Universität Duisburg-Essen, Düsseldorf SS 07

Forschungsseminar: Brauer groups and Artin stacks

Organisation: Jochen Heinloth, Marc Levine, Stefan Schröer
Place and time: Thursdays, 14-16 Uhr ct, T03 R03 D26 (?)

Overview

The goal of this workshop is to understand two theorems of de Jong on Brauer groups of schemes.

The first result states the equality $\text{per}(\alpha) = \text{ind}(\alpha)$, where $\alpha = [A]$ is the cohomology class of an Azumaya algebra A over a function field K of a surface [dJ1]. The *period* $\text{per}(\alpha)$ and *index* $\text{ind}(\alpha)$ are the two basic numerical invariants of such classes. In general, these numbers are related, but not equal. de Jong's theorem triggered some conjectures concerning their precise relations.

The second result states that on a quasiprojective scheme X, any torsion class α from the étale cohomology group $H^2(X, \mathcal{O}_X^\times)$ comes from an Azumaya algebra [dJ2]. This question goes back to Grothendieck, and Gabber apparently gave an unpublished proof for it. De Jong found a completely different proof, using the machinery of Artin stacks. Namely, he translated the existence problem for an Azumaya algebra on X to an existence problem for a certain locally free sheaf on an Artin stack. Here the Artin stack in question is nothing but a gerbe $\mathfrak{G}$ representing the given torsion class α. Surprisingly, the latter problem can be solved with scheme-theoretic methods.

The program of the seminar is divided into two parts. In the first part (Talks 1-5) we recall the basic results on Azumaya algebras and the Brauer group which are needed to understand the first result. The proof of the first result is then a beautiful geometric construction, reminiscent of the proof of the proof of the theorem of Graber-Harris-Starr as given by de Jong and Starr, which we studied last semester.

In the second part of the seminar we want to see how the language of $\mathbb{G}_m$-gerbes helps in working with questions on the Brauer group. We will start with a short introduction, explaining the necessary background on stacks. The plan below is roughly the same as the talks at the workshop on vector bundles, but maybe it is helpful to get a different exposition of this theory. Of course we could also skip parts of this, if everybody feels comfortable with the topic. In particular talks 7 and 8 could be put into one session. Finally we use this theory to explain de Jong's proof of Gabber's theorem. In the end we might have some time left to go back to the question raised in the first part of the seminar and study a generalization of the result, given by de Jong and Starr, who used stack theoretic methods to generalize the theorem of de Jong and at the same time prove some further results on the existence of sections of rational fibrations.

We should use coherent notations throughout. Try to use curly letters $\mathcal{M}, \mathcal{X}$ or $\mathfrak{M}, \mathfrak{G}$ for stacks and straight ones for schemes. A gerbe on a scheme should be denoted by either $\mathcal{X} \to X$ or $\widetilde{X} \to X$, depending on the calligraphic capabilities of the speaker.

Program

Talk 1: Azumaya algebras and Brauer group 05.04 Alexander Schmitt

Discuss the basic structure theory of Azumaya algebras and the definition of the Brauer group. Reference: [Gr], Section 1,2,3,5.

First, explain the definition of Azumaya algebras and Brauer groups on a ringed space or ringed site, and the coboundary map

$$H^1(X, \mathrm{PGL}(n, \mathcal{O}_X)) \to H^2(X, \mathcal{O}_X^\times)$$

in terms of cocycles. Then explain the structure results for Azumaya algebras over fields. Finally, give their consequences for Azumaya algebra over schemes.

Talk 2: Period and index 12.04. Dzmitry Doryn

Discuss period and index for Brauer classes. References: [Li],[FD],[dJ1].

First, explain period $\mathrm{per}(\alpha)$ and index $\mathrm{ind}(\alpha)$ for Galois cohomology classes $\alpha \in H^i(\mathrm{Spec}(K), \mathcal{F})$ as in [Li], Section 2.1. Next, give Wedderburns structure result that an Azumaya algebra over a field is a matrix algebra over a division algebra ([FD], Theorem 1.11). Use loc. cit., Theorem 4.4. to prove that the index of an Azumaya K-algebra $A = \mathrm{Mat}(n, D)$ is the square root of $\dim_K(D)$. Finally, state de Jongs' result, and discuss the examples in [dJ1], Section 1 showing that the period of an Azumaya algebra usually differs from its index.

Talk 3: Elementary transformations and deformations 19.04 Georg Hein

Discuss elementary transformations and the obstruction to infinitesimal deformations for Azumaya algebras. Reference: [dJ1], Section 2 and 3.

Talk 4: Degeneration 26.04

Discuss Sections 4 and 5 from [dJ1]. Section 5 deals with Azumaya algebras on reducible surfaces. In section 4, de Jong constructs a degeneration of a given finite separable covering of a surface into several components. Alex Küronya

Talk 5: de Jong's proof for period=index 03.05

Discuss de Jong's proof. Reference: [dJ1]. First outline his strategy, as in the Introduction of loc. cit. Then go through the proof in Section 6 and 7. Stefan Kukulies

Talk 6: Definition of stacks and algebraic stacks 10.05

This talk should define what a stack is (preferably as sheaf of categories, but that is a matter of taste). Give some examples (I like the classifying stack of torsors BG and quotient stacks $[X/G]$, and of course any scheme defines a stack.). Note that $\mathrm{Hom}_{\mathrm{Stacks}}(X, \mathcal{M}) = \mathcal{M}(X)$ as categories. Manuel Blickle

Define the notion of representable morphism of stacks. Check at least one example in detail, e.g. $X \mapsto [X/G]$. Comment on the diagonal morphisms of a stack $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$, i.e. for $x : X \to \mathcal{M}, y : Y \to \mathcal{M}$ we have $(X \times Y) \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M} = X \times_{\mathcal{M}} Y = \mathrm{Isom}(x, y)$. So this morphism knows all about automorphism groups of objects. Define Artin stacks.

References: The approach via fibred categories: [LMB] Chap. 3, the original source [DM] is much more concise, if you need details, look at A. Kreschs's homepage (follow the link at http://www.uni-essen.de/∼mat907/Heinloth.htm). There you can find proofs for all of the above results. A nice overview can also be found in the first section of [EHKV].

Approach via sheaves of categories: See [Go] or (http://www.uni-essen.de/∼hm0002/dalklein.dvi resp. http://www.uni-essen.de/∼hm0002/stacks.pdf). (Choose the approach you find easier to follow. Since the references for stacks are not always immediately accessible, ask the organizers if you need more information.)

Talk 7: Gerbes and the relation to the Brauer group 24.05

Pedro-Luis del Angel

Another example of stacks are gerbes over schemes. Define what a gerbe is and what an $A-$gerbe for an abelian group A is. Or do the latter explicitely in the example $\mathbb{G}_m$. Explain why Azumaya algebras define $\mathbb{G}_m$-gerbes. Isomorphism classes of A-Gerbes on X are classified by elements in $H^2(X, A)$ (either do this with cocycles, or in Giraud's way, checking that if one takes this as a Definition of H^2 one gets a δ-functor). A gerbe is neutral if it has a section. Any $\mathbb{G}_m$-gerbe is an Artin stack. This is probably not enough for one talk. On the other hand we need to get used to the notions, so take your time.

References: [Li] Section 2.2, [dJ2] Section 2. The precise references to Giraud's book are in Lieblich's article. Another cocyclic reference is in http://www.uni-essen.de/∼hm0002/stacks.pdf, which contains further references. Again an overview is in section 3 of [EHKV].

Talk 8: Coherent sheaves on stacks and Azumaya algebras 31.05 Florian Ivorra

Define what a (quasi)-coherent sheaf on an Artin-stack is ([LMB] 13.2), for our purposes you may use the definition of cartesian sheaf from the start). Give examples, in particular note that this is the same as to give bundles together with descent data on a presentation ([LMB] 12.8.2). Decompose the category of sheaves on a $\mathbb{G}_m$-gerbe according to weights. ([Li]. Prove Proposition 15.4 of [LMB], which turns out to be surprisingly useful. Explain the relation between vector bundles of weight 1 and Azumaya algebras [Li].

Talk 9: Gabber's Theorem for affine schemes 14.06 Franziska Heinloth

This is Corollary 3.1.5.2 in Lieblich's thesis [Li]. With the technique presented in the previous talks this is very easy, thus one should take some time to explain the details. (E.g. one should try not to skip results like Lemma 3.1.5.3). The time should suffice to give a complete proof. (There is a more detailed version of Lieblich's thesis - a printed version is available.)

Talk 10: Gabber's Theorem for quasi-projective schemes 21.06 Kay Rülling

This is de Jong's article [dJ2]. We already did section 2 of the article, so only the last part is left and this looks very well written.

Talk 11: Existence of rational sections 28.06 Stefan Schröer

The theorem of the article [dJS] also resumes the questions discussed in the seminar last term, discussing the existence of rational sections of fibrations. State the main theorem of the article (Section 2.1) and give the applications (Sections 4 and 3).

Talk 12: Almost proper GIT-Stacks 05.07 Hô Hai Phùng oder Jochen Heinloth?

This contains the proof of the theorem (section 2) in [dJS], mixing GIT and stacks in a beautiful way in order to generalize the deformation argument in part 1 of our seminar.

Bibliography

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