FAMILIES OF ABELIAN VARIETIES OVER CURVES WITH MAXIMAL HIGGS FIELD

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Throughout this note, Y will denote a non-singular complex projective curve, and $f : A \to Y$ a family of abelian varieties, with A non-singular. We write $U \subset Y$ for an open dense subscheme, with

$$f : A_0 = f^{-1}(U) \longrightarrow U$$

smooth, $S = Y \setminus U$, and $\Delta = f^{-1}(S)$. Consider the weight 1 variation of Hodge structures given by $f : A_0 \to U$, i.e. $R^1 f_* \mathbb{Z}_{A_0}$. We will assume throughout this note, that the monodromy of $R^1 f_* \mathbb{Z}_{A_0}$ around all points in S is unipotent. We write

$$(E, \theta) = (E^{1,0} \oplus E^{0,1}, \theta_{1,0})$$

for the Higgs-bundles induced by the Deligne extension of $(R^1 f_* \mathbb{Z}_{A_0}) \otimes \mathcal{O}_U$. Hence $E^{1,0} = f_* \Omega^1_{A/Y}(\log \Delta)$ and $E^{0,1} = R^1 f_* \mathcal{O}_A$. The Higgs field is given by the edge morphisms

$$f_* \Omega^1_{A/Y}(\log \Delta) \to R^1 f_* \mathcal{O}_A \otimes \Omega^1_Y(\log S)$$

of the tautological sequence

$$0 \to f^* \Omega^1_Y(\log S) \to \Omega^1_A(\log \Delta) \to \Omega^1_{A/Y}(\log \Delta)) \to 0.$$

By [9] $E^{1,0}$ is a direct sum $F^{1,0} \oplus N^{1,0}$ with $F^{1,0}$ ample and $N^{1,0} = \mathrm{Ker}(\theta_{1,0})$ flat.

Correspondingly, we have $E^{0,1} = F^{0,1} \oplus N^{0,1}$ and E is the direct sum of the Higgs bundles

$$(F = F^{1,0} \oplus F^{0,1}, \theta_{1,0}) \quad \text{and} \quad (N^{1,0} \oplus N^{0,1}, 0). \tag{0.0.1}$$

For $g_0 = \mathrm{rank}(F^{1,0})$ the Arakelov inequalities ([11], [7]) say

$$2 \cdot \deg(F^{1,0}) \leq g_0 \cdot \deg(\Omega^1_Y(\log S)). \tag{0.0.2}$$

In this note we will try to understand the geometry of families $f : A \to Y$, for which (0.0.2) is an equality, or as we will say shortly, of families reaching the Arakelov bound.

A family of abelian varieties is reaching the Arakelov bound if and only if the Higgs field is maximal (see 1.2, i), i.e. if $\theta_{1,0} : F^{1,0} \to F^{0,1} \otimes \Omega^1_Y(\log S)$ is an isomorphism.

For families of elliptic curves, the maximality of the Higgs field implies that the family is modular:

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Proposition 0.1. *Let $h : E \to Y$ be a semi stable family of elliptic curves, smooth over $U \subset Y$ with $U \neq Y$. If $E \to Y$ is reaching the Arakelov bound, $E \to Y$ is modular, i.e. U is the quotient of the upper half plane $\mathcal{H}$ by a subgroup of $Sl_2(\mathbb{Z})$ of finite index, and the morphism $U \to \mathbb{C} = \mathcal{H}/Sl_2(\mathbb{Z})$ is given by the j-invariant of the fibres.*

In the higher dimensional case one could hope, that the general fibre of a family with maximal Higgs field is quite special, and that the base curve is again a Shimura curve. The corresponding question has been considered in [18] for families of $K3$-surfaces, and methods and results of [18] have been our motivation to study the case of abelian varieties.

Theorem 0.2. *Let $f : A \to Y$ be a family of abelian varieties smooth over $Y \setminus S$, and such that the local monodromies around $s \in S$ are unipotent. If $S \neq \emptyset$, and if $f : A \to Y$ reaches the Arakelov bound, then there exists an étale covering $\pi : Y' \to Y$ such that $f' : A' = A \times_Y Y' \to Y'$ is isogenous over Y' to a product*

$$B \times E \times_{Y'} ... \times_{Y'} E,$$

where B is abelian variety defined over $\mathbb{C}$ of dimension $g - g_0$, and where $h : E \to Y'$ is a family of elliptic curves reaching the Arakelov bound.

We do not know whether for all g there are families of Jacobians among the families of abelian varieties considered in 0.2, i.e. whether one can find a family $\varphi : Z \to Y$ of curves of genus g such that $f : J(Z/Y) \to Y$ reaches the Arakelov bound.

For $Y = \mathbb{P}^1$ the Arakelov inequality implies $\#S \geq 4$. Our hope, that a family with $\#S = 4$ can not be a family of Jacobians, hence that the Jacobian of a family of curves over $\mathbb{P}^1$ must have more than 5 singular fibres, was destroyed by an example of a family of genus 2 curves over the modular curve $X(3)$ in [8], whose Jacobian is isogenous to the product of a fixed elliptic curve B with the modular curve $E(3) \to X(3)$ (see Section 5).

As mentioned already, this note owes a lot to the recent work of the second named author with Xiao-Tao Sun and Sheng-Li Tan. We thank Ernst Kani for explaining his beautiful construction in [8], and for sharing his view about higher genus analogs of families of curves with splitting Jacobians. It is also a pleasure to thank Hélène Esnault for her interest and help, and Ngaiming Mok, for explaining us differential geometric properties of base spaces of families.

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1. Splitting of $\mathbb{C}$-local systems

We will frequently use C. Simpson's correspondence between polystable Higgs bundles of degree zero and representations of the fundamental group $\pi_1(U, s)$.

Theorem 1.1 (C. Simpson [14])**.** *There exists a natural equivalence between the category of direct sums of stable filtered regular Higgs bundles of degree zero, and of direct sums of stable filtered local systems of degree zero.*

We will not recall the definition of a "filtered regular" Higgs bundle ([14], p. 717), just remark that for a Higgs bundle corresponding to a local system with unipotent monodromy around the points in S, the filtration is trivial.

For example, 1.1 implies that the splitting of Higgs bundles (0.1.1) corresponds to a decomposition over $\mathbb{C}$

$$(R^1 f_* \mathbb{Z}_{A_0}) \otimes \mathbb{C} = \mathbb{V} \oplus \mathbb{U}_1$$

where $\mathbb{V}$ corresponds to the Higgs bundle $(F = F^{1,0} \oplus F^{0,1}, \theta)$ and $\mathbb{U}_1$ to $(N = N^{1,0} \oplus N^{0,1}, \theta_N = 0)$. Let $\Theta(N,h)$ denote the curvature of the Hodge metric h on $E^{1,0} \oplus E^{0,1}$ restricted to N, then by [6], chapter II we have

$$\Theta(N, h|_N) = -\theta_N \wedge \bar{\theta}_N - \bar{\theta}_N \wedge \theta_N = 0.$$

This means that $h|_N$ is a flat metric. Hence, $\mathbb{U}_1$ is a unitary local system.

The local system $\mathbb{V}$ on $Y \setminus S$ is a variation of Hodge structures with unipotent local monodromies around $s \in S$. Hence by Deligne's theorem [4], one obtains a decomposition

$$\mathbb{V} = \bigoplus_i \mathbb{V}_i, \tag{1.1.1}$$

for irreducible local systems $\mathbb{V}_i$.

Restricting the Hodge filtration of $\mathbb{V}$ to $\mathbb{V}_i$, one obtains a Hodge filtration on $\mathbb{V}_i$, which in general is not polarized, and (1.1.1) is a decomposition of $\mathbb{C}$-variations of Hodge structures. Taking the grading of the Hodge filtration, one obtains a decomposition of the Higgs bundle $(F = F^{1,0} \oplus F^{0,1}, \theta)$ as a direct sum of sub Higgs bundles, as stated in 1.1.

As a typical application of Simpson's correspondence one finds

Proposition 1.2. *Keeping the notations from the Introduction, assume that $f : A \to Y$ reaches the Arakelov bound. Then*

i) *The sheaf $F^{1,0}$ is poly-stable. Namely there is a decomposition*

$$F^{1,0} \simeq \bigoplus_i \mathcal{A}_i$$

with $\mathcal{A}_i$ stable, and

$$\frac{\deg \mathcal{A}_i}{\mathrm{rank} \mathcal{A}_i} = \frac{\deg F^{1,0}}{\mathrm{rank} F^{1,0}}.$$

Moreover, $\theta_{1,0} : F^{1,0} \to F^{0,1} \otimes \Omega^1_Y(\log S)$ is an isomorphism.

ii) *If $\deg \Omega^1_Y(\log S)$ is even, there exists a decomposition*

$$\mathbb{V} \simeq \mathbb{L} \otimes \mathbb{T},$$

such that $\mathbb{T}$ is a unitary local system, and $\mathbb{L}$ is a rank-2 local system. For some invertible sheaf $\mathcal{L}$ the Higgs bundle corresponding to $\mathbb{L}$ is $(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau)$, with $\tau|_{\mathcal{L}^{-1}} = 0$ and $\tau|_{\mathcal{L}}$ given by an isomorphism

$$\tau^{1,0} : \mathcal{L} \to \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S).$$

Proof. i) Let $\mathcal{A} \subset F^{1,0}$ be a subsheaf, and let $\mathcal{B} \otimes \Omega^1_Y(\log S)$ be its image under $\theta_{1,0}$. Then $\mathcal{A} \oplus \mathcal{B}$ is a Higgs subbundle of $F^{1,0} \oplus F^{0,1}$, and applying 1.1 one finds $\deg(\mathcal{A}) + \deg(\mathcal{B}) \leq 0$. Hence

$$\begin{aligned}\deg(\mathcal{A}) = \deg(\mathcal{B}) + \operatorname{rank}(\mathcal{B}) \cdot \deg(\Omega^1_Y(\log S))\\ \leq -\deg(\mathcal{A}) + \operatorname{rank}(\mathcal{A}) \cdot \deg(\Omega^1_Y(\log S)),\end{aligned}$$

and (0.0.2) implies that

$$\frac{\deg(\mathcal{A})}{\operatorname{rank}(\mathcal{A})} \leq \frac{1}{2} \deg(\Omega^1_Y(\log S)) = \frac{\deg(F^{1,0})}{g_0}.$$

By 1.1 the Higgs bundle $(F^{1,0} \oplus F^{0,1}, \theta)$ splits as a direct sum of stable Higgs bundles of degree zero. If

$$\frac{\deg(\mathcal{A})}{\operatorname{rank}(\mathcal{A})} = \frac{\deg(F^{1,0})}{g_0},$$

the degree of $\mathcal{A} \oplus \mathcal{B}$ is zero, $\operatorname{rank}(\mathcal{A}) = \operatorname{rank}(\mathcal{B})$, and $(\mathcal{A} \oplus \mathcal{B}, \theta|_{\mathcal{A}\oplus\mathcal{B}})$ is a direct factor of $(F^{1,0} \oplus F^{0,1}, \theta)$. In particular, $\mathcal{A}$ is a direct factor of $F^{1,0}$.

For $\mathcal{A} = F^{1,0}$ one finds $\theta_{1,0}$ to be injective. By (0.0.2) it must be an isomorphism.

ii) Taking the determinant of

$$\theta^{1,0} : F^{1,0} \xrightarrow{\simeq} F^{0,1} \otimes \Omega^1_Y(\log S),$$

one obtains an isomorphism

$$\det \theta^{1,0} : \det F^{1,0} \xrightarrow{\simeq} \det F^{0,1} \otimes \Omega^1_Y(\log S)^{\otimes g_0},$$

Since $F^{1,0} \simeq F^{0,1\vee}$,

$$(\det F^{1,0})^{\otimes 2} \simeq \Omega^1_Y(\log S)^{\otimes g_0}.$$

By assumption there exists an invertible sheaf $\mathcal{L}$ with $\mathcal{L} = \Omega^1_Y(\log S)^{1/2}$. For some invertible sheaf $\mathcal{N}$ of order two in $\operatorname{Pic}(Y)$, one finds $\mathcal{N} \otimes \det F^{1,0} = \mathcal{L}^{\otimes g_0}$. By part i) the sheaf

$$\mathcal{T} = F^{1,0} \otimes \mathcal{L}^{-1}$$

is poly-stable of degree zero. 1.1 implies that the Higgs bundle $(\mathcal{T}, 0)$ corresponds to a local system $\mathbb{T}$, necessarily unitary.

Choose $\mathbb{L}$ to be the local system corresponding to the Higgs bundle

$$(\mathcal{L} \oplus \mathcal{L}^{-1}, \tau), \quad \text{with} \quad \tau^{1,0} : \mathcal{L} \xrightarrow{\simeq} \mathcal{L}^{-1} \otimes \Omega^1_Y(\log S).$$

The isomorphism

$$\theta^{1,0} : \mathcal{T} \otimes \mathcal{L} = F^{1,0} \xrightarrow{\simeq} F^{0,1} \otimes \Omega^1_Y(\log S) \xrightarrow{\simeq} F^{0,1} \otimes \mathcal{L}^{\otimes 2}$$

induces an isomorphism

$$\phi : F^{0,1} \xrightarrow{\simeq} \mathcal{T} \otimes \mathcal{L}^{-1},$$

such that $\theta^{1,0} = \operatorname{id}_{\mathcal{T}} \otimes \tau^{1,0}$. Hence the Higgs bundles $(F^{1,0} \oplus F^{0,1}, \theta)$ and $(\mathcal{T} \otimes (\mathcal{L} \oplus \mathcal{L}^{-1}), \operatorname{id}_{\mathcal{T}} \otimes \tau)$ are isomorphic, and $\mathbb{V} \simeq \mathbb{T} \otimes \mathbb{L}$. □

Remark 1.3.

i) If $\deg \Omega^1_Y(\log S)$ is odd, hence $S \neq \emptyset$, and if the genus of Y is not zero, one has to replace Y by an étale two to one cover, in order to be able to apply 1.2, ii).

ii) For $Y = \mathbb{P}^1$ and for families reaching the Arakelov bound, $\#S$ is always even. This, together with the decomposition 1.2, ii), for $\mathbb{U} = \mathbb{C}^{g_0}$, can easily obtained in the following way. By 1.2, i), $F^{1,0}$ must be the direct sum of invertible sheaves $\mathcal{L}_i$, all of the same degree, say ν. Since $\theta^{1,0}$ is an isomorphism, the image $\theta^{1,0}(\mathcal{L}_i)$ is $\mathcal{O}_{\mathbb{P}^1}(2-s+\nu)\otimes\Omega$. Since $F^{0,1}$ is dual to $F^{1,0}$ one obtains $-\nu = 2-s+\nu$, and writing $\mathcal{L}_i^{-1} = \theta^{1,0}(\mathcal{L}_i)$,
$$(F^{1,0}\oplus F^{0,1},\theta) \simeq (\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(\nu)\oplus\mathcal{O}_{\mathbb{P}^1}(-\nu), \bigoplus_i \tau).$$

Consider now the endomorphism $\mathcal{E}nd(\mathbb{V})$ of $\mathbb{V}$, which is a weight zero variation of Hodge structures. The Higgs bundle
$$(F^{1,0}\oplus F^{0,1},\theta)$$
for $\mathbb{V}$ induces the Higgs bundle
$$(F^{1,-1}\oplus F^{0,0}\oplus F^{-1,1},\theta)$$
corresponding to $\mathcal{E}nd(\mathbb{V}) = \mathbb{V}\otimes\mathbb{V}^\vee$, by choosing
$$F^{1,-1} = F^{1,0}\otimes F^{0,1\vee}, \quad F^{0,0} = F^{1,0}\otimes F^{1,0\vee}\oplus F^{0,1}\otimes F^{0,1\vee}$$
$$\text{and} \quad F^{-1,1} = F^{0,1}\otimes F^{1,0\vee}.$$
The Higgs field is given by
$$\theta_{1,-1} = -\mathrm{id}\otimes{\tau_{1,0}}^\vee \oplus \tau_{1,0}\otimes\mathrm{id} \quad \text{and} \quad \theta_{0,0} = \tau_{1,0}\otimes\mathrm{id}\oplus -\mathrm{id}\otimes{\tau_{1,0}}^\vee.$$

Lemma 1.4. *Assume that $f : A \to Y$ reaches the Arakelov bound or equivalently that the Higgs field of $\mathbb{V}$ is maximal. Let*
$$F^{0,0}_u := \mathrm{Ker}(\tau_{0,0}) \quad \textit{and} \quad F^{0,0}_m = \mathrm{Im}(\tau_{1,-1}).$$
Then there is a splitting of the Higgs bundle
$$(F^{1,-1}\oplus F^{0,0}\oplus F^{-1,1},\theta) = (F^{1,-1}\oplus F^{0,0}_m\oplus F^{-1,1},\theta)\oplus(F^{0,0}_u,0),$$
which corresponds to a splitting of the local system over $\mathbb{C}$
$$\mathcal{E}nd(\mathbb{V}) = \mathbb{W}\oplus\mathbb{U}.$$
Moreover, $\mathbb{U}$ is unitary of rank g_0^2, and $\mathbb{W}$ is a $\mathbb{C}$ variation of Hodge structures with maximal Higgs field, i.e.
$$\tau_{1,-1} : F^{1,-1} \to F^{0,0}_m\otimes\Omega^1_Y(\log S) \quad \textit{and} \quad \tau_{0,0} : F^{0,0}_m \to F^{-1,1}\otimes\Omega^1_Y(\log S)$$
are both isomorphisms.

Proof. By definition, $(F^{0,0}_u,0)$ is a sub Higgs bundle of $(F^{1,-1}\oplus F^{0,0}\oplus F^{-1,1},\theta)$. We have an exact sequence
$$0 \to F^{0,0}_u \to F^{0,0} \to F^{-1,1}\otimes\Omega^1_Y(\log S) \to \mathcal{C}$$
where $\mathcal{C}$ is a skyscraper sheaf. Hence
$$\deg(F^{0,0}_u) \geq \deg(F^{0,0}) - \deg(F^{-1,1}) - \mathrm{rank}(F^{-1,1})\cdot\deg(\Omega^1_Y(\log S).$$

Note that if this inequality is an equality then $\mathcal{C}$ is necessarily zero. Since $\deg(F^{0,0}) = 0$ and since, by the Arakelov equality,

$$\deg(F^{-1,1}) = g_0 \cdot \deg(F^{0,1}) + g_0 \cdot \deg(F^{1,0\vee}) = g_0^2 \cdot \deg(\Omega^1_Y(\log S))$$

one finds $\deg(F^{0,0}_u) \geq 0$. By 1.1 the degree of $F^{0,0}_u$ can not be strictly positive, hence it is zero. Again by 1.1 $(F^{0,0}_u, 0)$ being a Higgs subbundle of degree zero with trivial Higgs field, it corresponds to a unitary local subsystem $\mathbb{U}$ of $\mathrm{End}(\mathbb{V})$. The exact sequence

$$0 \to F^{0,0}_u \to F^{0,0} \to F^{-1,1} \otimes \Omega^1_Y(\log S) \to 0$$

splits, and one obtains a direct sum decomposition of Higgs bundles

$$(F^{1,-1} \oplus F^{0,0} \oplus F^{-1,1}, \theta) = (F^{1,-1} \oplus F^{0,0}_m \oplus F^{-1,1}, \theta) \oplus (F^{0,0}_u, 0),$$

which induces the splitting on $\mathcal{E}nd(\mathbb{V})$ with the desired properties. □

Remark 1.5. Using 1.2, ii), the splitting in 1.4 can be made more precise. We know that $\mathbb{V} = \mathbb{T} \oplus \mathbb{L}$ with $\mathbb{T}$ unitary and $\mathbb{L}$ a weight two variation of Hodge structures with maximal Higgs field. One obtains

$$\mathcal{E}nd(\mathbb{V}) = \mathbb{T} \otimes \mathbb{T}^\vee \otimes \mathbb{L} \otimes \mathbb{L}^\vee.$$

Applying 1.4 to $\mathbb{L}$ instead of $\mathbb{V}$, we obtain a decomposition

$$\mathcal{E}nd(\mathbb{L}) = \mathbb{L} \otimes \mathbb{L}^\vee = \mathbb{C} \oplus \mathbb{S}$$

where the $\mathbb{C}$ factor acts by multiplication on $\mathbb{L}$ and where $\mathbb{S}$ has a maximal Higgs field. So $\mathbb{T} \otimes \mathbb{T}^\vee$ is a direct factor of $\mathbb{V}$ of rank g_0^2. Its complement $\mathbb{W} = \mathbb{T} \otimes \mathbb{T}^\vee \otimes \mathbb{S}$ has again a maximal Higgs field.

Remark 1.6. If one replaces $\mathrm{End}(\mathbb{V})$ by the isomorphic locally constant system $(\mathbb{V} \otimes \mathbb{V}) \otimes_{\mathbb{Z}} \mathbb{C}$, one obtains the same decomposition. However, it is more natural to shift the weights by two, and to consider this as a variation of Hodge structures of weight 2.

A statement similar to 1.4 holds true for $\wedge^2(\mathbb{V})$. Here the Higgs bundle is given by

$$F'^{2,0} = F^{1,0} \wedge F^{1,0}, \quad F^{1,1} = F'^{1,0} \otimes F^{0,1} \quad \text{and} \quad F'^{0,2} = F^{0,1} \wedge F^{0,1}.$$

2. Splitting over $\bar{\mathbb{Q}}$

Up to now, we tried to describe the local systems of $\mathbb{C}$-vector spaces $\mathbb{V}$ induced by the family of abelian varieties. We say that such a local system $\mathbb{V}$ is defined over a subfield K of $\mathbb{C}$, if there exists a local system $\mathbb{V}_K$ of K-vector spaces with $\mathbb{V} = \mathbb{V}_K \otimes_K \mathbb{C}$. In this section we want to show, that the splitting $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$ considered in the last section are defined over $\bar{\mathbb{Q}}$, i.e. that there exists a number field K and local systems $\mathbb{V}_K$, $\mathbb{W}_K$ and $\mathbb{U}_K$ with

$$\mathbb{V} = \mathbb{V}_K \otimes \mathbb{C}, \quad \mathbb{W} = \mathbb{W}_K \otimes \mathbb{C}, \quad \mathbb{U} = \mathbb{U}_K \otimes \mathbb{C}, \quad \text{and with} \quad \mathbb{V}_K = \mathbb{W}_K \oplus \mathbb{U}_K.$$

We start with a simple observation. Suppose that $\mathbb{V}$ is a local system defined over a number field K. Choosing a base point $p \in Y \setminus S$ the local system $\mathbb{V}_K$ is given by a representation $\rho : \pi_1(Y \setminus S, p) \to \mathrm{Gl}(V_K)$ for the fibre V_K of $\mathbb{V}_K$ over p.

Fixing a positive integer r, let $\mathcal{G}(r,\mathbb{V})$ denote the set of all rank-r sub local systems of $\mathbb{V}$. Then $\mathcal{G}(r,\mathbb{V})$ is the subvariety of the Grassmann variety $\mathrm{Grass}(r,V_K)$ consisting of the $\pi_1(Y\setminus S,p)$ invariant points. In particular, it is a projective variety defined over K. An L-valued point of $\mathcal{G}(r,\mathbb{V})$ corresponds to a sub local system of $\mathbb{V}_L=\mathbb{V}_K\otimes_K L$. One obtains the following well known property.

Lemma 2.1. *If $[\mathbb{W}]\in\mathcal{G}(r,\mathbb{V})$ is an isolated point, then $\mathbb{W}$ is defined over $\bar{\mathbb{Q}}$.*

Lemma 2.2. *Let $\mathbb{V}$ be the underlying local system of an variation of Hodge structures defined over a real number field K, and suppose that there is a splitting*

$$\mathbb{V}=\mathbb{W}\oplus\mathbb{U}, \tag{2.2.1}$$

such that $\mathbb{U}$ is a unitary, and such that $\mathbb{W}$ has a generically maximal Higgs field $\theta^{p,q}$, i.e

$$\theta^{p,q}:F^{p,q}_{\mathbb{W}}\to F^{p-1,q+1}_{\mathbb{W}}\otimes\Omega^1_Y(\log S)$$

is generically isomorphic for all (p,q) with $p>0$. Then this splitting can be defined over $\bar{\mathbb{Q}}\cap\mathbb{R}$, and is orthogonal with respect to the polarization.

Proof. Consider a family $\{\mathbb{W}_t\}_{t\in\Delta}$ of local subsystems of $\mathbb{V}$ defined over a disk Δ with $\mathbb{W}_0=\mathbb{W}$. For $t\in\Delta$ let $(F_{\mathbb{W}_t},\theta_t)$ denote the Higgs bundle corresponding to $\mathbb{W}_t$. Hence $(F_{\mathbb{W}_t},\theta_t)$ is obtained by restricting the F-filtration of $\mathbb{V}\otimes\mathcal{O}_U$ to $\mathbb{W}_t\otimes\mathcal{O}_U$ and by taking the corresponding graded sheaf. So the Higgs map

$$\theta^{p,q}:F^{p,q}_t\to F^{p-1,q+1}_t\otimes\Omega^1_Y(\log S)$$

will again generically isomorphic for t sufficiently closed to 0 and $p>0$. If the projection

$$\rho:\mathbb{W}_t\to\mathbb{V}\to\mathbb{W}\oplus\mathbb{U}\to\mathbb{U}$$

is non-zero, the complete reducibility of local systems coming from variations of Hodge structures (due to Deligne [1]) implies that $\mathbb{W}_t$ contains a non-trivial unitary direct factor, say $\mathbb{U}_t$. Restricting again the F filtration and passing to the corresponding graded sheaf, we obtain a non-trivial splitting sub Higgs bundle $(F_{\mathbb{U}_t},0)$ of $(F_{\mathbb{W}_t},\theta_t)$, contradicting the generic maximality of the Higgs field $\oplus\theta^{p,q}$. Hence ρ is zero and $\mathbb{W}_t=\mathbb{W}$.

Thus $\mathbb{W}$ is rigid as a sub local system of $\mathbb{V}$, and by Lemma 2.1 $\mathbb{W}$ is defined over $\bar{\mathbb{Q}}$.

By assumption $\mathbb{V}=\mathbb{V}_{\mathbb{R}}\otimes\mathbb{C}$ and the complex conjugation defines an involution ι on $\mathbb{V}$. Let $\bar{\mathbb{W}}$ denote the image of $\mathbb{W}$ under ι. Then $\bar{\mathbb{W}}$ has again generically isomorphic Higgs maps $\theta^{p,q}$, for $p>0$. If $\bar{\mathbb{W}}\neq\mathbb{W}$, repeating the argument used above, one obtains a non-trivial map $\bar{\mathbb{W}}\to\mathbb{U}$, contradicting again the generic maximality of the Higgs field.

So enlarging the real algebraic number field K, if necessary, we may assume that $\mathbb{W}=\mathbb{W}_K\otimes_K\mathbb{C}$ for some $\mathbb{W}_K\subset\mathbb{V}_K$. The polarization on $\mathbb{V}$ restricts to a non-degenerated intersection form on $\mathbb{V}_K$. Choosing for $\mathbb{U}_K$ the orthogonal complement of $\mathbb{W}_K$ in $\mathbb{V}_K$ we obtain a splitting

$$\mathbb{V}_K=\mathbb{W}_K\oplus\mathbb{U}_K$$

inducing over $\mathbb{C}$ the one in (2.2.1). □

Corollary 2.3. *Let $f : A \to Y$ be a family of abelian varieties reaching the Arakelov bound, and let $\mathbb{U}_1$ be a unitary local system with*

$$R^1 f_*(\mathbb{Z}_{A_0}) \otimes \mathbb{C} = \mathbb{V} \oplus \mathbb{U}_1,$$

such that $\mathbb{V}$ has a maximal Higgs field.

i) *Then this splitting can be defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$, and is it orthogonal with respect to the polarization.*
ii) *The splitting $\mathcal{E}nd(\mathbb{V}) = \mathbb{W} \oplus \mathbb{U}$ constructed in Lemma 1.4 can be defined over $\bar{\mathbb{Q}} \cap \mathbb{R}$, and is orthogonal with respect to the polarization.*

3. Splitting over $\mathbb{Q}$

Lemma 3.1. *Assume that $S \neq \emptyset$ and let $\mathbb{V}_{\mathbb{Q}}$ be a $\mathbb{Q}$-variation of Hodge structures of weight k. Assume that over some number field K there exists a splitting*

$$\mathbb{V}_K = \mathbb{V}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K = \mathbb{W}_K \oplus \mathbb{U}_K$$

where $\mathbb{U} = \mathbb{U}_K \otimes_K \mathbb{C}$ is unitary and where the Higgs field of $\mathbb{W} = \mathbb{W}_K \otimes_K \mathbb{C}$ is maximal. Then $\mathbb{W}$, $\mathbb{U}$ and the decomposition $\mathbb{V} = \mathbb{W} \oplus \mathbb{U}$ are defined over $\mathbb{Q}$. Moreover, $\mathbb{U}$ extends to a local system over Y.

Proof. Let $\mathbb{T}$ be a sub local constant system of $\mathbb{W}$. Writing

$$\Big(\bigoplus_{p+q=k} F^{p,q}_{\mathbb{T}}, \bigoplus_{p+q=k} \theta_{p,q}\Big),$$

for the Higgs bundle corresponding to $\mathbb{T}$, the maximality of the Higgs field for $\mathbb{W}$ implies that the Higgs field for $\mathbb{T}$ is maximal, as well. In particular, for all $s \in S$ and for $p > 0$ the residue maps

$$\mathrm{res}_s(\theta_{p,q}) : F^{p,q}_{\mathbb{T},s} \longrightarrow F^{p-1,q+1}_{\mathbb{T},s}$$

are isomorphisms. By [14] the residues of the Higgs field at s are defined by the nilpotent part of the local monodromy matrix around s. Hence if γ is a small loop around s in Y, and if $\rho_{\mathbb{T}}(\gamma)$ denotes the image of γ under a representation of the fundamental group, defining $\mathbb{T}$, the nilpotent part $N(\rho_{\mathbb{T}}(\gamma)) = \log \rho_{\mathbb{T}}(\gamma)$ of $\rho_{\mathbb{T}}(\gamma)$ has to be non-trivial

We may assume that K is a Galois extension of $\mathbb{Q}$. The Galois group $\mathrm{Gal}(K/Q)$ acts on $\mathbb{V}_K$. For $\sigma \in \mathrm{Gal}(K/Q)$ consider the composite

$$p : \sigma(\mathbb{U}_K) \to \mathbb{V}_K = \mathbb{W}_K \oplus \mathbb{U}_K \to \mathbb{W}_K,$$

and the induced map $\sigma(\mathbb{U}) = \sigma(\mathbb{U}_K) \otimes_K \mathbb{C} \to \mathbb{W}$.

Let γ be a small loop around $s \in S$, and let $\rho_{\mathbb{U}}(\gamma)$ and $\rho_{\sigma(\mathbb{U})}$ be the images of γ under the representations defining $\mathbb{U}$ and $\sigma(\mathbb{U})$ respectively. Since $\mathbb{U}$ is unitary and unipotent, the nilpotent part of the monodromy matrix $N(\rho_{\mathbb{U}}(\gamma)) = 0$. This being invariant under conjugation, $N(\rho_{\sigma(\mathbb{U})}(\gamma))$ is zero, as well as $N(\rho_{p(\sigma(\mathbb{U}))}(\gamma))$.

Therefore $p(\sigma(\mathbb{U})) = 0$, hence $\sigma(\mathbb{U}) = \mathbb{U}$, and $\mathbb{U}$ is defined over $\mathbb{Q}$. Taking again the orthogonal complement, one obtains the $\mathbb{Q}$-splitting asked for in 3.1.

Since $N(\rho_{\mathbb{U}}(\gamma)) = 0$, the residues of $\mathbb{U}$ are zero in all points $s \in S$, hence $\mathbb{U}$ extends to a local system on Y. □

Corollary 3.2. *Suppose that $S \neq \emptyset$. Then the splittings in Corollary 2.3 can be defined over $\mathbb{Q}$.*

4. $\mathbb{Z}$-structures and isogenies

Proposition 4.1. *Let $f : A \to Y$ be a family of abelian varieties with unipotent local monodromies around $s \in S$, and reaching the Arakelov bound. If $S \neq \emptyset$ there exists a finite étale cover $\pi : Y' \to Y$ with*

i) $\pi^*(R^1 f_*(\mathbb{Z}_{A_0})) \supset \mathbb{V}'_{\mathbb{Z}} \oplus \mathbb{Z}^{g-g_0}$, $\quad \pi^*(R^1 f_*(\mathbb{Z}_{A_0})) \otimes \mathbb{Q} = (\mathbb{V}'_{\mathbb{Z}} \oplus \mathbb{Z}^{g-g_0}) \otimes \mathbb{Q}$, *where $\mathbb{V}'_{\mathbb{Z}}$ is an $\mathbb{Z}$-variation of Hodge structures of weight* 1 *with maximal Higgs field.*

ii) $\mathcal{E}nd(\mathbb{V}'_{\mathbb{Z}}) \supset \mathbb{W}'_{\mathbb{Z}} \oplus \mathbb{Z}^{g_0^2}$, $\quad \mathcal{E}nd(\mathbb{V}'_{\mathbb{Z}}) \otimes \mathbb{Q} = (\mathbb{W}'_{\mathbb{Z}} \oplus \mathbb{Z}^{g_0^2}) \otimes \mathbb{Q}$, *where $\mathbb{W}'_{\mathbb{Z}}$ is an $\mathbb{Z}$-variation of Hodge structures of weight 0 with maximal Higgs field, and where $\mathbb{Z}^{g_0^2}$ is a constant $\mathbb{Z}$-sub local system of type* $(0,0)$.

Proof. i) By 3.2 we already have the $\mathbb{Q}$-splitting

$$R^1 f_*(\mathbb{Z}_{A_0}) \otimes \mathbb{Q} = \mathbb{V}_{\mathbb{Q}} \oplus \mathbb{U}_{1\mathbb{Q}}.$$

A $\mathbb{Z}$-structure on $\mathbb{V}_{\mathbb{Q}}$ and $\mathbb{U}_{1\mathbb{Q}}$ can be defined by

$$\mathbb{V}_{\mathbb{Z}} = R^1 f_*(\mathbb{Z}_{A_0}) \cap \mathbb{V}_{\mathbb{Q}}, \quad \mathbb{U}_{1\mathbb{Z}} = R^1 f_*(\mathbb{Z}_{A_0}) \cap \mathbb{U}_{1\mathbb{Q}}.$$

Obviously

$$\mathbb{V}_{\mathbb{Z}} \otimes \mathbb{Q} = \mathbb{V}_{\mathbb{Q}}, \quad \text{and} \quad \mathbb{U}_{1\mathbb{Z}} \otimes \mathbb{Q} = \mathbb{U}_{1\mathbb{Q}}.$$

Since $\mathbb{U}_1$ is unitary and admits a $\mathbb{Z}$-structure, the monodromy group of $\mathbb{U}_1$ is a finite group. Since the local monodromies of $\mathbb{U}_1$ around S are trivial, $\mathbb{U}_1$ extends to a local system

$$\rho_{\mathbb{U}_1} : \pi_1(Y, p) \to \mathrm{Gl}(U_1),$$

where U_1 is the fibre of $\mathbb{U}_1$ in p. After passing to the finite étale cover of $\pi : Y' \longrightarrow Y$ corresponds to $\rho_{\mathbb{U}_1}$ we obtain a trivial $\mathbb{Z}$-sub local system of $\pi^*(R^1 f_*(\mathbb{Z}_{A_0}))$ of rank $g - g_0$. Together with $\pi^*\mathbb{V}_{\mathbb{Z}}$ we have

$$\pi^*(R^1 f_*(\mathbb{Z}_{A_0})) \supset \pi^*\mathbb{V}_{\mathbb{Z}} \oplus \mathbb{Z}^{g-g_0},$$

such that

$$\pi^*(R^1 f_*(\mathbb{Z}_{A_0})) \otimes \mathbb{Q} = (\pi^*\mathbb{V}_{\mathbb{Z}} \oplus \mathbb{Z}^{g-g_0}) \otimes \mathbb{Q}.$$

ii) By and Lemma 2.3, ii) and by Lemma 3.2 one has a $\mathbb{Q}$-splitting

$$\mathcal{E}nd(\mathbb{V}_{\mathbb{Z}}) \otimes \mathbb{Q} = \mathbb{W}_{\mathbb{Q}} \oplus \mathbb{U}_{\mathbb{Q}},$$

where $\mathbb{U}$ is a rank-g_0^2 unitary local system of (0,0)-type, and where $\mathbb{W}$ has a maximal Higgs field. So ii) follows from the same argument used to prove i). □

Proof of Theorem 0.2. Let Y' be the étale covering constructed in 4.1, ii). So using the notations introduced there,

$$R^1 f'_*(\mathbb{Z}_{A'_0}) \otimes \mathbb{Q} = V'_{\mathbb{Q}} \oplus \mathbb{Z}^{g-g_0} \quad \text{and} \quad \mathcal{E}nd(\mathbb{V}'_{\mathbb{Q}}) = \mathbb{W}'_{\mathbb{Q}} \oplus \mathbb{Z}^{g_0^2}. \tag{4.1.1}$$

The left hand side of (4.1.1) implies that $f' : A' \to Y'$ is isogenous to a product of a family of g_0 dimensional abelian varieties with a constant abelian variety B. By abuse of notations we will assume from now on, that B is trivial, hence $g = g_0$ and $R^1 f'_*(\mathbb{Z}_{A'_0}) \otimes \mathbb{Q} = V'_{\mathbb{Q}}$, and we will show that under this assumption

$f' : A' \to Y'$ is isogenous to a g-fold product of a modular family of elliptic curves.

Let us write $\mathrm{End}(*) = H^0(Y', \mathcal{E}nd(*))$ for the global endomorphisms. As explained in [12], for example, $\mathrm{End}(\mathbb{V}'_{\mathbb{Q}}) = \mathbb{Q}^{g^2}$ is a $\mathbb{Q}$ Hodge structure of weight zero, in our case the Hodge filtration is trivial, i.e. $\mathrm{End}(\mathbb{V}'_{\mathbb{Q}})^{0,0} = \mathrm{End}(\mathbb{V}'_{\mathbb{Q}})$.

If $A_\eta = A' \times_{Y'} \mathrm{Spec}(\mathbb{C}(Y'))$ denotes the general fibre of f', one obtains

$$\mathrm{End}(A_\eta) \otimes \mathbb{Q} = \mathrm{End}(\mathbb{V}'_{\mathbb{Q}})^{0,0} = \mathrm{End}(\mathbb{V}'_{\mathbb{Q}}).$$

By the complete reducibility of abelian varieties there exists simple abelian varieties $B_1, \ldots, B_r$ of dimension g_i, respectively, which are pairwise non isogenous, and such that $\mathbb{A}_\eta$ is isogenous to the product

$$B_1^{\times \nu_1} \times \cdots \times B_r^{\times \nu_r}.$$

Moreover, since $\mathbb{V}$ has no flat part, none of the B_i can be defined over $\mathbb{C}$. Let us assume that $g_i = 1$ for $i = 1, \ldots, r'$ and $g_i > 1$ for $i = r' + 1, \ldots, r$.

Let us write $D_i = \mathrm{End}(B_i) \otimes \mathbb{Q}$. By [10], p. 201, Each D_i is a division algebra of finite rank over $\mathbb{Q}$ with center K_i. Let us write d_i^2 for the rank of D_i over K_i and e_i for the rank of K_i over $\mathbb{Q}$. Hence $e_i \cdot d_i^2$ is the rank of D_i over $\mathbb{Q}$.

By [10], p. 202, or by [3], p. 141, either $d_i \leq 2$ and $e_i \cdot d_i$ divides g_i, or else $e_i \cdot d_i^2$ divides $2 \cdot g_i$. In both cases, the rank $e_i \cdot d_i^2$ is smaller than or equal to $2 \cdot g_i$. If $i \leq r'$, hence if B_i is an elliptic curve, not defined over $\mathbb{C}$, we have $e_i = d_i = 1$.

Writing $M_{\nu_i}(D_i)$ for the $\nu_i \times \nu_i$ matrices over D_i, one finds ([10], p. 174)

$$\mathrm{End}(A_\eta) \otimes \mathbb{Q} = M_{\nu_1}(D_1) \oplus \cdots \oplus M_{\nu_r}(D_r)$$

hence

$$g^2 = \dim_{\mathbb{Q}}(\mathrm{End}(A_\eta) \otimes \mathbb{Q}) = \Big(\sum_{i=1}^{r} \nu_i \cdot g_i\Big)^2 = \sum_{i=1}^{r} (e_i \cdot d_i^2) \cdot \nu_i^2 \leq$$
$$\sum_{i=1}^{r'} \nu_i^2 + \sum_{i=r'+1}^{r} \nu_i^2 \cdot 2 \cdot g_i \leq \sum_{i=1}^{r} \nu_i^2 \cdot g_i^2.$$

Obviously this implies that $r = 1$ and that $g_1 \leq 2$. If $g_1 = 1$, we are done. In fact, the isogeny extends all over $Y \setminus S$ and, since we assumed the monodromies to be unipotent, $E := B_1$ is the general fibre of a semi-stable family of elliptic curves. The Higgs field for this family is again maximal.

Before excluding the case $g_1 = 2$, let us compare this construction with the one in Remark 1.3, for $Y = \mathbb{P}^1$ and in Remark 1.5 in general.

Remark 4.2. Writing $\mathbb{T}'$, $\mathbb{L}'$ and $\mathbb{S}'$ for the pullbacks of $\mathbb{T}$, $\mathbb{L}$ and $\mathbb{S}$, respectively, one finds as remarked in 1.5 decompositions

$$\mathcal{E}nd(\mathbb{V}') = \mathbb{T}' \otimes \mathbb{T}'^{\vee} \otimes \mathbb{L}' \otimes \mathbb{L}'^{\vee}.$$

and

$$\mathcal{E}nd(\mathbb{L}') = \mathbb{L}' \otimes \mathbb{L}'^{\vee} = \mathbb{C} \oplus \mathbb{S}'.$$

So $\mathbb{T}' \otimes \mathbb{T}'^\vee$ is a direct factor of $\mathbb{V}$ of rank g_0^2, by construction unitary. Its complement $\mathbb{T}' \otimes \mathbb{T}'^\vee \otimes \mathbb{S}'$ has again a maximal Higgs field, hence no global section, and $\mathbb{T}' \otimes \mathbb{T}'^\vee = \mathbb{C}^{g_0^2}$. Altogether one finds

$$\mathcal{E}nd(\mathbb{V}') = \mathcal{E}nd(\mathbb{T}') = \bigoplus^{g_0^2} \mathcal{E}nd(\mathbb{L}').$$

Using this description, one finds again that $\mathbb{A}_\eta$ can not be the product of different non-isogenous abelian varieties.

End of the proof of 0.2. It remains to exclude the case that $g_1 = 2$, and that $e_1 \cdot d_1^2 = 4$. If the center K_1 is not a totally real number field, e_1 must be lager than 1 and one finds

I. $d_1 = 1$ and $D_1 = K_1$ is a quadratic imaginary extension of a real quadratic extension of $\mathbb{Q}$.

If K_1 is a real number field, looking again to the classification of endomorphisms of simple abelian varieties in [10] or [3], one finds that e_1 divides g_1, hence the only possible case is

II. $d_1 = 2$ and $e_1 = 1$, and D_1 is a quaternion algebra over $\mathbb{Q}$.

The abelian surface B_1 over $\mathrm{Spec}(\mathbb{C}(Y'))$ extends to a non-isotrivial family of abelian varieties $B' \to Y'$, smooth outside of S and with unipotent monodromies for all $s \in S$. This family again has a maximal Higgs field, and thereby the local monodromies in $s \in S$ are non-trivial. As we will see below, in both cases, I and II, the moduli scheme of abelian surfaces with the corresponding type of endomorphisms turns out to be a compact subvariety of the moduli scheme of polarized abelian varieties, a contradiction.

I. By [3], Example 6.6 in Chapter 9, there are only finitely many g_1 dimensional abelian varieties with a given type of complex multiplication, i.e. with D_1 a quadratic imaginary extension of a real number field of degree g_1 over $\mathbb{Q}$.

II. By [3], Exercise (1) in Chapter 9, there is no abelian surface for which D_1 is a totally definite quaternion algebra. Hence it remains to show the compactness of the moduli scheme of abelian surfaces B with a totally indefinite $D_1 = \mathrm{End}(B) \otimes \mathbb{Q}$, i.e. of the moduli scheme of false elliptic curves. Such abelian surfaces and there moduli have been studied in [16], and there it is shown, that the moduli scheme is a compact Shimura curve. This also follows from the construction of the moduli scheme in [3], §8 in Chapter 9, as a quotient of the upper half plane $\mathcal{H}$, and from [17], Chapter 9. □

Proof of Proposition 0.1. Let $\pi : Y' \to Y$ be an étale covering, $S' = \pi^{-1}(S)$ and let $g : E \to Y'$ be a semi-stable family of elliptic curves, reaching the Arakelov bound, and with $E_0 = g^{-1}(Y' \setminus S')$ smooth, for example the family occurring in 0.2. Hence $\mathbb{L}_\mathbb{Z} = R^1 g_* \mathbb{Z}_{E_0}$ is a $\mathbb{Z}$-variation of Hodge structures of weight one and of rank two. Writing $\mathcal{L}$ for the $(1,0)$ part, we have an isomorphism

$$\tau_{1,0} : \mathcal{L} \longrightarrow \mathcal{L}^{-1} \otimes \Omega_{Y'}(\log S'). \tag{4.2.1}$$

Since $\mathcal{L}$ is ample, $\Omega_{Y'}(\log S')$ is ample, hence the universal covering of $U' = Y' \setminus S'$ is the upper half plane $\mathcal{H}$. One obtains a commutative diagram

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\tilde{\varphi}} & \mathcal{H} \\ {\scriptstyle\psi'}\downarrow & & {\scriptstyle\psi}\downarrow \\ U' & \xrightarrow{j} & \mathbb{C} \end{array}$$

where j is given by the j-invariant of the fibres of $E_0 \to U'$, where ψ is the quotient map $\mathcal{H} \to \mathcal{H}/\mathrm{Sl}_2(\mathbb{Z})$, and where $\tilde{\varphi}$ is the period map. Since the tangent sheaf of the period domain $\mathcal{H}$ is given by the sheaf of homomorphisms from the $(1,0)$ part to the $(0,1)$ part of the variation of Hodge structures, the isomorphism $\tau_{1,0}$ implies that $\tilde{\varphi}$ is a local diffeomorphism. Note that the Hodge metric on the Higgs bundle corresponding to $\mathbb{L}_{\mathbb{Z}}$ has logarithmic growth at S and bounded curvature by Schmid [13]. By the remark after Prop. 9.8 together with the remark after Prop. 9.1 in [15]

$$\tilde{\varphi} : \tilde{U}' \to \mathcal{H}$$

is a covering map, hence an isomorphism.

Since $\tilde{\varphi}$ is an equivariant isomorphism with respect to the $\pi_1(U', *)-$action on $\tilde{U}'$ and the $P\rho_{\mathbb{L}_{\mathbb{Z}}}(\pi_1(U', *))-$action on $\mathcal{H}$, the homomorphism

$$\rho_{\mathbb{L}_{\mathbb{Z}}} : \pi_1(U', *) \to P\rho_{\mathbb{L}_{\mathbb{Z}}}(\pi_1(U', *)) \subset \mathrm{PSl}_2(\mathbb{Z})$$

must be injective, hence an isomorphism.

This in turn implies, that

$$\varphi : U' \to \mathcal{H}/\rho_{\mathbb{L}_{\mathbb{Z}}}(\pi_1(U', *))$$

is an isomorphism, $\rho_{\mathbb{L}_{\mathbb{Z}}}(\pi_1(U', *)) \subset \mathrm{Sl}_2(\mathbb{Z})$ is of finite index, and $E_0 \to U'$ is a modular curve. □

5. Family of curves and Jacobians

Let Y be a curve, let $h : Z \to Y$ be a semi-stable non-isotrivial family of curves of genus $g > 1$, smooth over V, and let $f : J(Z/Y) \to Y$ be a compactification of the Neron model of the Jacobian of $h^{-1}(V) \to V$. Let us write S for the points in $Y - V$ with $f^{-1}(y)$ singular and Γ for the other points in $Y \setminus V$, i.e. for the points y with $h^{-1}(y)$ singular but $f^{-1}(y)$ smooth. As usual we write $U = Y \setminus S$.

Let us first consider families of curves over $\mathbb{P}^1$. S.-L. Tan [19] has shown that $h : Z \to \mathbb{P}^1$ must have at least 5 singular fibres, hence

$$\#S + \#\Gamma \geq 5.$$

Moreover, he and Beauville [1] gave examples of families with exactly 5 singular fibres for all $g > 1$. In those examples one has $\Gamma = \emptyset$.

On the other hand, for $A = J(Z/Y)$ and for the ample sheaf $F^{1,0}$ introduced in (0.0.1) the Arakelov inequality (0.0.2) implies that

$$2 \cdot g_0 \leq 2 \cdot \deg(F^{1,0}) \leq g_0 \cdot (\#S - 2),$$

hence $\#S \geq 4$. For $\#S = 4$, the family $f : J(Z/Y) \to Y$ reaches the Arakelov bound, hence by 0.2 it is isogenous to a product of a constant abelian variety with a product of modular elliptic curves, again with 4 singular fibres. By [2] there are just 8 types of such families, among them the universal family $E(3) \to X(3)$ of elliptic curves with a level 3-structures.

Being optimistic one could hope, that those families can not occur as families of Jacobians, hence that there is no family of curves $h : Z \to P^1$ with $\#S = 4$. However, a counterexample has been constructed by E. Kani in [8].

Example 5.1. Let B be a fixed elliptic curve, defined over $\mathbb{C}$. Consider the Hurwitz functor $\mathcal{H}_{B,N}$ defined in [8], i.e. the functor from the category of complex schemes to the category of sets with

$$\mathcal{H}_{B,N}(T) = \{f : C \to B \times T;\ f \text{ is a normalized covering of degree } N \\ \text{and } C \text{ a smooth family of curves of genus 2 over } T\}.$$

The main result of [8] says that for $N \geq 3$ this functor is represented by an open subscheme $V = H_{B,N}$ of the modular curve $X(N)$ parameterizing elliptic curves with a level N-structure.

The smooth universal curve $\mathcal{C} \to H_{B,N}$ extends to a semi-stable curve $Z \to X(N)$ whose Jacobian is isogenous to $B \times E(N)$. Hence writing S for the cusps, $J(Z/X(N))$ is smooth outside of S, whereas $Z \to X(N)$ has singular semi-stable fibres outside of $H_{B,N}$. Theorem 6.2 in [8] gives an explicit formula for the number of points in $\Gamma = X(N) \setminus (H_{B,N} \cup S)$.

Evaluating this formula for $N = 3$ one finds $\#\Gamma = 3$. For $N = 3$ the modular curve $X(3)$ is isomorphic to $\mathbb{P}^1$ with 4 cusps. So the number of singular fibres is 4 for $J(Z/\mathbb{P}^1) \to \mathbb{P}^1$ and 7 for $Z \to \mathbb{P}^1$.

We do not know whether similar examples exist for $g > 2$. For $g > 7$ the constant part B in Theorem 0.2 can not be of codimension one. In fact, the irregularity $q(Z)$ of the total space of a family of curves of genus g over a curve of genus q satisfies by [20], p. 461, the inequality

$$q(Z) \leq \frac{5 \cdot g + 1}{6} + g(Y).$$

If $J(Z/Y) \to Y$ reaches the Arakelov bound, hence if it is isogenous to a product

$$B \times E \times_Y \dots \times_Y E,$$

one finds

$$\dim(B) \leq \frac{5 \cdot g + 1}{6}.$$

As explained in [5] it is not known, whether for $g \gg 2$ there are any curves C over $\mathbb{C}$ whose Jacobian is isogenous to the product of elliptic curves. We are even asking for families of curves whose Jacobian is isogenous to the product of the same elliptic curve, up to a constant factor.

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