ON THE ISOTRIVIALITY OF FAMILIES OF PROJECTIVE MANIFOLDS OVER CURVES

ECKART VIEHWEG AND KANG ZUO*

Let Y be a non-singular curve, X a manifold, both projective and defined over $\mathbb{C}$, and let $f : X \to Y$ be a surjective morphism with connected general fibre F. We fix a reduced divisor S on Y which contains the discriminant divisor of f, i.e. a reduced divisor with

$$f_0 = f|_{X_0} : X_0 \longrightarrow Y_0$$

smooth, for $Y_0 = Y - S$ and $X_0 = f^{-1}(Y_0)$. Recall that f is birationally isotrivial, if $X \times_Y \operatorname{Spec} \overline{\mathbb{C}(Y)}$ is birational to $F \times \operatorname{Spec}\overline{\mathbb{C}(Y)}$.

Theorem 0.1. *Assume that f is not birationally isotrivial, and that one of the following conditions holds true:*

a) *$\kappa(F) = \dim(F)$.*

b) *F has a minimal model F' with a semi-ample canonical $\mathbb{Q}$-divisor $K_{F'}$.*

Then f has at least

i) *three singular fibres if $Y = \mathbb{P}^1$.*

ii) *one singular fibre if Y is an elliptic curve.*

If $\kappa(F) = \dim(F) = 2$ or if the canonical sheaf ω_F is ample, part ii) of 0.1 has been shown by L. Migliorini [12] and part i) is due to S. Kovács [9], [10]. If $\dim(F) = 2$ and $\kappa(F) \leq 1$, both i) and ii) have recently been shown by K. Oguiso and the first named author [14]. The proof is based on the observation that the additional singular fibres, showing up in certain cyclic coverings of X, can be neglected in vanishing theorems.

The same principle is exploited in the proof of 0.1, replacing global vanishing theorems by the negativity of the kernel of the morphism of Hodge bundles, given by the cup-product with the Kodaira Spencer class. Such morphisms, which we will call Kodaira Spencer maps, have been studied by J. Jost and the second named author in [7] (see also [16]).

Since the total space X of a smooth isotrivial morphism to an elliptic curve has at most Kodaira dimension $\dim(X)-1$, part ii) in 0.1 implies that there are no smooth morphisms from a manifold of general type to an elliptic curve. A similar argument shows that a surjective morphism from a manifold of general type to $\mathbb{P}^1$ must have at least three singular fibres, an affirmative answer to a question posed by F. Catanese and M. Schneider. In fact, one has a stronger result. The assumptions a) or b) in 0.1 are just needed to guarantee that for some $\nu \gg 0$ the determinant of $f_*\omega^{\nu}_{X/Y}$ is ample. For $Y = \mathbb{P}^1$ this necessarily holds true, if X has a non-negative Kodaira dimension.

This work has been partly supported by the DFG Forschergruppe “Arithmetik und Geometrie”.

* Supported by a Heisenberg-fellowship, DFG.

Theorem 0.2. *Let X be a complex projective manifold of non-negative Kodaira dimension. Then a surjective morphism $X \to \mathbb{P}^1$ has at least 3 singular fibres.*

If X is a curve, this is just the well-known fact that a morphism to $\mathbb{P}^1$ from a curve of genus larger than or equal to one has to ramify over at least three points. For surfaces 0.2 follows from the bounds of A. Parshin [15] and A. Arakelov [1]. For threefolds of general type, or if ω_X is ample, 0.2 is an easy corollary of the results in [10] (see also [2]). For threefolds of lower Kodaira dimension one can use [14] instead.

As a byproduct, the proof of 0.1 gives some explicit bounds, generalizing the ones obtained by A. Parshin [15] and A. Arakelov [1] for families of curves, by E. Bedulev, if the general fibre F of f is an elliptic surface, and by E. Bedulev and the first author [2] under the assumption that F is canonically polarized, or a surface of general type. We write $s = \deg(S)$, and $n = \dim(F)$. The genus of Y is denoted by g, and δ is the number of singular fibres of f, which are not semistable, i.e. not reduced normal crossing divisors.

Theorem 0.3. *Assume that f is not birationally isotrivial, that ω_F is semi-ample, and that $h(t)$ is the Hilbert polynomial for a polarization of F. Then there exist constants ν and e, depending only on h, with*

$$\frac{\deg(f_*\omega^\nu_{X/Y})}{\operatorname{rank}(f_*\omega^\nu_{X/Y})} \leq (n \cdot (2g - 2 + s) + \delta) \cdot \nu \cdot e.$$

In particular, if f is semistable,

$$\frac{\deg(f_*\omega^\nu_{X/Y})}{\operatorname{rank}(f_*\omega^\nu_{X/Y})} \leq n \cdot (2g - 2 + s) \cdot \nu \cdot e.$$

As explained in [2], section 4, such bounds imply a generalization of the Shafarevich conjecture, saying that for given Y and S there are finitely many deformation types, if there exists a nice compactification of the corresponding moduli scheme. Unfortunately such a compactification has only been constructed for surfaces of general type.

We thank Keiji Oguiso for helpful remarks and comments, and Osamu Fujino for pointing out some ambiguities in the first version of section three. Sandor Kovács informed us, immediately after he got a copy of this article, that by different methods he obtained results, partly overlapping with the ones presented here (see [11]).

1. Hodge bundles and the Kodaira Spencer map

Let Y be a complex manifold and let S be a normal crossing divisor. A variation $\mathbb{V}_0$ of polarized Hodge structures of weight k on $Y_0 = Y - S$ gives rise to

$$E_0 = \operatorname{gr}_F(\mathbb{V} \otimes \mathcal{O}_{Y_0}) = \bigoplus_{p+q=k} E_0^{p,q},$$

together with a Higgs structure $\theta_0 = \oplus\theta_{p,q} : E_0 \to E_0 \otimes \Omega^1_{Y_0}$.

Lemma 1.1. *If $\mathcal{N} \subset E_0^{p,q}$ is a sub-bundle with $\theta_{p,q}(\mathcal{N}) = 0$, then the curvature of the restricted Hodge metric on $\mathcal{N}$ is negative semidefinite.*

In fact, the negativity of the curvature of the restricted Hodge metric on $\det(\mathcal{N})$, which will be the only case used in this note, as well as the next lemma 1.2, follow from [18] and can also be found in [7], Lemma 1. For the convenience of the reader we sketch the proof.

Proof. Let $\Theta(E_0, h)$ denote the curvature form of the Hodge metric h on E_0. Then by [6], chapter II, we have

$$\Theta(E_0, h) + \theta \wedge \bar{\theta}_h + \bar{\theta}_h \wedge \theta = 0,$$

where $\bar{\theta}_h$ is the complex conjugation of θ with respect to h. h restricts to a metric $h|_{\mathcal{N}}$, and induces a $\mathcal{C}^\infty-$decomposition $E_0 = \mathcal{N} \oplus \mathcal{N}^\perp$. One obtains

$$\Theta(\mathcal{N}, h) = \Theta(E_0, h)|_{\mathcal{N}} + \bar{A}_h \wedge A = -\theta \wedge \bar{\theta}_h|_{\mathcal{N}} - \bar{\theta}_h \wedge \theta|_{\mathcal{N}} + \bar{A}_h \wedge A,$$

where $A \in A^{1,0}(\mathrm{Hom}(\mathcal{N}, \mathcal{N}^\perp))$ is the second fundamental form of the sub-bundle $\mathcal{N} \subset E_0$, and $\bar{A}_h$ is its complex conjugate with respect to h.

Since $\theta(\mathcal{N}) = 0$, we have $\bar{\theta}_h \wedge \theta|_{\mathcal{N}} = 0$, hence

$$\Theta(\mathcal{N}, h) = -\theta \wedge \bar{\theta}_h|_{\mathcal{N}} + \bar{A}_h \wedge A.$$

$\theta \wedge \bar{\theta}_h$ is positive semidefinite and $\bar{A}_h \wedge A$ is negative semidefinite, so $\Theta(\mathcal{N}, h)$ is negative semidefinite. □

Suppose the local monodromy of $\mathbb{V}_0$ around the components of S are unipotent and let $\mathcal{V}$ be the Deligne extension of $\mathbb{V}_0 \otimes \mathcal{O}_{Y_0}$. By [17] the F-filtration extends to a filtration of $\mathcal{V}$ by subbundles, hence there exists a canonical extension E of E_0 to Y, and θ_0 extends to

$$\theta = \bigoplus_{p+q=k} \theta_{p,q} : E = \bigoplus_{p+q=k} E^{p,q} \longrightarrow E \otimes \Omega^1_Y(\log S) = \bigoplus_{p+q=k} E^{p,q} \otimes \Omega^1_Y(\log S).$$

Lemma 1.2. *Keeping the assumptions made above, suppose Y is a smooth projective curve. If $\mathcal{N} \subset E^{p,q}$ is a sub-bundle with $\theta_{p,q}(\mathcal{N}) = 0$, then* $\deg(\mathcal{N}) \leq 0$.

Proof. Let $\mathcal{N}^\vee$ be the dual of $\mathcal{N}$. We have the projection $E^{p,q\vee} \to \mathcal{N}^\vee$. Note that $E^{p,q\vee} = E^{\vee q,p}$ as a Hodge bundle of the system of Hodge bundles corresponding to the dual variation of Hodge structures $\mathbb{V}_0^\vee$. The monodromy of $\mathbb{V}_0^\vee$ around S is again unipotent. We have the projection

$$F^{\vee q} \to E^{\vee q,p} \to \mathcal{N}^\vee,$$

where $F^{\vee q}$ is the q-th subbundle in the extended Hodge filtration of $\mathcal{V}^\vee$.

This presentation of $\mathcal{N}^\vee$ as a quotient of a subbundle of a variation of Hodge structures allows to apply [8], 5.20. So the Chern forms of the induced Hodge metric on $(\mathcal{N}|_{Y_0})^\vee$ represent the corresponding Chern classes of $\mathcal{N}^\vee$. From 1.1 we get in particular $\deg(\mathcal{N}^\vee) \geq 0$, and hence $\deg(\mathcal{N}) \leq 0$. □

Let $g : Z \to Y$ be a surjective morphism between a projective n-dimensional manifold Z and a non-singular curve Y, both defined over the complex numbers. Let $S \subset Y$ be a divisor such that g is smooth outside of $\Pi = g^{-1}(S)$. We will assume Π to be a normal crossing divisor. The smooth projective morphism

$$g_0 : Z_0 = Z - \Pi \longrightarrow Y - S$$

obtained by restricting g gives rise to variations of Hodge structures $\mathbb{V}_0 = R^k g_{0*}\mathbb{C}_{Z_0}$. As explained in [20], p. 423, the primitive decomposition of $\mathbb{V}_0$ allows to define a polarization on $\mathbb{V}_0$. If the fibres of g are connected and if g is semistable, i.e. if Π is reduced, the local monodromies around points in S are unipotent.

Using the notations introduced above we find

$$E^{p,q} = R^q g_* \Omega^p_{Z/Y}(\log \Pi).$$

$\theta_{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S)$, which we will call the Kodaira Spencer map, is the edge-morphism induced by the tautological exact sequence

$$(1.2.1) \quad 0 \to g^*\Omega^1_Y(\log S) \otimes \Omega^{p-1}_{Z/Y}(\log \Pi) \to \Omega^p_Z(\log \Pi) \to \Omega^p_{Z/Y}(\log \Pi) \to 0.$$

$\theta_{p,q}$ is given by the cup product with the Kodaira Spencer class, induced by g, and the induced map from the tangent sheaf of Y to the homomorphism from $E^{p-1,q+1}$ to $E^{p,q}$ is part of the infinitesimal period map.

Proposition 1.3. *Let $\mathcal{N}$ be an invertible subsheaf of $E^{p,q}$ with $\theta_{p,q}(\mathcal{N}) = 0$. Then* $\deg(\mathcal{N}) \leq 0$.

Proof. If the fibres of g are connected and if g is semistable, this is nothing but 1.2.

In general, let L be a finite extension of the function field $\mathbb{C}(Y)$, containing the Galois hull of the algebraic closure of $\mathbb{C}(Y)$ in $\mathbb{C}(Z)$, and let $Y' \to Y$ be the normalization of Y in L. Consider the normalization $\tilde{Z}$ of $Z \times_Y Y'$, a desingularization $\varphi' : Z' \to \tilde{Z}$ and the induced morphisms

$$\begin{tikzcd}
Z' \arrow[r, "\varphi'"] \arrow[dr, "g'"'] & \tilde{Z} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\tilde{g}"] & Z \times_Y Y' \arrow[r, "p_1"] \arrow[dl, "p_2"] & Z \arrow[dl, "g"] \\
 & Y' \arrow[r, "\psi"] & Y &
\end{tikzcd}$$

$\varphi = \tilde{\varphi} \circ \varphi'$ and $\psi' = p_1 \circ \varphi$. We will enlarge S such that $Y' \to Y$ is étale over $Y - S$, hence for $S' = \psi^* S$

$$\psi^*\Omega^1_Y(\log S) = \Omega^1_{Y'}(\log S').$$

If one chooses L large enough, Z' will be the disjoint union of semistable families over Y', hence $\Pi' = g'^* S'$ is a reduced normal crossing divisor, and $\varphi|_{g'^{-1}(Y'-S')}$ is an isomorphism. By the generalized Hurwitz formula [5], 3.21,

$$\psi'^*\Omega^p_Z(\log \Pi) \subset \Omega^p_{Z'}(\log \Pi'),$$

and by [3], Lemme 1.2,

$$R^q\varphi'_*\Omega^p_{Z'}(\log \Pi') = \begin{cases} \tilde{\varphi}^* p_1^* \Omega^p_Z(\log \Pi) & \text{for } \ q = 0 \\ 0 & \text{for } \ q > 0. \end{cases}$$

The exact sequence (1.2.1), for Z' instead of Z, and induction on p allow to show the same for the relative differential forms, i.e.

$$R^q\varphi'_*\Omega^p_{Z'/Y'}(\log \Pi') = \begin{cases} \tilde{\varphi}^* p_1^* \Omega^p_{Z/Y}(\log \Pi) & \text{for } \ q = 0 \\ 0 & \text{for } \ q > 0. \end{cases}$$

Hence the pullback of the exact sequence (1.2.1) to $\tilde{Z}$ is isomorphic to

$$(1.3.1)\quad 0 \longrightarrow {\varphi'}_*({g'}^*\Omega^1_{Y'}(\log S') \otimes \Omega^{p-1}_{Z'/Y'}(\log \Pi')) \\ \longrightarrow {\varphi'}_*\Omega^p_{Z'}(\log \Pi') \longrightarrow {\varphi'}_*\Omega^p_{Z'/Y'}(\log \Pi') \longrightarrow 0.$$

Writing

$$E'^{p,q} = R^q g'_* \Omega^p_{Z'/Y'}(\log \Pi'),$$

and $\theta'_{p,q}$ for the edge-morphism, we find

$$E'^{p,q} = R^q p_{2*}(\tilde{\varphi}_* \mathcal{O}_{\tilde{Z}} \otimes p_1^* \Omega^p_{Z/Y}(\log \Pi)).$$

Moreover, the inclusion $\mathcal{O}_{Z\times_Y Y'} \to \tilde{\varphi}_*\mathcal{O}_{\tilde{Z}}$ and flat base change give an inclusion

$$\psi^* E^{p,q} = R^q p_{2*}(p_1^* \Omega^p_Z(\log \Pi)) \longrightarrow E'^{p,q}$$

and the diagram

$$\begin{array}{ccc} \psi^* E^{p,q} & \xrightarrow{\psi^*\theta_{p,q}} & \psi^* E^{p-1,q+1} \otimes \Omega^1_Y(\log S) \\ \subset\big\downarrow & & \subset\big\downarrow \\ E'^{p,q} & \xrightarrow{\theta'_{p,q}} & E'^{p-1,q+1} \otimes \Omega^1_{Y'}(\log S') \end{array}$$

commutes. In particular, if $\mathcal{N}$ lies in the kernel of $\theta_{p,q}$, the sheaf $\psi^*\mathcal{N}$ lies in the kernel of $\theta'_{p,q}$. Since we already know 1.3 for semistable morphisms with connected fibres, we find $\deg(\mathcal{N}) \leq 0$. $\square$

2. Positivity of direct image sheaves

As in [2] or [14] a second ingredient in the proof of 0.1 and 0.3 will be explicit bounds for the positivity of certain direct image sheaves.

Definition 2.1. Let $\mathcal{E}$ be a locally free sheaf and $\mathcal{A}$ an invertible sheaf on the curve Y.

a) $\mathcal{E}$ is nef if for all finite morphisms $\pi : Z \to Y$ and all invertible quotients $\mathcal{Q}$ of $\pi^*\mathcal{E}$ the degree of $\mathcal{Q}$ is non-negative.

b) For $\alpha, \beta \in \mathbb{N} - \{0\}$ we write

$$\mathcal{E} \succeq \frac{\alpha}{\beta}\mathcal{A}$$

if $S^\beta(\mathcal{E}) \otimes \mathcal{A}^{-\alpha}$ is nef. This is well-defined, since obviously the latter holds true if and only if

$$S^{\beta\cdot\mu}(\mathcal{E}) \otimes \mathcal{A}^{-\alpha\cdot\mu}$$

is nef, for some $\mu > 0$.

Nef locally free sheaves on curves, have already been used in [4] to study the height of points of curves over function fields. In [19], §2, and in the higher dimensional birational classification theory, one needs positive coherent torsionfree sheaves over higher dimensional manifolds, and there one often considers weakly positive sheaves, instead of nef sheaves. In the one-dimensional case, both notions coincide, and all the properties of weakly positive sheaves, listed in [13] or [19] carry over to nef sheaves on curves. Let us recall one property:

Lemma 2.2. *Given $d \in \mathbb{N}$, assume that for all $\mu \in \mathbb{N}$, sufficiently large and divisible, there exists a covering $\tau : Y' \to Y$ of degree μ such that $\tau^*\mathcal{E} \otimes \mathcal{H}$ is nef, for one, hence for all invertible sheaves of degree d. Then $\mathcal{E}$ is nef.*

Proof. Let $\pi : Z \to Y$ and $\mathcal{Q}$ be as in 2.1, a), and let Z' be a component of the normalization of $Z \times_Y Y'$. If

$$\begin{array}{ccc} Z' & \xrightarrow{\tau'} & Z \\ {\scriptstyle \pi'}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{\tau} & Y \end{array}$$

are the induced morphisms, then

$$\begin{aligned} 0 \leq \deg({\tau'}^*\mathcal{Q} \otimes {\pi'}^*\mathcal{H}) = \deg(\tau') \cdot \deg(\mathcal{Q}) + \deg(\pi') \cdot d \\ \leq \mu \cdot \deg(\mathcal{Q}) + \deg(\pi) \cdot d. \end{aligned}$$

This, for all $\mu \in \mathbb{N} - \{0\}$, implies that $\deg(\mathcal{Q}) \geq 0$. □

Next recall the definition of the (algebraic) multiplier sheaves. We consider a surjective morphism $f : X \to Y$, with connected general fibre F, where X is an (n+1)-dimensional complex projective manifold, and Y a non-singular projective curve. If Γ is an effective divisor on X,

$$\omega_{X/Y}\left\{-\frac{\Gamma}{N}\right\} = \tau_*\left(\omega_{X'/Y}\left(-\left[\frac{\Gamma'}{N}\right]\right)\right)$$

where $\tau : X' \to X$ is any blowing up with $\Gamma' = \tau^*\Gamma$ a normal crossing divisor (see for example [5], 7.4, or [19], section 5.3).

Fujita's positivity theorem (today an easy corollary of Kollár's vanishing theorem) says that $f_*\omega_{X/Y}$ is nef. A direct consequence is the following.

Lemma 2.3. *Let $\mathcal{N}$ be an invertible sheaf on X and Γ an effective divisor. Assume that for some $N > 0$ there exists a nef locally free sheaf $\mathcal{E}$ on Y and a surjection $f^*\mathcal{E} \to \mathcal{N}^N(-\Gamma)$. Then*

$$f_*\left(\mathcal{N} \otimes \omega_{X/Y}\left\{-\frac{\Gamma}{N}\right\}\right)$$

is nef.

Proof. Let $p \in Y$ be a point. Then $\mathcal{E} \otimes \mathcal{O}_Y(N \cdot p)$ is ample, hence

$$\mathcal{N}^N(-\Gamma) \otimes f^*\mathcal{O}_Y(N \cdot p)$$

is semi-ample. By [5], 7.16, the sheaf

$$f_*\left(\mathcal{N} \otimes \omega_{X/Y}\left\{-\frac{\Gamma}{N}\right\}\right) \otimes \mathcal{O}_Y(p)$$

is nef. Since the same holds true over all Y', finite over Y and unramified in S, one obtains 2.3 from 2.2. □

As an application of 2.3 one obtains, as explained in [13],

Lemma 2.4.

i) *$f_*\omega^{\nu}_{X/Y}$ is nef, for all $\nu \geq 0$.*

ii) *If* $\lambda_\nu = \det(f_*\omega^\nu_{X/Y})$ *is ample, for some* $\nu > 1$*, then there exists a positive rational number* η *with* $f_*\omega^\nu_{X/Y} \succeq \eta \cdot \lambda_\nu$*.*

As in [2], we will need an explicit bound for the rational number η in 2.4, ii). To this aim recall the following definition, used in [4], [5], § 7 and [19], section 5.3.

Definition 2.5. Let $\mathcal{L}$ be an invertible sheaf on F with $H^0(F, \mathcal{L}) \neq 0$, and let Γ be an effective divisor. Then

$$e(\Gamma) = \text{Min } \left\{ N \in \mathbb{N} - \{0\};\ \omega_F \left\{ -\frac{\Gamma}{N} \right\} = \omega_F \right\} \qquad \text{and}$$

$$e(\mathcal{L}) = \text{Max } \left\{ e(\Gamma);\ \Gamma \text{ the zero set of } \sigma \in H^0(F, \mathcal{L}) - \{0\} \right\}.$$

Notations 2.6. For $f : X \to Y$, we choose $\nu > 1$ with $f_*\omega^\nu_{X/Y} = \mathcal{E} \neq 0$, and a blowing up $\tau : X' \to X$ such that the fibres of $f' = f \circ \tau$ are normal crossing divisors, such that

$$\mathcal{L} = \text{Im}(f'^* f'_* \omega^\nu_{X'/Y} = f'^* \mathcal{E} \longrightarrow \omega^\nu_{X'/Y})$$

is invertible and $\omega^\nu_{X'/Y} = \mathcal{L}(B)$, for a normal crossing divisor B. Let F' be a general fibre of f'. We define:

$e = e(\mathcal{L}|_{F'})$

$r = \text{rank}(f'_*\omega^\nu_{X'/Y}) = \text{rank}(\mathcal{E})$

$\lambda = \det(f'_*\omega^\nu_{X'/Y}) = \det(\mathcal{E})$.

Proposition 2.7. *If* λ *is ample,* $f_*\omega^\nu_{X/Y} \succeq \frac{1}{r \cdot e} \cdot \lambda$*.*

Proof. If $e = 1$, the sheaf $\mathcal{L}|_{F'}$ is trivial, hence $f_*\omega^\nu_{X/Y} = \lambda$ and 2.7 obviously holds true. Hence we will assume $e \geq 2$. For some $\mu \gg 0$ there exists an effective divisor Σ_1, disjoint from S with $\lambda^\mu = \mathcal{O}_Y(\Sigma_1)$. By 2.2 and by flat base change, we are free to replace Y by any Y', finite over Y and unramified over a neighborhood of S. Hence we are allowed to assume that $\Sigma_1 = (\nu - 1) \cdot e \cdot \mu \cdot \Sigma$ or that

$$\lambda = \mathcal{O}_Y((\nu - 1) \cdot e \cdot \Sigma).$$

Consider the r-fold fibre product

$$f^r : X^r = X' \times_Y X' \ldots \times_Y X' \longrightarrow Y.$$

f^r is flat and Gorenstein and smooth over some open subscheme. Let

$$\pi : X^{(r)} \longrightarrow X^r$$

be a desingularization such that the general fibre $F^{(r)}$ of $f^{(r)} = f^r \circ \pi$ is isomorphic to $F \times \ldots \times F$. For

$$\mathcal{N} = \pi^* \bigotimes_{i=1}^{r} pr_i^* \mathcal{L} \subset \pi^* \omega^\nu_{X^r/Y},$$

using flat base change, and the natural maps

$$\mathcal{O}_{X^r} \to \pi_* \mathcal{O}_{X^{(r)}} \quad \text{and} \quad \pi_* \omega_{X^{(r)}} \to \omega_{X^r},$$

one finds

$$\bigotimes^r f'_*\omega^\nu_{X'/Y} = \bigotimes^r \mathcal{E} = \bigotimes^r f'_*\mathcal{L} \longrightarrow f^{(r)}_*\mathcal{N} \quad \text{and} \tag{2.7.1}$$

$$f^{(r)}_*((\pi^*\omega^{\nu-1}_{X^r/Y}) \otimes \omega_{X^{(r)}/Y}) \longrightarrow f^r_*\omega^\nu_{X^r/Y} = \bigotimes f'_*\omega^\nu_{X'/Y}, \tag{2.7.2}$$

and both are isomorphism over some open dense subset of Y. (2.7.1) induces a surjection

$$f^{(r)*}\bigotimes^r \mathcal{E} = \pi^*\bigotimes_{i=1}^r pr_i^* f'^*\mathcal{E} \longrightarrow \pi^*\bigotimes_{i=1}^r pr_i^*\mathcal{L} = \mathcal{N}.$$

In particular, since $\lambda \subset \otimes^r\mathcal{E}$, the sheaf $f^{(r)^*}\lambda$ is a subsheaf of $\mathcal{N}$. Let Γ denote the divisor with $\mathcal{N}(-\Gamma) = f^{(r)*}\lambda$. For some divisor C, supported in fibres of $f^{(r)}$ one has

$$\pi^*\omega_{X^r/Y} = \omega_{X^{(r)}/Y}(C).$$

Blowing up $X^{(r)}$ with centers in fibres of $f^{(r)}$ we find a normal crossing divisor D with $\mathcal{N}(D) = \omega_{X^{(r)}/Y}(C)^\nu$ and such that the divisor

$$D - \pi^*\sum_{i=1}^r pr_i^* B$$

is effective. For

$$\nabla = e\cdot(\nu-1)\cdot D + \nu\cdot\Gamma + e\cdot\nu\cdot(\nu-1)\cdot f^{(r)*}(\Sigma)$$

one obtains

$$\omega_{X^{(r)}/Y}(C)^{e\cdot\nu\cdot(\nu-1)}(-\nabla) = \mathcal{N}^{e\cdot(\nu-1)}(-\nu\cdot\Gamma)\otimes f^{(r)*}\lambda^{-\nu} = \mathcal{N}^{e\cdot(\nu-1)-\nu}.$$

Since we assumed $e, \nu \geq 2$, the exponent of $\mathcal{N}$ is non-negative. By 2.4, i), the sheaf $\otimes^r\mathcal{E}$ is nef and 2.3 implies that

$$\mathcal{F} = f^{(r)}_*\left(\omega_{X^{(r)}/Y}(C)^{\nu-1}\otimes\omega_{X^{(r)}/Y}\left\{-\frac{\nabla}{e\cdot\nu}\right\}\right)$$

is nef. $\mathcal{F}$ is contained in

$$\mathcal{F}' = f^{(r)}_*(\pi^*\omega^{\nu-1}_{X^r/Y}\otimes\omega_{X^{(r)}/Y})\otimes\mathcal{O}_Y(-(\nu-1)\cdot\Sigma),$$

and using (2.7.2) ones finds

$$\mathcal{F} \subseteq (\bigotimes^r f'_*\omega^\nu_{X'/Y})\otimes\mathcal{O}_Y(-(\nu-1)\cdot\Sigma).$$

On the other hand, $\mathcal{F}$ contains

$$\mathcal{F}'' = f^{(r)}_*\left(\omega_{X^{(r)}/Y}(C)^{\nu-1}\otimes\omega_{X^{(r)}/Y}\left\{-\frac{\nabla - e\cdot D}{e\cdot\nu}\right\}\right).$$

Over some sufficiently small open dense subset $U \subset Y$

$$\mathcal{F}''|_U = f^{(r)}_*\left(\omega^{\nu-1}_{X^{(r)}/Y}\otimes\omega_{X^{(r)}/Y}\left\{-\frac{\nu\cdot\Gamma}{\nu\cdot e}\right\}\otimes\mathcal{O}_{X^{(r)}}(-D)\right)\Big|_U.$$

By definition $e = e(\mathcal{L}|_F)$, and from [5] or [19], 5.21, one has

$$e = e(\mathcal{L}|_F) = e(\bigotimes_{i=1}^r pr_i^*\mathcal{L}|_F) = e(\mathcal{N}|_{F^{(r)}}).$$

The semicontinuity of e in [19], 5.14, implies that for U small enough,

$$\mathcal{F}''|_U = f_*^{(r)}(\omega^\nu_{X^{(r)}/Y} \otimes \mathcal{O}_{X^{(r)}}(-D)) = \mathcal{F}'|_U.$$

Hence $\mathcal{F}, \mathcal{F}'$ and $\mathcal{F}''$ have the same rank and $\mathcal{F}'$ is nef. We obtain

$$\bigotimes^r f'_*\omega^\nu_{X'/Y} \succeq \mathcal{O}_Y((\nu-1)\cdot\Sigma) = \frac{1}{e}\lambda$$

or $f'_*\omega^\nu_{X'/Y} \succeq \frac{1}{r\cdot e}\lambda$, as claimed. □

3. Differential forms and cyclic coverings

Let Y be a non-singular projective curve of genus g, and let $S \subset Y$ be a reduced divisor of degree s. We consider again an $(n+1)$-dimensional manifold X and a surjective morphism $f : X \to Y$ with connected general fibre F. For $\Delta = f^{-1}(S)$ we will assume that $f_0 = f|_{X-\Delta}$ is smooth.

For some $\nu > 1$, with $f_*\omega^\nu_{X/Y} \neq 0$, we choose, as in 2.6, a blowing up $\tau : X' \to X$ and a normal crossing divisor B. Hence writing $f' = f \circ \tau$ and $\mathcal{L} = \omega^\nu_{X'/Y}(-B)$ one has $f'_*\mathcal{L} = f'_*\omega^\nu_{X'/Y}$ and $f'^*f'_*\mathcal{L} \twoheadrightarrow \mathcal{L}$. Choose $S' \supset S$, such that f' is smooth outside of $\Delta' = f'^{-1}(S')$, and such that $B - (B \cap \Delta')$ is a relative normal crossing divisor over $Y - S'$. Blowing up with centers in Δ', we may also assume $B + \Delta'$ to be a normal crossing divisor.

We will assume throughout this section that there exists an ample invertible sheaf $\mathcal{A}$ with

$$f'_*\omega^\nu_{X'/Y} \otimes \mathcal{A}^{-\nu} = f_*\omega^\nu_{X/Y} \otimes \mathcal{A}^{-\nu} \qquad \text{ample.} \tag{3.0.3}$$

Remark 3.1. Using the notations introduced in 2.6, the existence of such a sheaf $\mathcal{A}$ implies that $\deg(\lambda) > \nu \cdot r \cdot \deg(\mathcal{A})$. On the other hand, if $\deg(\lambda)$ is strictly larger than $\nu \cdot r \cdot e$ one can choose an ample invertible sheaf $\mathcal{A}$ with $\deg(\lambda) > \nu \cdot r \cdot e \cdot \deg(\mathcal{A})$. By 2.7 such a sheaf will satisfy the condition (3.0.3).

The condition (3.0.3) implies that for $N = \nu \cdot \mu$ and μ sufficiently large, and for $\mathcal{M} = \omega_{X'/Y} \otimes f'^*\mathcal{A}^{-1}$ the sheaf

$$\mathcal{M}^N(-\mu \cdot B) = \mathcal{L}^\mu \otimes f'^*\mathcal{A}^{-N}$$

is generated by global sections. For a general section σ with zero divisor $V(\sigma)$, the divisor $M = \mu B + V(\sigma)$ as well as $M + \Delta'$ are normal crossing divisors. Enlarging S', if necessary, and replacing X' by a blowing up with centers in Δ', we may assume that all the fibres of

$$f \circ \tau : X' - \Delta' \longrightarrow Y - S'$$

intersect M transversely.

As explained in [5], § 2, σ defines a cyclic covering Z of X'. In explicit terms, writing $\mathcal{M}^{(-i)} = \mathcal{M}^{-i}([\frac{i \cdot M}{N}])$, the covering is given by

$$\gamma : Z = \mathbf{Spec}\left(\bigotimes_{i=0}^{N-1} \mathcal{M}^{(-i)}\right) \longrightarrow X'.$$

Let us write $g = f' \circ \gamma$ for the induced morphism from Z to Y. The variety Z is normal, and the discriminant of γ is $M' = (M - N \cdot [\frac{M}{N}])_{\mathrm{red}}$. Let us define

$$\Omega^p_Z(\log \gamma^{-1}(\Delta' + M')) \quad \text{and} \quad \Omega^p_{Z/Y}(\log \gamma^{-1}(\Delta' + M'))$$

to be the reflexive hulls of the corresponding sheaves on the smooth locus of Z. One has natural isomorphisms

$$\begin{aligned} &\Omega^p_Z(\log\gamma^{-1}(\Delta'+M')) \cong \gamma^*\Omega^p_{X'}(\log(\Delta'+M')) \\ \text{and}\quad &\Omega^p_{Z/Y}(\log\gamma^{-1}(\Delta'+M')) \cong \gamma^*\Omega^p_{X'/Y}(\log(\Delta'+M')). \end{aligned} \tag{3.1.1}$$

In fact, as explained in [5], §2, both isomorphisms exist on the smooth locus of Z, and since the sheaves on the right hand sides are locally free, they extend to Z. In particular, the sheaves $\Omega^p_Z(\log\gamma^{-1}(\Delta'+M'))$ and $\Omega^p_{Z/Y}(\log\gamma^{-1}(\Delta'+M'))$ are both locally free, and $\Omega^p_{X'}(\log(\Delta'+M'))\otimes\mathcal{M}^{(-1)}$ is a direct factor of $\gamma_*\Omega^p_Z(\log\gamma^{-1}(\Delta'+M'))$. The image of the induced map

$$\gamma^*\Omega^p_{X'}(\log(\Delta'+M'))\otimes\mathcal{M}^{(-1)} \longrightarrow \Omega^p_Z(\log\gamma^{-1}(\Delta'+M'))$$

lies in the subsheaf $\Omega^p_Z(\log\gamma^{-1}(\Delta'))\tilde{\ }$, where $(\quad)\tilde{\ }$ denotes the reflexive hull. Again, similar inclusions hold true for the relative differential forms over Y.

Kawamata's covering construction (see [19], 2.2) allows to choose a finite covering $Z''\to Z$ with Z'' non singular. Blowing up centers in fibres over Y, we obtain a non-singular variety Z' and a generically finite map $\eta: Z'\to Z$, such that all the fibres of

$$g' = g\circ\eta : Z' \longrightarrow Y$$

are normal crossing divisors. By abuse of notations we will add some points to S', hence some fibres to Δ', and assume that g' is smooth outside of $\Pi' = {g'}^{-1}(S')$ and that η is finite outside of Π'. Let $\gamma': Z'\to X'$ be the induced map. Since γ' factors through a non-singular variety finite over X', the higher direct images $R^q\gamma'_*\mathcal{O}_{Z'}$ are zero for $q>0$.

Let M'' denote the proper transform of M' in Z'. Since γ' is finite outside of Π' one obtains natural inclusions (see [5], 3.20, for example)

$$\begin{aligned} \eta^*\Omega^p_Z(\log\gamma^{-1}(\Delta'+M')) &\longrightarrow \Omega^p_{Z'}(\log(\Pi'+M'')) \quad\text{and} \\ \eta^*\Omega^p_{Z/Y}(\log\gamma^{-1}(\Delta'+M')) &\longrightarrow \Omega^p_{Z'/Y}(\log(\Pi'+M'')). \end{aligned} \tag{3.1.2}$$

(3.1.1) and (3.1.2) together induce inclusions

$$\begin{aligned} \gamma'^*(\tau^*\Omega^p_{X/\bullet}(\log\Delta))\otimes\mathcal{M}^{(-1)} &\longrightarrow \gamma'^*\Omega^p_{X'/\bullet}(\log(\Delta'+M'))\otimes\mathcal{M}^{(-1)} \\ &\longrightarrow \eta^*\Omega^p_{Z/\bullet}(\log\gamma^{-1}(\Delta'+M')) \longrightarrow \Omega^p_{Z'/\bullet}(\log(\Pi'+M'')), \end{aligned} \tag{3.1.3}$$

where $\bullet$ stands for Y or for $\mathrm{Spec}\mathbb{C}$, respectively.

Again, the image of composite of the injections in 3.1.3 must have trivial residues along the components of M'', hence one finds a natural map

$$\iota_\bullet : \gamma'^*(\tau^*\Omega^p_{X/\bullet}(\log\Delta))\otimes\mathcal{M}^{(-1)} \xrightarrow{\subset} \Omega^p_{Z'/\bullet}(\log\Pi').$$

Consider the tautological sequence

(3.1.4)

$$0\longrightarrow f^*\Omega^1_Y(\log S)\otimes\Omega^{p-1}_{X/Y}(\log\Delta)\longrightarrow\Omega^p_X(\log\Delta)\longrightarrow\Omega^p_{X/Y}(\log\Delta)\longrightarrow 0.$$

Pulling it back to X' and tensorizing with $\mathcal{M}^{(-1)}$ one obtains

$$\begin{aligned} &0\longrightarrow f'^*\Omega^1_Y(\log S)\otimes\tau^*\Omega^{p-1}_{X/Y}(\log\Delta)\otimes\mathcal{M}^{(-1)} \\ &\longrightarrow\tau^*\Omega^p_X(\log\Delta)\otimes\mathcal{M}^{(-1)}\longrightarrow\tau^*\Omega^p_{X/Y}(\log\Delta)\otimes\mathcal{M}^{(-1)}\longrightarrow 0. \end{aligned} \tag{3.1.5}$$

The inclusions ι_Y, $\iota_{\mathrm{Spec}\mathbb{C}}$ and

$$\iota : \Omega^1_Y(\log S) \hookrightarrow \Omega^1_Y(\log S')$$

induce a morphism from the pullback of this exact sequence to

(3.1.6)
$$0 \longrightarrow g'^*\Omega^1_Y(\log S') \otimes \Omega^{p-1}_{Z'/Y}(\log \Pi') \longrightarrow \Omega^p_{Z'}(\log \Pi') \longrightarrow \Omega^p_{Z'/Y}(\log \Pi') \longrightarrow 0$$

Let us define

$$E^{p,q} = R^q g'_* \Omega^p_{Z'/Y}(\log \Pi')$$
$$\text{and} \qquad F^{p,q} = R^q f'_*((\tau^*\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)}).$$

The inclusion ι_Y gives a map

$$R^q g'_*(\gamma'^*(\tau^*\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)}) \longrightarrow R^q g'_*\Omega^p_{Z'/Y}(\log \Pi').$$

Since the first sheaf is isomorphic to

$$R^q f'_*((\gamma'_*\gamma'^*\mathcal{O}_{Z'}) \otimes (\tau^*\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)})$$

we obtain thereby a morphism $\rho_{p,q} : F^{p,q} \to E^{p,q}$. Obviously

$$\rho_{n,0} : f'_*((\tau^*\Omega^n_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)}) \longrightarrow g'_*\Omega^n_{Z'/Y}(\log \Pi')$$

is injective, and

$$\rho_{0,n} : R^n f'_*(\mathcal{M}^{(-1)}) \longrightarrow R^n f'_*(\gamma_*\mathcal{O}_Z) \longrightarrow R^n f'_*(\gamma'_*\mathcal{O}_{Z'}) = R^n g'_*\mathcal{O}_{Z'}$$

gives $F^{0,n}$ as a direct factor of $E^{0,n}$. The edge-morphism

$$E^{p,q} \xrightarrow{\theta_{p,q}} E^{p-1,q+1} \otimes \Omega^1_Y(\log S')$$

of the exact sequence (3.1.6) is the Kodaira Spencer map, studied in §1. Since the pullback of (3.1.5) to Z' is a subsequence of (3.1.6) $\theta_{p,q}$ commutes with the edge-morphism

$$R^q g'_*(\gamma'^*(\tau^*\Omega^p_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)})$$
$$\longrightarrow R^{q+1} g'_*(\gamma'^*(\tau^*\Omega^{p-1}_{X/Y}(\log \Delta)) \otimes \mathcal{M}^{(-1)}) \otimes \Omega^1_Y(\log S).$$

So the edge-morphism of the exact sequence (3.1.5), denoted by

$$F^{p,q} \xrightarrow{\tau_{p,q}} F^{p-1,q+1} \otimes \Omega^1_Y(\log S),$$

is compatible with $\theta_{p,q}$.

Lemma 3.2. *Assume for an ample invertible sheaf $\mathcal{A}$, and for $\nu > 1$ (3.0.3) holds true. Then, using the notations introduced above,*

i) *the Kodaira Spencer map for $g' : Z' \to Y$ and the edge-morphism of the exact sequence (3.1.5) induce a commutative diagram*

$$\begin{array}{ccc} E^{p,q} & \xrightarrow{\theta_{p,q}} & E^{p-1,q+1} \otimes \Omega^1_Y(\log S') \\ {\scriptstyle \rho_{p,q}}\uparrow & & \uparrow{\scriptstyle \rho_{p-1,q+1}\otimes\iota} \\ F^{p,q} & \xrightarrow{\tau_{p,q}} & F^{p-1,q+1} \otimes \Omega^1_Y(\log S). \end{array}$$

ii) *$\rho_{n,0}$ is injective.*

iii) *$F^{0,n}$ is a direct factor of $E^{0,n}$.*

iv) *the sheaf $(F^{0,n})^\vee$ is ample.*

v) *the sheaf $F^{n,0}$ is invertible of degree*

$$\deg(F^{n,0}) \geq \deg(\mathcal{A}) - \delta,$$

where δ denotes the number of non-semistable fibres.

Proof. It remains to verify iv) and v). Comparing the first Chern classes of the sheaves in (3.1.4) one finds $\Omega^n_{X/Y}(\log \Delta) = \omega_{X/Y}(\Delta_{\rm red} - \Delta)$. Hence for some invertible sheaf $\mathcal{H}$ of degree δ one has an inclusion

$$\Omega^n_{X/Y}(\log \Delta) \supset \omega_{X/Y} \otimes f^*\mathcal{H}^{-1}.$$

Recall that $\mathcal{M} = \omega_{X'/Y} \otimes f'^*\mathcal{A}^{-1}$, and that

$$\mathcal{M}^{(-1)} = \mathcal{M}^{-1}\left(\left[\frac{M}{N}\right]\right) = \mathcal{M}^{-1}\left(\left[\frac{B}{\nu}\right]\right),$$

where B is the relative fix locus of $\omega^\nu_{X'/Y}$. In particular, if E denotes the effective divisor with $\omega_{X'/Y}(-E) = \tau^*\omega_{X/Y}$, the divisor B is larger than $\nu \cdot E$.

In order to prove that $\mathcal{A} \otimes \mathcal{H}^{-1}$ is a subsheaf of $F^{n,0}$, and that the latter is invertible, we just have to show that

$$f'_*(\tau^*\omega_{X/Y} \otimes \mathcal{M}^{(-1)}) = f'_*\left((f'^*\mathcal{A}) \otimes \mathcal{O}_{X'}\left(\left[\frac{B}{\nu}\right] - E\right)\right)$$

is isomorphic to $\mathcal{A}$, or that $\mathcal{O}_Y \simeq f'_*\mathcal{O}_{X'}\left(\left[\frac{B}{\nu}\right] - E\right)$. This is obvious since $0 \leq \left[\frac{B}{\nu}\right] - E \leq B$ and since B is the relative fix locus of an invertible sheaf.

1.3 says in particular, that $E^{0,n} = \mathrm{Ker}(\tau_{0,n})$ has no invertible subsheaf of positive degree. Using iii) one obtains the same for $F^{0,n} = R^n f'_*\mathcal{M}^{(-1)}$ and $(F^{0,n})^\vee$ is nef. Serre duality and the projection formula imply

$$(F^{0,n})^\vee = \mathcal{A}^{-1} \otimes f'_*(\omega^2_{X'/Y} \otimes \mathcal{O}_{X'}(-[\frac{B}{\nu}])).$$

To prove the ampleness, claimed in iv), choose some $\eta > 0$ such that

$$S^\eta(f_*\omega^\nu_{X/Y} \otimes \mathcal{A}^{-\nu}) \otimes \mathcal{A}^{-1}$$

is ample and consider a finite covering $\varphi : Y' \to Y$ of degree $\nu \cdot \eta$, étale over a neighborhood of the discriminant divisor S'. Then $\varphi^*\mathcal{A} = \mathcal{A}'^{\nu\cdot\eta}$, for some ample invertible sheaf $\mathcal{A}'$ on Y', and both, $f_*\omega^\nu_{X/Y}$ and $F^{0,n}$ are compatible with pullbacks.

Replacing Y by Y', we may assume thereby that $\mathcal{A} = \mathcal{A}'^{\nu\cdot\eta}$, for some $\mathcal{A}'$, and that $f_*\omega^\nu_{X/Y} \otimes \mathcal{A}^{-\nu} \otimes \mathcal{A}'^{-\nu}$ is ample. Repeating the argument used above, for $\mathcal{A} \otimes \mathcal{A}'$ instead of $\mathcal{A}$, one finds

$$\mathcal{A}^{-1} \otimes \mathcal{A}'^{-1} \otimes f'_*(\omega^2_{X'/Y} \otimes \mathcal{O}_{X'}(-[\frac{B}{\nu}]))$$

to be nef, hence $\mathcal{A}^{-1} \otimes f'_*(\omega^2_{X'/Y} \otimes \mathcal{O}_{X'}(-[\frac{B}{\nu}]))$ to be ample. □

4. The proof of 0.1, 0.2 and 0.3

Recall that starting from $f : X \to Y$ we constructed a family $g' : Z' \to Y$, with discriminant locus $S' \supset S$. As in §1, we consider the Hodge bundles

$$E^{p,q} = R^q g'_* \Omega^p_{Z'/Y}(\log \Pi').$$

In the last section we obtained a morphism

$$\rho_{p,q} : F^{p,q} = R^q f'_*(\tau^* \Omega^p_{X/Y}(\log \Delta) \otimes \mathcal{M}^{(-1)}) \longrightarrow E^{p,q},$$

compatible with the Kodaira Spencer map

$$\theta_{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S').$$

Moreover, by 3.2 the image of $\rho_{p-1,q+1} \otimes \iota$ is contained in $E^{p-1,q+1} \otimes \Omega^1_Y(\log S)$.

Proposition 4.1. *Let δ denote the number of those singular fibres of f which are not reduced normal crossing divisors and let $\nu > 1$ be an integer with $f_*\omega^\nu_{X/Y} \neq 0$. If $\mathcal{A}$ is an invertible sheaf, with $\deg(\mathcal{A}) > \delta$, and such that $f_*\omega^\nu_{X/Y} \otimes \mathcal{A}^{-\nu}$ is ample, then*

a) $(2g - 2 + s) \geq 0$ *implies that* $\deg(\mathcal{A}) \leq n \cdot (2g - 2 + s) + \delta$.
b) $(2g - 2 + s) > 0$ *implies that* $\deg(\mathcal{A}) < n \cdot (2g - 2 + s) + \delta$.

Proof. To handle both cases at once, define $\epsilon = 1$, if $2g - 2 + s = 0$ and $\epsilon = 0$, otherwise. Assume that

$$\deg(\mathcal{A}) \geq n \cdot (2g - 2 + s) + \epsilon + \delta.$$

Then by 3.2, v), $F^{n,0}$ is an invertible subsheaf of $E^{n,0}$ of degree

$$\deg(F^{n,0}) \geq n \cdot (2g - 2 + s) + \epsilon,$$

in particular it is ample. For $0 \leq i \leq n$, we will construct by induction an invertible subsheaf $\tilde{F}^{n-i,i}$ of $\rho_{n-i,i}(F^{n-i,i}) \subset E^{n-i,i}$ of degree

$$\deg(\tilde{F}^{n-i,i}) \geq (n - i) \cdot (2g - 2 + s) + \epsilon.$$

If $i < n$, the sheaf $\tilde{F}^{n-i,i}$ is ample, and by 1.3 it can not lie in the kernel of $\theta_{n-i,i}$. On the other hand, by 3.2, i),

$$\theta_{n-i,i}(\tilde{F}^{n-i,i}) \subseteq \theta_{n-i,i}(\rho_{n-i,i}F^{n-i,i}) \subseteq \rho_{n-i-1,i+1}(F^{n-i-1,i+1}) \otimes \Omega^1_Y(\log S).$$

The invertible sheaf

$$\tilde{F}^{n-i-1,i+1} = \theta_{n-i,i}(\tilde{F}^{n-i,i}) \otimes \Omega^1_Y(\log S)^{-1}$$

thereby is a subsheaf of

$$\rho_{n-i-1,i+1}(F^{n-i-1,i+1}) \subseteq E^{n-i-1,i+1}$$

of degree $\deg(\tilde{F}^{n-i-1,i+1}) \geq (n - i - 1) \cdot (2g - 2 + s) + \epsilon$.

For $i = n$ we obtain a subsheaf $\tilde{F}^{0,n}$ of degree $\deg(\tilde{F}^{0,n}) \geq \epsilon \geq 0$, contradicting 3.2, iv). $\square$

Now everything is set to prove 0.1, 0.3, and 0.2. We will proceed in the following way: Adding one or two points to S, if necessary, hence declaring some of the smooth fibres to be "singular", we are allowed to assume

$$(2g - 2 + s) = \deg(\Omega^1_Y(\log S)) \geq 0. \quad (4.1.1)$$

If $2g - 2 + s = 0$ and if f is semistable, one finds the degree of λ_ν to be zero. Then the assumptions a) and b) in 0.1 imply that f is isotrivial. For $Y = \mathbb{P}^1$, independently of the additional assumptions on F, this will imply that $\kappa(X) = -\infty$.

For $2g-2+s > 0$ both inequalities stated in 0.3 will hold true, independently of the semi-ampleness of ω_F, whenever $\deg(\lambda_\nu) > 0$.

Proposition 4.2. *Let Y_0 be either an elliptic curve or $\mathbb{C}^*$, and let $\phi : Y' \to Y$ be a finite morphism, étale over Y_0, such that $X \times_Y Y' \to Y'$ has a semistable model $f' : X' \to Y'$. Then, for all $\nu > 1$ with $H^0(F, \omega_F^\nu) \neq 0$, the degree of $\det(f'_*\omega^\nu_{X'/Y'})$ is zero.*

Proof. The sheaf $\lambda' := \det({f'}_*\omega^\nu_{X'/Y'})$ is nef, hence if 4.2 does not hold true, it is ample. Since $f' : X' \to Y'$ is semistable, λ' is compatible with further pullbacks. Replacing Y' by a covering, we may assume that $\deg(\lambda') > \nu \cdot e \cdot r$. By Proposition 2.7

$$f'_*\omega^\nu_{X'/Y'} \succeq \frac{1}{r \cdot e} \cdot \lambda',$$

hence $f'_*\omega^\nu_{X'/Y'} \otimes \mathcal{O}_{Y'}(-\nu \cdot y')$ is ample, for a point $y' \in Y'$. 4.1, a) implies that $\deg(\mathcal{O}_{Y'}(y')) \leq 0$, obviously a contradiction. □

Proposition 4.3. *Assume that $f_*\omega^\nu_{X/Y} \neq 0$, and that $\lambda = \det(f_*\omega^\nu_{X/Y})$ is ample, for some $\nu > 1$. Let δ denote the number of non-semistable fibres, $r = \mathrm{rank}(f_*\omega^\nu_{X/Y})$, and let e be the constant introduced in 2.6. If $2g-2+s > 0$, then $\deg(\lambda) \leq (n \cdot (2g - 2 + s) + \delta) \cdot \nu \cdot e \cdot r$.*

Proof. Choose an invertible sheaf $\mathcal{A}$ on Y of degree $n \cdot (2g - 2 + s) + \delta$. If

$$\deg(\lambda) > (n \cdot (2g - 2 + s) + \delta) \cdot \nu \cdot e \cdot r,$$

one finds $\deg(\mathcal{A}^{\nu \cdot e \cdot r}) < \deg(\lambda)$. By Proposition 2.7

$$f_*\omega^\nu_{X/Y} \succeq \frac{1}{r \cdot e} \cdot \lambda,$$

hence $f_*\omega^\nu_{X/Y} \otimes \mathcal{A}^{-\nu}$ is ample. Proposition 4.1, b) implies that

$$\deg(\mathcal{A}) < n \cdot (2g - 2 + s) + \delta,$$

contradicting the choice of $\mathcal{A}$. □

Proof of 0.1. Assume $f : X \to Y$ is a morphism, smooth over Y_0. If Y_0 is an elliptic curve, f is smooth. For $Y_0 = \mathbb{C}^*$, there exists a finite covering $\phi : Y' \to Y$, étale over $\mathbb{C}^*$, and a semistable family $f' : X' \to Y'$, birational to the pullback of f. Hence to show that f is isotrivial, 4.2 allows to assume that $\deg(\det(f_*\omega^\nu_{X/Y})) = 0$, for all $\nu > 1$.

The experts will have noticed that the assumption a) or b) in 0.1 are exactly those needed by Kawamata, Kollár and the first named author to prove the additivity of the Kodaira dimension, and even the stronger statement $Q_{n,1}$, saying that for non-isotrivial morphisms f

$$\kappa(\omega_{X/Y}) = \kappa(F) + 1.$$

Using 2.4, this is equivalent to the ampleness of $\det(f_*\omega^\nu_{X/Y})$, for all positive multiples ν of some $\nu_0 \gg 1$ (see [13], 7.2 and 7.6, and the references given there). In particular, the morphism $f : X \to Y$, considered above, is isotrivial. $\square$

Proof of 0.2. Assume there exists a morphism $X \to \mathbb{P}^1$, smooth over $\mathbb{C}^*$, and with $\kappa(X) \geq 0$. Let

$$X \xrightarrow{f} Y' \xrightarrow{\phi} \mathbb{P}^1$$

be the Stein factorization. ϕ must be smooth over $\mathbb{C}^*$, hence $Y' = \mathbb{P}^1$ and $\phi^{-1}(S) = \{0, \infty\}$.

Altogether we find a morphism, denoted by $f : X \to \mathbb{P}^1$, which is smooth over $\mathbb{C}^*$ and whose general fibre F is connected.

For a finite morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d, étale over $\mathbb{C}^*$, let

$$\varphi : Z \to X \times_{\mathbb{P}^1} \mathbb{P}^1$$

be a desingularization, and $g = pr_2 \circ \varphi : Z \to \mathbb{P}^1$. Then $\varphi^* pr_1^* \omega_X^\nu$ is a subsheaf of ω_Z^ν, hence $\kappa(Z) \geq \kappa(X) \geq 0$. In particular, for some ν_0 and for all positive multiples ν of ν_0 the sheaf

$$g_* \omega_Z^\nu = \mathcal{O}_{\mathbb{P}^1}(-2 \cdot \nu) \otimes g_* \omega_{Z/\mathbb{P}^1}^\nu$$

has a non trivial section. Therefore $g_* \omega_{Z/\mathbb{P}^1}^\nu$ contains a non-trivial ample subsheaf. On the other hand, $g_* \omega_{Z/\mathbb{P}^1}^\nu$ is nef, hence $\det(g_* \omega_{Z/\mathbb{P}^1}^\nu)$ must be ample. Since g is smooth over $\mathbb{P}^1 - \{0, \infty\}$, and semistable for d sufficiently large and divisible, this contradicts 4.2. $\square$

Proof of 0.3. By 4.3 both inequalities in 0.3 hold true under the slightly more general assumptions a) and b) stated in 0.1. However, the constant e, defined in 2.6 might depend on the general fibre of f and not just on the Hilbert polynomial $h(t)$. For this reason we have to require in 0.3 the sheaf ω_F to be semi-ample.

Under this assumption, there exists a quasi-projective moduli scheme M_h, parameterizing pairs $(F, \mathcal{H})$ where F is a manifold with ω_F semi-ample and with $\mathcal{H}$ a polarization with Hilbert polynomial h (see [19]). Seshadri and Kollár constructed a finite covering Z of M_h, which carries a universal family (see [19], 9.25), containing all $(F, \mathcal{H})$ with Hilbert polynomial $h(t)$ as fibres. So we find some $\nu > 0$, such that for all such $(F, \mathcal{H})$, ω_F^ν is generated by global sections, hence we may choose the sheaf $\mathcal{L}$ in 2.6 in such a way that $\mathcal{L}|_F = \omega_F^\nu$). Finally, by [19], 5.17, $e(\mathcal{L}|_F) = e(\omega_F^\nu)$ is upper semicontinous for the Zariski topology. Hence there exists some e, with $e \geq e(\omega_F^\nu)$ for all F. Altogether, we can choose ν and e in 0.3, to depend just on M_h, hence just on $h(t)$. $\square$

References

[1] Arakelov, A.: Families of algebraic curves with fixed deneracies. Math. U.S.S.R. Izv. **5** (1971) 1277 - 1302

[2] Bedulev, E., Viehweg, E.: On the Shafarevich conjecture for surfaces of general type over function fields. Invent. Math. **139** (2000) 603 - 615

[3] Esnault, H., Viehweg, E.: Revêtement cycliques. Algebraic Threefolds, Proc. Varenna 1981. Springer Lect. Notes in Math. **947** (1982) 241 - 250

[4] Esnault, H.; Viehweg, E.: Effective bounds for semipositive sheaves and for the height of points on curves over complex function fields. Compositio Math. **76** (1990) 69 - 85.

[5] Esnault, H., Viehweg, E.: Lectures on Vanishing Theorems. DMV-Seminar **20** (1992), Birkhäuser, Basel-Boston-Berlin

[6] Griffiths, P.: Topics in transcendental algebraic geometry. Ann of Math. Stud. **106**, Princeton Univ. Press, Princeton, NJ. (1984)

[7] Jost, J, Zuo, K.: Arakelov type inequalities for Hodge bundles over algebraic varieties, Part 1: Hodge bundles over algebraic curves. preprint (1999), to appear in J. Alg. Geometry

[8] Kollár, J.: Subadditivity of the Kodaira Dimension: Fibres of general type. Algebraic Geometry, Sendai, 1985. Advanced Studies in Pure Mathematics **10** (1987) 361-398

[9] Kovács, S.: On the minimal number of singular fibres in a family of surfaces of general type. Journ. Reine Angew. Math. **487** (1997) 171 - 177

[10] Kovács, S.: Algebraic hyperbolicity of fine moduli spaces. J. Alg. Geom. **9** (2000) 165 - 174

[11] Kovács, S.: Logarithmic Kodaira-Akizuki-Nakano Vanishing and Arakelov-Parshin Boundedness for singular varieties. preprint (2000)

[12] Migliorini, L.: A smooth family of minimal surfaces of general type over a curve of genus at most one is trivial. J. Alg. Geom. **4** (1995) 353 - 361

[13] Mori, S.: Classification of higher dimensional varieties. In: Algebraic Geometry. Bowdoin 1985, Proc. Symp. Pure Math. **46** (1987) 269 - 331

[14] Oguiso, K., Viehweg, E.: On the isotriviality of families of elliptic surfaces. preprint (1999), to appear in J. Alg. Geom.

[15] Parshin, A.: Algebraic curves over function fields I. Math. U.S.S.R. Isv. **2** (1968) 1145 - 1170

[16] Peters, C.: Arakelov-type inequalities for Hodge bundles. preprint (1999)

[17] Schmid, W.: Variation of Hodge structure: The singularities of the period mapping. Invent. math. **22** (1973) 211 - 319

[18] Simpson, C.: Harmonic bundles on noncompact curves. J. Am. Math. Soc. 3 **3** (990) 713 - 770

[19] Viehweg, E.: Quasi-projective Moduli for Polarized Manifolds. Ergebnisse der Mathematik, 3. Folge **30** (1995), Springer Verlag, Berlin-Heidelberg-New York

[20] Zucker, S.: Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric. Annals of Math. **109** (1979) 415-476

Universität GH Essen, FB6 Mathematik, 45117 Essen, Germany
E-mail address: `viehweg@uni-essen.de`

The Chinese University of Hong Kong, Department of Mathematics, Shatin, Hong Kong
E-mail address: `kzuo@math.cuhk.edu.hk`