Seminar Sommersemester 2008

Canonical compactifications of moduli spaces for abelian varieties

For a long time, it was not clear how to define a moduli theoretic compactification of the moduli space of abelian varieties with only 'minor' singularities along the boundary. A major break through in this direction is the work [Alex] of Alexeev. Its review by J. Kollár contains a concise but very readable historical overview, and points out the main new ideas. A further improvement is provided by recent work of Olsson, [Ols2], who added log structures to Alexeev's moduli.
A first conceptual step of Alexeev is to consider no longer the moduli $\mathcal{A}_{g,d}$ of abelian varieties A of fixed dimension g with a polarization of degree d, but moduli $\mathcal{AP}_{g,d}$ of pairs (A, Θ) where $A \in \mathcal{A}_{g,d}$ and Θ is an ample divisor defining the polarization. In the principally polarized case $d = 1$ the two agree. In general Alexeev's moduli $\mathcal{AP}_{g,d}$ form a covering of $\mathcal{A}_{g,d}$ with the fibers being the choices of possible divisors. The key new insight of Alexeev is to describe the boundary of his moduli, as a moduli space $\overline{\mathcal{AP}}_{g,d}$. It parameterizes semiabelic 'pairs' which are triples $(G \curvearrowright P, \theta)$ where G is a semiabelian variety of dimension g, P is a projective variety with an ample divisor θ of degree d, G acts on P with finitely many orbits, and P has a few further nice properties. (The 'pair' is (P, θ).)
Alexeev's construction has two drawbacks. The first is that in the non-principally polarized case it might be nice to remove the choice of ample divisor at the end of the day. The second is that the moduli constructed by Alexeev have several components, and only one of them contains the open substack of abelian varieties (with a choice of ample divisor). So a natural compactification is obtained by singling out this one 'main' component.
Solutions to both problems are presented by M. Olsson in [Ols2]. Adding log-structures into Aleexev's moduli, he is able to directly single out the main component in the second problem. He gives a solution to the first problem, as well. However it is not clear to the authors of this program **(or is it?)** to what extend Olsson's approach is based on Alexeev's moduli $\overline{\mathcal{AP}}_{g,d}$. The program will focus on the principally polarized case.

Our main sources are [Alex] by V. Alexeev, [Ols2] by M. Olsson and [Mum2] by D. Mumford. The reader will note that after the initial lectures 1-4 much of the program is centered around the construction of families describing semiabelic pairs (and an additional log-structure). This should not come as a surprise since, when studying the representability of a moduli functor, such families arise at three crucial points: (a) to provide smooth charts of the stack, (b) to describe infinitesimally all deformations, (c) to describe all possible degenerations.

lecture 1: $\mathcal{A}_g$ the moduli space of abelian varieties
10.04.08, Gebhard Böckle
We review Mumford's construction of moduli given in [Mum1], §1–6.

lecture 2: $\overline{\mathcal{A}_g}$ Mumford's compactification of $\mathcal{A}_g$
17.04.08, Eckart Viehweg
Here the compactification $\mathcal{A}_g$ constructed by Mumford and its properties are explained.

lecture 3: logarithmic algebraic geometry
24.04.08, Franziska Heinloth
The aim of this talk is to present the material of sections 1–3 of Kato's article [Kato]. (We are only interested in fine log structures)

- define (pre) log structures
- construction of the associated log structure of a pre-log structure
- morphisms of (pre) log structures
- f_* and f^*
- Examples 1.5
- log differentials, explain why $(0, 1 \otimes a) = d\log(a)$.
- definition of charts and their existence (étale locally)
- definition of log-smooth/étale, and the basic examples 3.7.1 and 3.7.2
- characterisation of log-smoothness/étaleness (Theorem 3.5)
- connection of log-smoothness and $\omega^1_{X/Y}$.

(The numbers refer to Kato's article.) The main emphasis should be on the examples and definitions, because the time is limited to 90 minutes.

lecture 4: Artin stacks
8.05.08, Georg Hein
We recall the definition of Artin stacks and explain why they naturally appear in the theory of moduli spaces.

lecture 5: Semiabelic pairs and linearization of torus actions
15.05.08, Alex Küronya
This talk is based on [Alex]. Start by giving the precise definition of *seminormal variety* and of *semiabelic pair* (also in the relative situation) from [Alex], §1.1. Explain briefly why in the case that the toric part is trivial, a semiabelic pair of degree 1 is simply an abelian variety. (cf. [Alex], Cor 3.0.9.)
The main part of the talk should cover Ch.4 of Alexeev's work. The key result is Thm. 4.3.1. It says that if a torus T acts on a pair (P, L) with P a 'nice' proper scheme and L ample on P, there is a cover $(\tilde{P}, \tilde{L})$ with a simultaneous action by the torus T and its character group such that $(\tilde{P}, \tilde{L})/X \cong (P, L)$ **and** such that now T acts on $\tilde{P}$ **and** on $\tilde{L}$, i.e., the action of T on $\tilde{L}$ is linear. (this goes back to constructions of Mumford from [Mum2]).
Along the way there appears a version of the theorem of the square for semiabelian varieties (4.1.6, 4.1.7 4.1.18) which probably needs to be stated without proof, and its consequence 4.1.22 which is needed in proving the desired correspondence.

lecture 6:
29.05.08, Manuel Blickle
The main aim of the present lecture is to reveal the usefulness of the content of the previous one. Again the source is [Alex]. Abelic pairs $(G \curvearrowright P, L)$ with a linear torus action possess a useful combinatorial description – in spirit similar to that of toric varieties. In this talk, the semiabelian variety G and a polarization of its abelian quotient A are fixed. Moreover one should stick to principally polarized A.
The talk starts with §5.2 of Alexeev. It seems very useful to describe the entire section in some detail. Many later results are based on it. The second half of the talk should cover §5.3 where the non-linearized case is described using the results from the previous talk. It culminates in Thm. 5.3.8. Unfortunately the precise combinatorial description is kind of a mess. It requires definitions from 1.1.16 to 1.1.29, §2.1 and §2.2 and §5.1. For simplicity,

it should be assumed that $\rho : |\widetilde{\Delta}| \to X_{\mathbb{R}}$ is injective. – For injective ρ, Alexeev's rather technical description of the (co-)homology of cell complexes with values in local systems should simplify considerably. Perhaps it is enough to give the audience an idea of the combinatorics without laying it all out in front of us.
At the very end, the extensions Thm 5.4.1 and 5.4.3 to the case of semiabelic pairs, $(G \curvearrowright P, \theta)$, could be mentioned.

lectures 7 and 8: Degenerating abelian varieties over complete rings
5.06.08 and 12.06.08, Juan Cerviño & Kay Rülling
These two talks should explain the content of [Mum2]. At the end it would be good to come back to Alexeev's work and integrate (in some way) [Alex], §5.6.

lecture 9: $\overline{\mathcal{AP}}_{g,d}$ is a moduli stack
19.06.08, Stefan Kukulies
This lecture presents §5 of Alexeev's work [Alex]. The main result is Thm. 5.10.1. We suggest to give a detailed proof of properness, which is Theorem 5.7.1. This should be the main part of the talk. In the remaining time. the speaker could go over (without going into many details) the further results needed for the representability of a functor by an algebraic stack. A brief review of Artin's axioms might help. Whatever time is left could be spend on saying something about §5.9 in which charts of the stack are constructed. The construction 'is as in' [Mum2], cf. talks 7 and 8.

lecture 10: Olsson's standard family
26.06.08, Philipp Gross, Felix Schüller, Holger Partsch
Cover 3.1 from [Ols2]. As in talk 6, the combinatorics are not light to digest. Perhaps this is intrinsic to the rather difficult subject. The new feature is the appearance of log structures. Olsson's central combinatorial object is the monoid H_S. To give some insight into H_S, examples seem more useful than proofs. We recommend $X = \mathbb{Z}$, for the paving S obtained by the quadratic function $a(n) = n^2$, and $X = \mathbb{Z}^2$ for the pavings S obtained by the quadratic functions $a_t(n, m) = n^2 + tmn + m^2$ for $t \in \{-1, 0, 1\}$. It is not recommended to give proofs for 3.1.4–3.1.9 but to point them out in the example. To understand the notation at the beginning of §3.1 it may be inevitable to consult §2.1. A comparison to Alexeev's combinatorics as described in talk 6 would be wonderful (but is not expected).
Once H_S is 'understood', the family in 3.1.10 is easy to describe. Perhaps it is useful to prove 3.1.11 and 3.1.13. The content of 3.1.14-3.1.21 was essentially covered in talks 5 and 6. So one should be able to move rapidly from 3.1.13 to 3.1.22, i.e., to defining the standard family. It would be nice if the talk ends by stating Olsson's moduli problem described in §3.6 and explaining how the the standard families constructed in 3.1.22 fit. (they describe the strata of certain degeneration types over fields and over local Artin rings).
Note: In 3.1.1 there appears twice a Q which should be replaced by $X_{\mathbb{R}}$.

lecture 11: Deformations and automorphisms
3.07.08, Christian Liedtke
It would be good if the talk was given by someone with a good conceptual understanding of deformation theory.
We search for an audacious speaker, willing to cover 3.2–3.4 from [Ols2] in one talk. We suggest to give precise statements of the results on automorphisms as given in §3.2 and §3.4, i.e. of Prop. 3.2.2 and Prop 3.4.2, omitting proofs. The remaining (hopefully longer) portion of the talk should explain the deformation theory of §3.3. The key result is Prop.

3.3.3. It would be good to briefly recall some generalities on deformation theory and a theory of obstructions and then to explain parts of the proof.
Alternativly, a introduction to deformation theory should be given.

lecture 12: Versal families and Alexeev's main component
10.07.08, Xiatao Sun
Cover 3.5 in [Ols2]. Describe Olsson's versal family over $\mathcal{W}_3$ and the action of the group $\mathcal{G}$. Olsson stops short of redoing §5.9 from [Alex]. After explaining the objects, the main emphasis should be on relating W_3 (at least its infinitesimal versions) to the normalization $\widetilde{\mathcal{Q}}$ of Alexeev's main component $\mathcal{Q}$: 3.5.15-3.5.20. Since the families constructed here cover Olsson's space $\mathcal{K}_g$ this indicates that one might expect a morphism $\mathcal{K}_g \to \widetilde{\mathcal{Q}}$.

lecture 13: The morphism $\widetilde{\mathcal{Q}} \to \mathcal{K}_g$
17.07.08, N.N.
Again, the source is [Ols2]. A morphism $\widetilde{\mathcal{Q}} \to \mathcal{K}_g$ is defined and discussed in 3.7.4 and 3.7.5. This should be fully explained, since it might explain the occurrence of log structures in Olsson's work. Thereafter, the speaker can either cover §3.8 on approximation, or explain the proof of the main Thm. 3.6.2 of Ch.3, assuming the results from §3.8.
Alternatively, we might discuss the next program.

Gebhard Böckle (`boeckle@iem.uni-due.de`) Georg Hein (`georg.hein@uni-due.de`)
The material can be found at: `http://www.uni-essen.de/∼mat903/sem/ss08/`

References

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