Universität Duisburg-Essen SS 2006

Arbeitsgemeinschaft
'Über eine Vermutung von A. J. de Jong'

Organisation: G. Böckle, H. Esnault, J. Heinloth Di 14:15–16:15, T03R03D26

Programme of the seminar

In 1996 A. J. de Jong made a conjecture on Galois representations of arithmetic fundamental groups of normal varieties over finite fields and gave a proof in the 2-dimensional case. More recently, Drinfeld has shown how to deduce from it some conjectures by Kashiwara about the behavior of perverse sheaves of D-modules in characteristic zero under direct image. Moreover Gaitsgory has given a conditional proof of de Jong's conjecture for representations of all characteristics different from 2, and thereby completed the proof of Kashiwara's conjectures. (Gaitsgory's proof relies on a theory of $\overline{\mathbb{F}_\ell((T))}$-adic étale sheaves including a formalism of trace formulas analogous to that of $\overline{\mathbb{Q}}_\ell$-adic étale sheaves over schemes in which ℓ is invertible; quite conceivably such a theory does exist; but details have not been worked out.)

In the first half, we plan to read de Jong's original article [deJ] together with a fundamental article [Dr1] by Drinfeld which is crucial for de Jong's proof in the 2-dimensional case. In the second half, we will try to understand the Drinfeld's conditional proof [Dr2] of Kashiwara's conjectures.

11.04.06 **1. An introduction to de Jong's conjecture**
State Deligne's conjecture of the algebraicity of Frobenius eigenvalues of lisse $\bar{\mathbb{Q}}_\ell$-sheaves and de Jong's analog. Give first consequences and reformulations of de Jong's analog. Discuss the abelian case. Reduction to the unramified situation. Idea of proof using automorphic forms. ([deJ] §1, 2, and 4.7–4.10)
Gebhard Böckle

18.04.06 **2. Universal deformation rings**
Consequences of de Jong's conjecture for universal deformation rings, [deJ], §3.
Henrik Russell

25.04.06 **3. Classical and adelic modular forms and Hecke operators**
The aim of this talk is to present the reformulation of the definition of modular forms in terms of functions on adele groups and to describe the Hecke operators in this setting. A good reference is [Bu], §3.2 and 3.6. Perhaps also the cuspidality condition could be discussed. In the end one should define unramified automorphic forms for GL_2 over function fields with Λ-coefficients, [deJ], p. 13, and Hecke operators on these, and give the proof of [deJ], Prop. 4.7.
Christian Rohde

02.05.06 **4. Translation to geometry**
The aim is to explain §1 of [Dr1]. Perhaps it is best to start with the two identifications for Bun_2 and Flag_2 on p. 89, 5th line from bottom, so that automorphic forms are immediately recognized as functions on a geometric object. (One of the indentification can be found in [Ha], §2.) Obviously Bun_2 is a quotient of Flag_2. Drinfeld's first observation is that it's easy to define a Hecke eigenform on Flag_2. This is described on the first page of §1 (the correspondence of the two descriptions should be taken on faith.) The 'multiplicity one' result

($\dim U = 1$) can be deduced for the corresponding result for Whittaker models, cf. [Bu], §3.5. The problem is thus "reduced" to showing that if the eigenvalues come from a Galois representation, then the eigenform 'descends' to Bun_2.

The definition of $r(D)$, middle of page 88, is important. Perhaps with a bit of 'handwaving' (explaining what happens in words) it's not necessary to write down formulas (4) and (5), and proceed directly to formula (6) for the function f on Flag_2 which one tries to descend to Bun_2. In the remainder of the talk, one should present as much as possible of the reduction steps given in §1.

Xiaotao Sun

09.05.06 **5. The values of f via the trace formula**

Perhaps this talk could sketch the argument for GL_1 given in [Lau], §1, as a motivation. Then continue with Drinfeld: The first main (but elementary) result is Prop. 2.1 which interpretes the values of f at a pair $(\mathfrak{L}, \mathfrak{A})$ via a trace formula. The proof of the second half of the proposition should be taken from [deJ], 4.13. After this, the strategy of Drinfeld is to investigate the variation of $f(\mathfrak{L}, \mathfrak{A})$ by defining a projective space over a family parametrizing $\mathfrak{A}$ (while $\mathcal{L}$ is fixed). The value of $f(\mathfrak{L}, \mathfrak{A})$ can be read of from the (derived) direct image of a suitable sheaf on the relative projective space. The precise definitions are given at the bottom of p. 93 and top of p.94, the result needed on the direct image is Prop. 2.4. Below Prop. 2.4, Drinfeld gives more details of the proof. The family defined above is an open part of a projective variety. The analysis of its boundary will be largely the content of the following talks. The talk could end by stating the vanishing cycle theorem I and giving a sketch of its proof.

Phùng Hô Hai

16.05.05 **6. Completion of the proof modulo some vanishing (cycles) theorems**

Recall the geometric setting from the previous talk and add some more notation in order to be able to describe a suitable local system on the boundary Δ_n. Assuming the second vanishing cycle theorem and Deligne's theorem given in the appendix, complete the proof of the main Theorem of Drinfeld and de Jong. (have a look at [deJ], 4.14.) In the end perhaps review Drinfeld's argument and what we have done so far. (The theorem by Nagata and Zariski is in [SGA1], X.3. A reference outside SGA 4, pp. 491–519, for Deligne's result mentioned in the introduction and used first on p.95, lines -8 to -6, is [SGA1], X.Cor.1.4 combined with [SGA1], XI.1.)

Björn Buth

23.05.06 **7. Fill in the gaps**

This should definitely contain a proof of Deligne's theorem given in the appendix to Drinfeld - and with the comments of [deJ], 4.17–4.20. In the remaining time, it would be great if something intersting could be said about the second vanishing cycle theorem. Maybe the main ideas and the geometry (unfortunately there is again more notation).

Hélène Esnault

30.05.06 **8. BBDG: Perverse sheaves**

Definition of perverse sheaves, perverse images, simple perverse sheaves ([BBDG], mainly out of section 2), simple perverse sheaves of geometric origin ([BBDG], 6.2.4). Then list the theorems which cover Kashiwara's conjectures [Dr2], section 1.2, 1-2-3.

Franziska Heinloth

13.06.06 **9. de Jong implies Kashiwara**

The talk explains Drinfeld's result [Dr2], Main Theorem 1.4. Presentation of the Theorem and Leitfaden of the proof as explained in section 2. One needs an explanation of why irreducible lisse sheaves on a complex open form a stack over $\mathbb{Z}$. This is where then the ℓ in de Jong's conjecture shows up, as the residue characteristic of closed points which enter via the Frobenius of the finite residue field the Galois action on this stack. Then one needs why, if we know that bad loci are constructible, it implies Theorem 1.4.

Jochen Heinloth

20.06.06 **10. Constructibility**

Section 3 of [Dr2]. It deals with the behavior of constructible sheaves of A modules both by going from A to its field of fraction and by specialization from A to a residue field.

Georg Hein

27.06.06 **11. Good models**

Sections 4 and 5 of [Dr2]. We are back to [BBDG] for the notion of good model with respect to the choice of finitely many constructible complexes (see section 4). For the "same" notion on vanishing cycles, Drinfeld quotes Deligne, [SGA7II Deligne]. All we need here is to believe that models are good (with the suitable notions) after suitable localization.

Manuel Blickle

04.07.06 **12. Proof**

Section 6 of [Dr2]. For the first 2 conjectures, preservation of semi-simplicity by direct images and hard Lefschetz, Drinfeld uses [BBDG] and what was discussed earlier. For the 3-rd one, he uses a Theorem of Gabber, mentioned in Beilinson-Bernstein: proof of Jantzen's conjectures, which I do not have here. Am not sure how much work it is to explain the statement. If fine, we do it, if too hard, we skip it.

Hélène Esnault

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