Y, D_Y + L'_Y)$ is klt. Write $\xi^* L_{\widetilde{V}} = L_Y + E$ with $E \geq 0$ and ξ-exceptional. Then $K_Y + D_Y + L'_Y \sim_{\mathbb{Q}} (K_Y + D_Y + L_Y) + E$ is big.

So we are allowed to apply the first part of Lemma 2.5 and we find a constant $M(b, N)$ such that $|(t_0+1)(K_Y + D_Y + L'_Y)|$ gives rise to a birational map for all $t_0 \in \mathbb{Z}_{>0}$ with $2Nb|(t_0+1)$ and $t_0 \geq M(b, N)$. Thus the same holds for $|(t_0+1)(K_V + D_V + L'_V)|$ with $(L_V \sim_{\mathbb{Q}})\ L'_V$ the pushdown of L'_Y.

So $(t_0+1)(K_V + D_V + L_V).\bar{\Gamma} \geq 1$ for any movable curve $\bar{\Gamma}$ on V. On $\widetilde{V}$, we take $\Gamma \cong \mathbb{P}^1$ with $\Gamma^2 = 0$ or 1 (when $\widetilde{V}$ is ruled or $\mathbb{P}^2$) such that $\bar{\Gamma} = \pi(\Gamma)$. Note that $\bar{\Gamma}.K_V = \Gamma.(K_{\widetilde{V}} + J) \geq \Gamma.K_{\widetilde{V}} \geq -3$.

If $e \leq 1/2$ there is nothing to show. Otherwise,

$$0 = \bar{\Gamma}.(K_V + D_V + eL_V) \geq -3 + e\bar{\Gamma}.L_V \geq -3 + \frac{1}{2}\bar{\Gamma}.L_V.$$

Then

$$6(1-e) \geq (1-e)\bar{\Gamma}.L_V = \bar{\Gamma}.(K_V + D_V + L_V) \geq \frac{1}{t_0+1}$$

gives an upper bound for e. $\square$

Lemma 2.11. *In Case* (2) *of Proposition* 2.7(f), *there is a constant* $\nu = \nu(N, b)$ *(depending only on* N, b*) such that the threshold* e *satisfies*

$$e \leq 1 - \frac{1}{4\nu} < 1.$$

Proof. Again it is sufficient to consider the case $e \geq 1/2$. We calculate

$$0 = \Gamma.\beta\Gamma = \Gamma.(K_V + D_V + eL_V) \geq -2 + \frac{1}{2}\Gamma.L_V = -2 + \frac{1}{2}\widetilde{\Gamma}.L_Y.$$

Here the fibre $\widetilde{\Gamma}$ is the pullback on Y of the general fibre Γ on V in Proposition 2.7(f), Case (2). Since $K_Y + D_Y + L_Y$ is big and $N\widetilde{\Gamma}.(K_Y + L_Y) \in \mathbb{Z}_{>0}$, we apply [FM, Proposition 6.3], obtain $\nu = \nu(N, b)$ satisfying the following and hence conclude the lemma (noting that $E_{\tau\sigma}$ is contained in fibres):

$$\nu \leq \widetilde{\Gamma}.(K_Y + D_Y + L_Y) = (1-e)\widetilde{\Gamma}.L_Y \leq 4(1-e). \qquad \square$$

3. The proof of Theorem 0.2 and Corollary 0.4

When $K_Y + D_Y$ is big, especially when $L_Y \equiv 0$, the statement of Theorem 0.2 has been verified in Lemma 2.5. If L_Y is big, the theorem follows from Lemmata 2.6, 2.10 and 2.11. So for Theorem 0.2, it remains to consider the case:

Assumption 3.1. L_Y is not numerically trivial, $\kappa(L_Y) \leq 1$, and $\kappa(K_Y + D_Y) \leq 1$.

By [8aut], for a nef $\mathbb{Q}$-divisor L_Y on a projective manifold Y there exists a *nef reduction*, i.e. an almost holomorphic dominant rational map $\varrho : Y \dashrightarrow T$ such that the restriction of L_Y to compact fibres is numerically trivial, and the restriction to general curves C with $\dim(\varrho(C)) > 0$ is of positive degree. The dimension of T is called the *nef dimension* and denoted as $n(L_Y)$. Obviously $\kappa(L_Y) \leq n(L_Y)$ and the first assumption in Assumption 3.1 implies that the nef dimension $n(L_Y)$ is one or two. In the first case, the reduction map ϱ is birational, whereas, in the second case, it is a morphism to a curve.

Recall that starting with Remark and Assumption 2.4, we had chosen Y such that the birational morphism $\sigma : Y \to W$, constructed in Lemma 2.2, is a minimal desingularization. As in Section 2, L_W and D_W are the direct images of L_Y and D_Y, respectively. We write $K_Y = \sigma^* K_W - J$ with J an effective σ-exceptional $\mathbb{Q}$-divisor.

Lemma 3.2.

(1) $0 \leq L_Y^2 \leq L_W^2$.

(2) *If* $n(L_Y) = 2$, *then* $\kappa(L_Y) \leq 0$ *and* $L_Y.K_Y \geq 0$.

(3) *Let* e *be the threshold from Proposition* 2.7 and let P_Y be the positive and N_Y the negative part in the Zariski decomposition

$$K_Y + D_Y + L_Y = P_Y + N_Y.$$

Then $P_Y - (1-e)\sigma^* L_W$ is pseudo-effective. Furthermore,

$$P_Y^2 \geq (1-e)^2 L_W^2 \geq (1-e)^2 L_Y^2. \tag{3.1}$$

Proof. (1) Recall that L_Y is nef, hence L_W is nef as well. Since $L_Y \leq \sigma^* L_W$, one obtains (1).

(2) Since L_Y is nef, but neither big nor numerically trivial, one finds that $L_Y^2 = 0$ and that $H^2(Y, mL_Y) = 0$ for m sufficiently large. Then for m sufficiently divisible

$$\dim(H^0(Y, mL_Y)) \geq -\frac{L_Y.K_Y}{2} \cdot m + \chi(\mathcal{O}_Y).$$

If $L_Y.K_Y < 0$, then $\kappa(L_Y) = 1$. However $\kappa(L_Y) = 1$ implies that the restriction of L_Y to the fibres of $\Phi_{|mL_Y|}$ is numerically trivial, contradicting $n(L_Y) = 2$.

(3) Using the notation from Lemma 2.2, $P_Y = \sigma^*(K_W + D_W + L_W)$ and $N_Y = E_\sigma$. Moreover $K_Y + D_Y + eL_Y$ is pseudo-effective by the choice of e. Then its σ-pushdown $K_W + D_W + eL_W$ is pseudo-effective as well and one obtains the first part of (3). Since P_Y and L_W are nef, the pseudo-effectivity of $P_Y - (1-e)\sigma^* L_W$ implies (3.1). □

Lemma 3.3. *Assume that L_Y is not numerically trivial, that $\kappa(L_Y) \le 1$ and that either $L_Y^2 > 0$ or $L_Y.K_Y \ge 0$. Then $|MK_X|$ is an Iitaka fibration for some $M = M(b, N, e)$ depending only on b, N, and e.*

Proof. Keeping the notation from Lemma 3.2 (3), one has:

Claim 3.4. $$P_Y^2 \ge \frac{(1-e)^2}{3(Nb)^2}.$$

Proof. Assume first that $L_Y^2 > 0$. Since NbL_Y is an integral Cartier divisor, $L_Y^2 \ge 1/(Nb)^2$ and the claim follows from (3.1) in Lemma 3.2 (3).

Assume next that $L_Y^2 = 0$. Since L_Y is nef, one finds by assumption that

$$L_W.K_W = L_Y.(K_Y + J) \ge L_Y.K_Y \ge 0.$$

If $L_Y.K_Y$ is positive, by Lemma 1.2 (5) it has to be larger than or equal to $1/Nb$. Applying Lemma 3.2 (3) one finds

$$\begin{aligned} P_Y^2 \ge P_Y.(1-e)\sigma^* L_W &= (1-e)(K_W + D_W + L_W).L_W \\ &\ge (1-e)L_W.K_W \ge (1-e)L_Y.K_Y \ge \frac{1-e}{Nb} \ge \frac{(1-e)^2}{3(Nb)^2}. \end{aligned}$$

If $L_Y.K_Y = 0$, consider first the case $L_Y.\sigma' D_W > 0$, where σ' stands for the proper transform. By Lemma 1.2, this intersection number is $\ge 1/(Nb)^2$. As above, one obtains

$$P_Y^2 \ge (1-e)L_W.D_W = (1-e)L_Y.\sigma^* D_W \ge (1-e)L_Y.\sigma' D_W \ge \frac{1-e}{(Nb)^2} \ge \frac{(1-e)^2}{3(Nb)^2}.$$

It remains to handle the worse case,

$$L_Y^2 = L_Y.K_Y = L_Y.\sigma' D_W = 0. \tag{3.2}$$

Since $K_Y + D_Y + L_Y$ is big and since L_Y is not numerically trivial,

$$0 < L_Y.(K_Y + D_Y + L_Y) = L_Y.D_Y.$$

Thus $L_Y.D_1 > 0$ for some irreducible curve D_1 in $\mathrm{Supp} D_Y \setminus \sigma' D_W$. Then D_1 lies in the exceptional locus of σ, hence $D_1 \cong \mathbb{P}^1$ and $D_1^2 \le -2$.

If $D_1^2 = -n$ with $n \geq 3$, then [Z, Lemma 1.7] implies that

$$J \geq \frac{n-2}{n} D_1 \geq \frac{1}{3} D_1,$$

and

$$\begin{aligned} P_Y^2 &\geq (1-e) L_W . K_W \\ &= (1-e) L_Y . (K_Y + J) \geq (1-e) L_Y . \frac{1}{3} D_1 \geq \frac{1-e}{3Nb} \geq \frac{(1-e)^2}{3(Nb)^2}. \end{aligned}$$

If $D_1^2 = -2$ consider the contraction $\sigma_1 : Y \to W_1$ of D_1. Then $\sigma_1^* L_{W_1} = L_Y + aD_1$ with $a = L_Y . D_1/2 \geq 1/2Nb$. Note that $0 = L_Y^2 = (\sigma_1^* L_{W_1} - aD_1)^2 = L_{W_1}^2 - 2a^2$, so $L_W^2 \geq L_{W_1}^2 = 2a^2 \geq 1/2(Nb)^2$ and

$$P_Y^2 \geq (1-e) L_W^2 \geq \frac{1-e}{2(Nb)^2} \geq \frac{(1-e)^2}{3(Nb)^2}. \qquad \square$$

As a next step in the proof of Lemma 3.3, consider two general points x_1, x_2 of Y. If the nef dimension $n(L_Y) = 1$, we may assume that the two points are not in the same fibre of the nef reduction. Thus for a very general curve C_t on Y containing x_1, x_2, one has

$$P_Y . C_t \geq (1-e) L_Y . C_t \geq \frac{1-e}{Nb}.$$

Then the adjoint linear system

$$|K_Y + \lceil s_0(K_Y + D_Y + L_Y) + L_Y \rceil|, \quad \text{for} \quad s_0 = b(\lceil \frac{5Nb}{1-e} \rceil + 1) - 1$$

separates the points x_1, x_2. In fact, the nef part of the divisor

$$\lceil s_0(K_Y + D_Y + L_Y) + L_Y \rceil$$

is larger than $s_0 P_Y$ and the inequalities

$$s_0 P_Y . C_t \geq s_0(1-e) L_Y . C_t \geq \frac{s_0(1-e)}{Nb} \geq 4 \quad \text{and} \quad (s_0 P_Y)^2 \geq s_0^2 \frac{(1-e)^2}{3(Nb)^2} > 8$$

allow us to apply [La, Th. 3.2]. Thus, by Lemma 1.2,

$$\begin{aligned} h^0(X, (s_0+1)K_X) &= h^0(Y, (s_0+1)(K_Y + D_Y + L_Y)) \\ &\geq h^0(X, K_Y + \lceil s_0(K_Y + D_Y + L_Y) + L_Y \rceil) \geq 2. \end{aligned}$$

Now by [Ko86, Th. 4.6], $\Phi_{|tK_X|}$ is an Iitaka fibration for $t = (s_0+1)(2M+1) + M$, where M is a constant as in [FM, Corollary 6.2], depending only on $A(b, N)$. $\square$

Recall that Assumption 3.1 implies that $n(L_Y)$ is one or two. In the second case, Lemma 3.2 (2) allows us to apply Lemma 3.3. So it remains to consider the case below.

Lemma 3.5. *Assume that $n(L_Y) = 1$, $L_Y^2 = 0$ and $L_Y.K_Y < 0$. Then Y is a ruled surface over a curve C of genus $q(Y)$ with general fibre $\Sigma \cong \mathbb{P}^1$. The $\mathbb{Q}$-divisor L_Y is $\mathbb{Q}$-linearly equivalent to a positive multiple of Σ, and $|MK_X|$ is an Iitaka fibration for some constant $M = M(b, N, q(Y))$ depending only on $b, N, q(Y)$.*

Proof. Since $n(L_Y) = 1$, the nef reduction is a map $\varrho : Y \to C$ to a curve. It is an easy exercise (whose solution can be found in [8aut, Proposition 2.11]) to see that ϱ is a morphism and that L_Y is numerically equivalent to a positive multiple of the general fibre Σ of ϱ. Since $L_Y.K_Y < 0$, one has $\Sigma.K_Y < 0$. Thus $2g(\Sigma) - 2 = \Sigma.K_Y < 0$ and $\Sigma \cong \mathbb{P}^1$. So $\varrho : Y \to C$ is a $\mathbb{P}^1$-fibration and Y is a ruled surface with $g(C) = q(Y)$.

Moreover the divisor NbL_Y on the ruled surface $Y \to C$ is numerically equivalent to some $\alpha\Sigma$. Considering the intersection of Σ with a section of $\varrho : Y \to C$ one sees that α is an integer. The numerically trivial sheaf $NbL_Y - \alpha\Sigma$ is linearly equivalent to the pullback of a numerically trivial sheaf on a relative minimal model of Y which in turn must be the pullback of a sheaf on C. Hence $NbL_Y \sim \varrho^*\Pi$ for some integral divisor Π on C of positive degree. Then $(2g(C)+1)\Pi$ is very ample and $(2g(C)+1)NbL_Y$ is linearly equivalent to the disjoint union H of smooth fibres in general position. For ($L_Y \sim_{\mathbb{Q}}$) $L'_Y = H/(2g(C)+1)Nb$ the pair $(Y, D_Y + L'_Y)$ is klt and $K_Y + D_Y + L'_Y$ is big. The coefficients of $D_Y + L'_Y$ lie in the DCC set $A(b, N) \cup \{1/(2g(C)+1)Nb\}$. Hence repeating the arguments used in the proof of Lemma 2.5 one finds a constant M depending only on b, N and $g(C)$ such that $|(s+1)(K_Y + D_Y + L_Y)|$ defines a birational map for all $s \geq M$ with $s + 1$ divisible by $(2g(C) + 1)Nb$. Now the lemma follows from Lemma 1.2 (4). $\square$

Lemma 3.6. *Keeping the assumptions made in Lemma* 3.5, either $K_Y + D_Y$ is big or $q(Y) \leq 1$.

Proof. If $K_Y + D_Y$ is not pseudo-effective we can apply Proposition 2.7(f). There, in Case 1 the irregularity is zero. So we only have to consider Case 2. Using the notation introduced there, $K_Y + D_Y + L_Y \equiv (1 - e)L_Y + \beta\Gamma + E_{\tau\sigma}$ is big, $\Gamma.L_Y > 0$ and hence, using the notation from Lemma 3.5, $\Gamma.\Sigma > 0$. Further $\Gamma \cong \mathbb{P}^1$. So there are two different $\mathbb{P}^1$-fibrations on Y with fibres Γ and Σ, and Y is rational.

Therefore, we may assume that $K_Y + D_Y$ is pseudo-effective with $\kappa(K_Y + D_Y) \leq 1$. Applying Construction 2.3 to the klt-pair (Y, D_Y), we get morphisms $\gamma : Y \to Z$ and $\psi_Z : Z \to B$ with γ birational, and an ample $\mathbb{Q}$-divisor H on B such that

$$E_\gamma = K_Y + D_Y - \gamma^*\psi_Z^*(H)$$

is an effective γ-exceptional $\mathbb{Q}$-divisor consisting of rational curves. By the assumption

$$\dim B = \kappa(H) = \kappa(K_Y + D_Y) \leq 1.$$

Consider the case $\dim B = 1$. So $\psi = \psi_Z \circ \gamma : Y \to B$ is a family of curves over a curve with general fibre Γ. By abuse of notation, Γ will also be considered as the general fibre of ψ_Z. For $\alpha = \deg H$, one has

$$K_Y + D_Y \sim_{\mathbb{Q}} \equiv \alpha\Gamma + E_\gamma \quad \text{and} \quad K_Z + D_Z \equiv \alpha\Gamma.$$

Since E_γ is contained in fibres

$$0 = \Gamma.(\alpha\Gamma + E_\gamma) = \Gamma.(K_Y + D_Y) \geq \Gamma.K_Y$$

and Γ is either $\mathbb{P}^1$ or an elliptic curve.

Since $K_Y + D_Y + L_Y \equiv \alpha\Gamma + E_\gamma + L_Y$ is big, $\Gamma.L_Y > 0$. Using the notation from Lemma 3.5, this implies that $\Gamma.\Sigma > 0$ where Y is ruled over C with general fibre Σ. So Γ dominates the base curve C and $q(Y) = g(C) \leq 1$, as claimed.

In case $\dim B = 0$, one has $K_Z + D_Z \equiv 0$ (indeed, $\sim_{\mathbb{Q}} 0$ by [Ko]). If $L_Y.E_\gamma = 0$, then $K_Y + D_Y + L_Y \equiv L_Y + E_\gamma$ is the Zariski decomposition and hence

$$2 = \kappa(K_Y + D_Y + L_Y) = \kappa(L_Y) \leq 1,$$

a contradiction.

Thus $L_Y.E_\gamma > 0$ and, using the notation from Lemma 3.5, one finds $\Sigma.E_\gamma > 0$. The divisor E_γ is exceptional for the birational morphism to the klt surface Z, whence all its components are isomorphic to $\mathbb{P}^1$. Since one of them intersects Σ, the base curve C in Lemma 3.5 is dominated by $\mathbb{P}^1$ and hence $g(C) = q(Y) = 0$. □

Proof of Theorem 0.2. As recalled at the beginning of this section it remains to verify the theorem under Assumption 3.1. Then the theorem follows from Lemmata 3.3, 3.5, and 3.6, using Lemmata 2.10 and 2.11. □

Proof of Corollary 0.4. When $\kappa(X) = 0$, one can take M_3 to be the Beauville number as in [Mo, §10]. When $\kappa(X) = 1$, the result is just [FM, Corollary 6.2]. When $\kappa(X) = 3$, we can take $M_3 = 126$ by [CC, Th. 1.1] (see also [HM] and [Ta]). So the only remaining case is the one where $\kappa(X) = 2$. Here the corollary follows from Theorem 0.2 for $n = 3$, $b = 1$, $B_{n-2} = 2$ and $N = N(B_{n-2}) = 12$. □

4. Some comments

Remark 4.1. Although the arguments used in Sections 2 and 3 are formulated just for surfaces Y, some can be easily extended to the higher dimensional case. In particular, the general minimal model program in [BCHM] extends the Zariski decomposition for pseudo-effective divisors to the case $\dim(Y) > 2$.

However, as pointed out by the referee, there is no replacement for the fact that on surfaces the direct image of a nef divisor is nef. Similarly, the proof of Claim 3.4, essential for Lemma 3.3 is done "case by case". In the "worst case" (3.2) one cannot extract any positivity from L_Y, nor from $K_Y + \sigma' D_W$. One has to get the positivity from exceptional components of D_Y. Again, there is little hope to do something similar for $\dim(Y) > 2$.

Remark 4.2. If the general fibre F of the Iitaka fibration is a good minimal model, and hence if $\omega_F^b \cong \mathcal{O}_F$, choose an ample invertible sheaf $\mathcal{A}$ on X. Writing h for the Hilbert polynomial of $\mathcal{A}|_F$ one obtains a morphism from the complement Y_0 of the discriminant locus to the moduli scheme M_h of polarized minimal models with Hilbert polynomial h. In [Vi06] we constructed a compactification $\overline{M}_h$ of M_h and a nef $\mathbb{Q}$-Cartier divisor λ on $\overline{M}_h$, which on each curve meeting M_h corresponds to the semistable part. Moreover, λ is ample with respect to M_h. In different terms, there is an effective $\mathbb{Q}$-Cartier divisor Γ on $\overline{M}_h$, supported in $\overline{M}_h \setminus M_h$, such that $\alpha\lambda - \Gamma$ is ample for all $\alpha \geq 1$.

If we choose Y such that $Y_0 \to M_h$ extends to a morphism $\varphi : Y \to \overline{M}_h$, one finds that $L_Y = \varphi^*(\lambda)$, and hence that $\alpha L_Y - \varphi^*(\Gamma)$ is semi-ample. For some constant $C > 0$, depending only on h, both CL_Y and $C(L_Y - \varphi^*(\Gamma))$ are divisors, and choosing C large enough, $C(\alpha L_Y - \varphi^*(\Gamma))$ will have many global sections for all $\alpha \in \mathbb{Z}_{>0}$ (provided that L_Y is not numerically trivial). Perhaps this allows us to answer Question 0.1 in the affirmative, assuming the existence of good minimal models. Perhaps to this aim one has to compare the image of the map $\varphi : Y \to \overline{M}_h$ with the log minimal model of Y.

Anyway, one still would need some argument which guarantees the independence of the constant M from the Hilbert polynomial h.

Remark 4.3. Gianluca Pacienza [Pa] recently gave an affirmative answer to Question 0.1 for $\kappa = n - 2$, or more precisely if the general fibre F of the Iitaka fibration has a good minimal model, assuming that Y is non-uniruled and that the morphism $Y_0 \to M_h$ in Remark 4.2 is generically finite over its image. Note that the last assumption implies that L_Y is big.

Finally, let us give a direct proof of the Corollary 0.4, without referring to Theorem 0.2, but using the existence of good minimal models in dimension three:

Proof of Corollary 0.4 (using the existence of minimal models). As before all cases are known, except the one where $\kappa(X) = 2$. Assume that X is a good minimal threefold. For some $m \gg 0$ the morphism $\sigma : X \to S$ associated with $|mK_X|$ has connected fibres and, by the Abundance theorem for threefolds, $K_X \sim_{\mathbb{Q}} \sigma^* G$ for an ample $\mathbb{Q}$-Cartier divisor G.

As in [Na, Proof of Corollary (0.4)] (S, Δ) is klt for the effective $\mathbb{Q}$-divisor

$$\Delta := \frac{1}{12} H' + \sum_j a_j D'_j + \sum_i (1 - \frac{1}{m_i}) \Gamma'_i \quad \text{and} \quad K_X \sim_{\mathbb{Q}} \sigma^*(K_S + \Delta).$$

Here H', D'_j, Γ'_i stand for the divisors $\mu_* H,\ \mu_* D_j,\ \mu_* \Gamma_i$ in the notation of [Na]. Moreover

$$a_j \in K_2 := \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}\}.$$

We remark that by [KM, Th. 3.5.2] S is the canonical model, denoted by W in Lemma 2.2, and that $\Delta = L_W + D_W$. Note that Δ has coefficients in the DCC set

$$\mathrm{Ell} := \{1 - \frac{1}{m} \,|\, m \in \mathbb{Z}_{\geq 2}\} \cup K_2 \cup \{\frac{1}{12}\}.$$

By [La, Th. 5.4] there exists a computable constant M, depending only on the DCC set Ell, such that the adjoint linear system $|K_S + \lceil t(K_S + \Delta) \rceil|$ gives rise to a birational map for all $t \geq M$. This adjoint divisor is smaller than or equal to $(t+1)(K_S + \Delta)$ provided that $12|t$. So $\Phi_{|(t+1)(K_S+\Delta)|}$ is birational and hence $\Phi_{|(t+1)K_X|}$ is an Iitaka fibration (see Lemma 1.2 (4)). $\square$

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