Forschungsseminar
Shimura varieties and their canonical models

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SS 09

Introduction

Shimura varieties arise in the effort to generalize the Kronecker-Weber theorem on abelian extensions of the field of rational numbers to arbitrary number fields. Kronecker's Jugendtraum from the 19th century does this for imaginary quadratic fields where the abelian extensions are generated by special values of certain elliptic functions. The study of elliptic functions later led to the introduction of modular curves, which are quotients $\Gamma\backslash\mathbb{H}$ of the upper half-plane $\mathbb{H}$ by an arithmetic subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Modular curves are the simplest examples of Shimura varieties.

The modern theory of Shimura varieties started in the 1950s with the theory of abelian varieties with complex multiplication, developed by Taniyama, Shimura and Weil, and the study of more general quotient spaces than $\Gamma\backslash\mathbb{H}$. These quotient spaces, which were given the name Shimura varieties by Langlands in 1976, were studied in the 1960s by Mumford, Shimura and others, and led to Shimura's proof of the existence of canonical models for certain classes of Shimura varieties, i. e. models of Shimura varieties, which are initially defined over the field of complex numbers, over number fields.

In the 1970s, in his two fundamental papers [DeTS] and [DeVS], Deligne reformulated Shimura's theory in the language of abstract reductive groups and extended Shimura's results on canonical models to more classes of Shimura varieties. The aim of this seminar is to learn the basics about Shimura varieties in the language introduced by Deligne in [DeVS] and about their canonical models. The main reference and guide will be [MiISV].

At the beginning there will be some preliminaries on algebraic groups (especially reductive ones) and hermitian symmetric domains. Then we will proceed to investigate the relation between certain $G(\mathbb{R})^+$-conjugacy classes of homomorphisms $\mathbb{S} \to G$, with G a real algebraic group, moduli of variations of Hodge structures and hermitian symmetric domains. Afterwards we define locally symmetric spaces, connected Shimura varieties and finally Shimura varieties in the adèlic description.

We will investigate certain classes of Shimura varieties and their moduli interpretation (if there is one), in particular we look at Siegel modular varieties, Shimura varieties of PEL - and of Hodge type and Shimura varieties arising from quaternion algebras over a totally real numberfield. Then we will define the canonical model of a Shimura variety in the sense of Deligne and we will prove their uniqueness.

It follows a short excursion into the world of CM Abelian varieties, including a proof of the fundamental theorem of complex multiplication (Taniyama-Shimura). And we use this to prove the existence of a canonical model in the case where the Shimura varieties admit a description as moduli spaces. Finally we will see that canonical models also exist in the case where the Shimura varieties arise from a quaternion algebra over a totally real number field, which not always admit a moduli description.

We will not discuss the general Theorem of Milne, Borovoi, that any Shimura variety admits a canonical model. Since this subject is quite technical, *there are lots of examples throughout the whole seminar and speakers are strongly encouraged to discuss them carefully even on the price of being short in the proofs.*

The Talks

1. Algebraic groups (16.04.09).

This is a warm up talk concerning some properties of algebraic groups and their Lie algebras, which will be frequently used throughout the seminar. The emphasis should be on examples rather than proofs. Assume throughout that the ground field k has characteristic 0.

The following things from [MiAGr] should be explained. First, mention the Jordan decomposition Theorem 10.1 and define semisimple and unipotent elements, Definition 11.5, and unipotent groups, Def. 11.25 using 11.24(c).

Then define semisimple and reductive groups, Definition 11.30, and mention the characterization of these notions via the (unipotent) radical, Proposition 11.31. The definition of the radical is on the same page, see also Theorem 11.26. Discuss the examples $\mathrm{SL}_n, \mathrm{Sp}_n$ and GL_n using Proposition 15.13 and Example 11.33.

Now we come to Lie algebras. Give their definition 12.3 and explain Remark 12.5. Discuss some of the examples 12.9–12.16. Then explain the adjoint map Ad on page 114, define the adjoint group G^{ad} to be $G/Z(G)$ and define a group to be adjoint iff its center is trivial (i.e. iff $G^{\mathrm{ad}} = G$). Conclude with Proposition 14.8. that if G is connected, then G^{ad} is the image of Ad. Give Theorem 14.23 and conclude, that for G semisimple G^{ad} is a direct product of simple groups.

Finally, define the derived group G^{der}, Definition 11.9, and discuss the structure of reductive groups, Theorem 15.1, by presenting the exact sequences [MiISV, p.303]. Discuss the example of GL_n on the same page and GSp_n from [MiISV, p.315].

Doan Trung Cuong

2. The classical theory of Shimura varieties (23.04.09).

Eckart will talk about the classical theory of Shimura varieties as moduli spaces of abelian varieties with extra structures, as studied by Mumford in [MuNSP] and [MuFAV], and their relevance in algebraic geometry. See also [BLCAV].

Eckart Viehweg

3. Hermitian symmetric domains and conjugacy classes (30.04.09).

In this talk we collect some facts about hermitian symmetric domains and describe their connection with $G(\mathbb{R})^+$-conjugacy classes of maps $u : U_1 \to G$, where G is a real adjoint algebraic group.

Go through [MiISV] page 267-278: Especially recall the notions of almost-complex structure, hermitian manifold, hermitian symmetric space, shortly explain the examples 1.1, (a) - (c) and the classification on page 271. Give Example 1.2, Theorem 1.3 (and the comments following the proof of 1.3). Proposition 1.7, Example 1.8, Theorem 1.9 (if time permits, it would be nice to say some words on the proof), Example 1.10. Explain *Cartan involution* on page 274 (this is important so be careful here.) Give Example 1.15, Theorem 1.16, Example 1.17 and Proposition 1.20. Finally state and prove Theorem 1.21 as detailed as possible.

Le Dang Thi Nguyen

4. Hodge structures and hermitian symmetric domains (07.05.09).

The aim of this lecture is to recall some notions about Hodge structures and to relate them to hermitian symmetric domains.

Go through [DeVS] 1.1.1 - 1.1.10 or [MiISV] page 280-284. The minimum to be explained is: Hodge structures (HS), example of HS of type $(-1,0),(0,-1)$, Tate HS, HS from geometry, especially those attached to Abelian varieties, the Hodge filtration, HS as representations of $\mathbb{S}$ (define $\mathbb{S}$), the weight homomorphism, μ_h, polarization of a HS, Flag varieties, and Variations of hodge structures (VHS).

Next for G a real algebraic group we investigate certain $G(\mathbb{R})^+$-conjugacy classes of homomorphisms $\mathbb{S} \to G$. This is [DeVS] 1.1.11-1.1.16. Motivate why one might want to consider such classes as in [DeVS, 1.1.11] (maybe complemented by [DeHCAV, §6, Proof of 6.1]). Then go through 1.1.12 - 1.1.14 and explain the proof of 1.1.14 as detailed as possible (see also [MiISV, Thm 2.14]). Give Corollaire 1.1.16 and Corollaire 1.1.17. Show how to deduce 1.1.17 from [MiISV, Thm 1.21] (which was proved in the previous talk) as explained under A.(1) and A.(2) in the proof of 1.1.17.

If time permits, give a short preview to the connection of 1.1.13 $(\alpha) - (\gamma)$ with the definition of a Shimura datum [DeVS, 2.1.1].

Abolfazl Mohajer

5. Locally symmetric varieties and connected Shimura varieties (14.05.09).

Define arithmetic subgroups ([MiISV, page 287]) and congruence subgroups ([MiISV, page 295]). Then define locally symmetric varieties [MiISV, page 293] and list the following properties (from [MiISV]) for them: Proposition 3.1, Theorem 3.3, (b), Example 3.4, Theorem 3.12, Theorem 3.14, Corollary 3.16, Theorem 3.21.

Now define the ring of finite adèles and for G an algebraic group explain the topology of $G(\mathbb{A}_f)$ and give an example of a compact open subgroup in it (p. 295-297). Define a connected Shimura datum as in Definition 4.22 and shortly explain why it is the same notion as in Definition 4.4. Give Proposition 4.8 and say some words about the proof (it follows from Theorem 1.21, which was explained in talk 2). Go through the definition of a connected Shimura variety 4.10-4.12 and carefully explain Example 4.14. Finally state and prove Proposition 4.18 and just state Proposition 4.19.

Dima Doryn

6. Shimura varieties and Siegel modular varieties (28.05.09).

Give Definition 5.5 from [MiISV], Example 5.6. Corollary 5.8. Notice that Proposition 5.9. follows from [DeVS, 1.1.14], which was proved in the 3. talk. State and prove Lemma 5.13 and give Definition 5.14. Say a word on morphisms (Definition 5.15, Theorem 5.16). State the additional Axioms SV4-SV6 (p. 311-312) and discuss example 5.24. Now explain the example of the Siegel modular variety, page 314-320 as detailed as possible (skip the part on Abelian varieties). Explain the moduli interpretation on page 320.

Georg Hein

7. Shimura varieties of Hodge - and of PEL type and of type A_1 (04.06.09).

Discuss the examples of Shimura varieties of Hodge - and of PEL type in [MiISV] page 320-328 and of simple Shimura varieties of type A_1 (page 332) as detailed as possible, including the moduli interpretation if it exists and make clear why there is no such interpretation in the case (c) of Shimura varieties of type A_1. If time permits you could also say some words about Shimura varieties of Abelian type.

Stefan Kukulies

8. Canonical models and their uniqueness (18.06.09).

The aim of this talk is to define the notion of a canonical model of a Shimura variety and to show that if they exist, then they are unique (up to isomorphism).

You may start with some motivating words as in [MiISV] page 332 - 333 ('Where we are headed'). Then jump to page 340 and give a short 'Review of class field theory' and the 'Convention for the (Artin) reciprocity map' on page 341. Then go through pages 343-346 and define a model, the reflex field (for the definition of the Weyl group in Lemma 12.1, see [MiAGr, 17.7]), special points, r_x and canonical models. At least examples 12.4 (c) and (d) and Example 12.7 should be done (other examples only if time allows). Then show the uniqueness of the canonical models (page 348-349) as detailed as possible. (For this last part see also [DeTS, 5.1- 5.5.], note that a 'modèle faiblement canonique' is defined in [DeTS, 3.13] and it is canonical iff $E = E(G, X)$.)

Andre Chatzistamatiou

9. The Fundamental Theorem of Complex Multiplication (25.06.09).

The aim of this talk is to present a proof of Theorem 3.10 in [MiFTCM].The talk is independent of all talks before.

From [MiFTCM] explain the notions of CM-algebra, CM pair, reflex field and reflex norm (page 2-3), further the notion of an Abelian variety with complex multiplication (page 4). Then state and prove Theorem 2.1 (the Shimura-Taniyama formula) and Theorem 3.2 (the Fundamental theorem in terms of ideals) and Theorem 3.10 (the fundamental theorem in terms of idèles). Be short on any additional material, which you need from the 1. Chapter. For us the most important one is Theorem 3.10, so try to explain this as detailed as possible. (See also [MiISV, page 333-342].)

Manuel Blickle

10. Existence of canonical models for Siegel modular varieties etc. (02.07.09).

Using the fundamental theorem of complex multiplication from the previous talk, we see in this talk that Siegel modular varieties, Shimura varieties of PEL- and of Hodge type admit a canonical model and we also see, that this canonical models still have a moduli interpretation.

Go through chapter 14 in [MiISV]. Be short on the preliminary part 'Descent of the base field' and 'Review of local systems and families of Abelian varieties'. Then try to explain as detailed as possible the case of Siegel modular varieties. In particular a proof of Proposition 14.10 should be given as in [MuNSP, §2, 2nd Prop.]. (Notice that what Mumford calls the Hodge group is now called the Mumford-Tate group see [DeHCAV, §3].) For the proof of Proposition 14.12 notice that we proved the fundamental theorem of complex multiplication in previous talk not only for a CM field but for any CM-algebra. Thus Proposition 14.12 follows indeed from what we saw in the previous lecture. (See 12.11 to see how to deduce 14.12 from the FToCM.) Then give the proof of the existence as on page 353-354. (Remark that the existence can also be shown by using GIT, see [DeTS, 4.16].)

Stefan Schröer

11. Existence of canonical models for simple Shimura varieties of type A_1 (16.07.09).

Simple Shimura varieties of type A_1 are those arising from quaternion algebras over a totally real number fields and were discussed in talk 6. To find a canonical model for them is particularly interesting since, if B ramifies in some places of F, then the weight is not defined over $\mathbb{Q}$ and thus the Shimura varieties arising in this way don't admit a moduli interpretation, which means that the method of constructing a canonical model from talk 9 does not work. The aim of this talk is to discuss the existence of a canonical model in this case, i.e. to explain Théorème 6.4 in [DeTS]. If time allows there could be some explanations on the strategy of proving the general theorem (due to Milne, Borovoi), that Shimura varieties always exist.

To prove Théorème 6.4 in [DeTS] it seems to be necessary to introduce the notion of a weak canonical model ('modèle faiblement canonique') otherwise the essential ingredient Proposition 5.10 can not be stated. Since Deligne uses in [DeTS] a slightly different language (in every respect) than our main reference [MiISV], the notion of a weak canonical model should be defined in the following way (see also [DeVS, 2.2.5]): In the situation of [MiISV, Def 12.8] choose a number field $L \supset E(G,X)$ and denote by $L(x)$ the composition field $L.E(x)$. Then a model $M_K(G,X)$ of $\mathrm{Sh}(G,X)$ over L is called weakly canonical iff the conditions in [MiISV, Def. 12.8] hold for $E(x)$ replaced by $L(x)$ (and $E(x)^{\mathrm{ab}}$ replaced by $L(x)^{\mathrm{ab}}$). (In particular a weak canonical model for $L = E(G,X)$ is a canonical model, by [MiISV, 12.3,(b)]). Also modify [MiISV, Def 12.10] appropriately. Show that (in the above situation) if $M(G,X)$ is a weak canonical model over L then the map

$$\mathrm{Sh}(T,x) \longrightarrow \mathrm{Sh}(G,X) = M(G,X) \times_{\mathrm{Spec}\, L} \mathrm{Spec}\, \mathbb{C}$$

induced by $(T,x) \subset (G,X)$ is defined over $L(x)$, cf. [MiISV, page 346-347] (this is the Definition of a weak canonical model in [DeTS]). Note, that weak canonical models are unique (the proof is the same as in talk 7). Now it should be possible to explain the main ingredients in the proof of Théorème 6.4 in [DeTS]. Also give the very interesting Remarque 6.6.

It should at least be mentioned at the end that Milne and Borovoi proved the existence of canonical models for all Shimura varieties (see [MiAACSV, 7.2]). The strategy (or history) seems to be the following (see also [MiISV, page 356]): Langlands made a conjecture on the action of automorphisms of $\mathbb{C}$ on Shimura varieties, callled Conjecture C (see [MiShCSV, 4. Conjecture C]). Milne and Shih proved in [MiShCSV, Prop. 7.14] that Conjecture C implies the existence of canonical models for all Shimura varieties. On the other hand they introduced an a priori weaker conjecture, called Conjecture C_0, which is a statement about the action of automorphisms of $\mathbb{C}$ only for connected Shimura varieties. Using (a variant of) the result of Deligne ([DeVS, 2.7.18]), that $\mathrm{Sh}(G,X)$ has a canonical model iff the connected Shimura variety $\mathrm{Sh}^0(G^{\mathrm{der}}, X^+)$ has a canonical model, Milne and Shih prove in [MiShCSV, Prop 9.4], that Conjecture C_0 implies Conjecture C. Then in [MiAACSV] Milne proves Conjecture C_0 using that it is true for Shimura varieties of type A (Deligne) and ideas of Borovoi.

Kay Rülling

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