Forschungsseminar
p-divisible Groups and crystalline Dieudonné theory

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WS 09/10

Introduction

Let A be an Abelian scheme over some base scheme S. Then the kernels $A[p^n]$ of multiplication by p^n on A (p a prime) form an inductive system of finite locally free commutative group schemes over S, $(A[p^n])_n$, via the inclusions $A[p^n] \hookrightarrow A[p^{n+1}]$. This inductive system is called the *p-divisible (or Barsotti-Tate) group of A* and is denoted $A(p) = \varinjlim A[p^n]$ (or $A(\infty)$ or $A[p^\infty]$). If none of the residue characteristics of S equals p, it is the same to give $A(p)$ or to give its Tate-module $T_p(A)$ viewed as lisse $\mathbb{Z}_p$-sheaf on $S_{\text{ét}}$ (since in this case all the $A[p^n]$ are étale over S). If S has on the other hand some characteristic p points, $A(p)$ carries much more information than just its étale part $A(p)^{\text{ét}}$ and in fact a lot of informations about A itself. For example we have the following classical result of Serre and Tate: Let S_0 be a scheme on which p is nilpotent, $S_0 \hookrightarrow S$ a nilpotent immersion (i.e. S_0 is given by a nilpotent ideal sheaf in $\mathcal{O}_S$) and let A_0 be an Abelian scheme over S_0. Then the Abelian schemes A over S, which lift A_0 correspond uniquely to the liftings of the p-divisible group $A_0(p)$ of A_0. It follows from this that ordinary Abelian varieties over a perfect field k admit a canonical lifting over the ring of Witt vectors $W(k)$.

To investigate these p-divisible groups of an Abelian scheme further, it is convenient to work just with an abstract category of p-divisible groups and try to describe their general properties. In general a p-divisible (or Barsotti-Tate) group over a scheme S is an inductive system $(G_n)_n$ of finite locally free commutative group schemes G_n over S, such that G_n is isomorphic to $\mathrm{Ker}(G_{n+1} \xrightarrow{p^n} G_{n+1})$. Now one of the most important theorems on p-divisible groups is the following:

Theorem 1 (Tate, de Jong)**.** *Let S be an integral normal scheme with function field K and G and H p-divisible groups on S. Then the natural map*

$$\mathrm{Hom_S(G,H)} \xrightarrow{\simeq} \mathrm{Hom_K(G_K,H_K)}$$

is an isomorphism.

Applied to the p-divisible groups of Abelian schemes one obtains as consequences for example: If R is a discrete valuation ring of characteristic $p > 0$ and A is an Abelian variety over the generic point of R, then $A(p)$ tells us when A has good or semistable reduction (or neither of both). Also when F is a finitely generated field over $\mathbb{F}_p$ and A and B are Abelian varieties over F, then

$$\mathrm{Hom}(A,B) \otimes \mathbb{Z}_p \cong \mathrm{Hom}(A(p), B(p)).$$

Theorem 1 was proved by Tate in 1966 in the case $\mathrm{char}(K) = 0$. The case $\mathrm{char}(K) = p > 0$ was proved by de Jong 30 years later using the classification of p-divisible groups via Dieudonné crystals, which was developed in the mean time by Dieudonné, Manin, Fontaine, Grothendieck, Messing, Mazur, Berthelot, Breen, ...

The first aim of the seminar is to understand p-divisible groups, the Theorem of Serre-Tate from the first paragraph above and its application to ordinary Abelian varieties. Then to understand the main methods used in the proof of Theorem 1, mainly in the characteristic $p > 0$ case. Let us give the logic of the proof of Theorem 1 we are following:

- Reduce to the case $S = \operatorname{Spec} R$, with R a complete discrete valuation ring with algebraically closed residue field. (Talk 5)
- Following Tate we prove the Theorem in the case $\operatorname{char}(K) = 0$. One essential ingredient here, is that a p-divisible group over K corresponds to a $\operatorname{Gal}(\bar{K}/K)$-action on $\mathbb{Z}_p^h$, some h. (Talk 5.)
- Classify p-divisible groups over a perfect field k of characteristic $p > 0$ as follows (Dieudonné, Manin, Fontaine): There is an equivalence of categories

$$M : (p-\text{div-groups}) \xrightarrow{\simeq} (\text{Dieudonné modules over } k),$$

$$G \mapsto M(G) = \operatorname{Hom}_k(G, CW).$$

  Here CW is the sheaf of Witt covectors (Talk 6) and a Dieudonné module over k is a module over $D_k = W(k)[F, V]$, which is free and finite as a $W(k)$-module. (Talk 7)

- Extend the previous classification to an equivalence of p-divisible groups over a perfect valuation ring R (i.e. a valuation ring with surjective Frobenius) and Dieudonné modules over R. (Talk 8)
- (Berthelot-Breen-Messing) For an arbitrary ring R of characteristic p define the crystaline Dieudonné functor

$$\mathbb{D} : (p-\text{div-groups over } R) \longrightarrow F-\text{Crystals}(R/\mathbb{Z}_p),$$

$$G \mapsto \mathcal{E}\text{xt}^1_{R/\mathbb{Z}_p}(G, \mathcal{O}_{A/\mathbb{Z}_p}),$$

  which satisfies the following properties:

  (i) $\mathbb{D}$ commutes with arbitrary base change.

  (ii) If R is perfect (either a field or a valuation ring), then $\mathbb{D}(G)$ equals the Frobenius twist of $M(G)$ constructed above.

  (iii) If $\pi : A \to \operatorname{Spec} R$ is an Abelian scheme, then $\mathbb{D}(A(p)) = R^1\pi_{\text{cris}*}\mathcal{O}_{A/\mathbb{Z}_p}$.

  In particular for R as in (ii), $\mathbb{D}$ is an equivalence of categories. (Talk 10)

  (Notice, that if R has a p-basis, then $\mathbb{D}$ is fully faithful. But we won't use this result. Instead we use the following trick:)

- By the first two points it is enough to prove Theorem 1 in the case $R = k[[t]]$ with $\bar{k} = k$, $\operatorname{char}(k) = p > 0$. Let G and H be p-divisible groups over R. Then

$$\operatorname{Hom}_R(\mathbb{D}(G), \mathbb{D}(H)) \xrightarrow{\simeq} \operatorname{Hom}_K(\mathbb{D}(G)_K, \mathbb{D}(H)_K)$$
$$\Longrightarrow \operatorname{Hom}_R(G, H) \xrightarrow{\simeq} \operatorname{Hom}_K(G_K, H_K).$$

  This Step uses that $\mathbb{D}$ is an equivalence over the perfect closures of R and K and that it is compatible with base change. (Talk 11)

- Thus we are reduced to a question about F-crystals over $R = k[[t]]$ (resp. $K = k((t))$). These can be described via finite free modules over $\Omega = W(k)[[t]]$ (resp. over the p-adic completion of $\Gamma = W(k)[[t]][1/t]$) together with a connection, a Frobenius and a Verschiebung. These are particular cases of de Jong's (F, θ)-modules over Ω (resp. Γ). Here F is a Frobenius and θ a derivation compatible with F in a certain way. Thus we are reduced to prove: Let M_1 and M_2 be two (F, θ)-modules over Ω, then the natural map

$$\mathrm{Hom}_{\Omega}^{(F,\theta)}(M_1, M_2) \xrightarrow{\simeq} \mathrm{Hom}_{\Gamma}^{(F,\theta)}(M_1 \otimes_{\Omega} \Gamma, M_2 \otimes_{\Omega} \Gamma)$$

  is an isomorphism. (Talk 11)

- The injectivity of the above map is clear and one easily sees that the surjectivity is implied by the following Theorem of de Jong

  Theorem 2. *Let M be an (F, θ)-module over Ω and $\varphi : M \to \Gamma$ an Ω-linear map, such that*

  (i) $\exists\, \ell \geq 0 : \varphi(F(m)) = p^{\ell} \sigma(\varphi(m)), \forall\, m \in M$ *(σ is the Frobenius on Γ).*

  (ii) $\varphi(\theta(m)) = \frac{d}{dt}\varphi(m), \forall\, m \in M$.

  Then $\varphi(M) \subset \Omega$.

  (Talk 11)

- Prove Theorem 2 by clever calculations. See [dJ-ICM][4., 5.] for details. (Talk 12, 13).

The above mentioned applications of Theorem 1 to the reduction behavior of Abelian schemes and their homomorphisms will be explained in talk 14.

Finally it should be mentioned, that de Jong proves a bit more: he proves that for two crystals E and E' over a DVR of characteristic p, which admits a p-basis, the natural map $\mathrm{Hom}(E, E') \to \mathrm{Hom}(E_K, E'_K)$ is an isomorphism, which by the fully faithfulness of $\mathbb{D}$ implies Theorem 1.

The Talks

1. Finite locally free group schemes (15.10.09).

Aim of the talk: Understand the definition and first properties of finite locally free group schemes over a base scheme S, introduce Frobenius and Verschiebung on them and explain the decomposition $G = G^{\mathrm{mult}} \times G^{\mathrm{bi}} \times G^{\text{ét}}$ over $S = \operatorname{Spec} k$.

Details: Explain as much as possible from [Gro, I, 5.] and [Gro, II, 1.-3.], but at least do the following: Introduce the notion of locally free finite group schemes over a base scheme S. Then explain how to construct the maps $F_{G/S} : G \to G^{(p)}$ and $V_{G/S} : G^{(p)} \to G$ as in [Gro, I, 5.] and [SGA 3-Exp VII_A, 4.2-4.3]. (Be short for $F_{G/S}$ and a bit more detailed for $V_{G/S}$. Assume S affine and G finite locally free over S, then the construction in [SGA 3-Exp VII_A, 4.3] simplifies a lot, see in particular [SGA 3-Exp VII_A, 4.3.3].) Then explain Cartier Duality ([Gro, II, 1.4], see also [Mum-AV, p.132- 135]), the different types of groups as in [Gro, II, 2.1] and the decomposition in [Gro, II, 3.2] for $S = \operatorname{Spec} k$. And of course give examples. (See also [Tate, §1].)

Nguyen Duy Tan

2. p-divisible groups (Barsotti-Tate groups) (22.10.09).

Aim of the talk: Understand the definition and first properties of p-divisible groups (Barsotti-Tate groups) over a base scheme S, and see examples.

Details: Go through [Tate, (2.1)] and [Gro, III, 5-7] (without Dieudonné theory). At least do the following: Give the definition of a p-divisible group as in [Tate, (2.1)] (cf. [Gro, III, 4.2]). Then do [Gro, III, 5.1-5.5] and give the Examples 6.1, 6.3, 6.4, 6.5 in [Gro, III] (6.2 only if time permits.) Finally 7., in particular 7.2.

Felix Schueller

3. The Theorem of Serre and Tate (after Katz (after Drinfeld)) (05.11.09).

Aim of the talk: Explain and prove the theorem of Serre and Tate, which says that the liftings of Abelian schemes over a nilpotent thickening are classified by the liftings of the corresponding p-divisible groups.

Details: Start by stating the following theorem of Grothendieck: *Let A_0 be an Abelian scheme over an affine base S_0 and let $S_0 \hookrightarrow S$ be a nilpotent immersion (i.e. a thickening). Then A_0 lifts to an Abelian scheme A over S (i.e. $A \times_S S_0 = A_0$).* If time allows, it would be nice to give the argument for this theorem as in the proof of [Ill-FGA, Thm 5.23] (see also [Ill-FGA, Rem 5.24, (a)]) admitting the existence of the obstruction theory o as granted. (for a more sophisticated argument see [Ill-DefBT, Prop. 3.2, a)] and [Ill-DefBT, Lem A.1.2.]). Then say shortly what a formal Lie group is ([Tate, (2.2)] or [Gro, III, 6.7.] or [Gro, VI, p.124]). But don't spend too much time with the above since the main part of the talk is the following: Prove the theorem of Serre-Tate as detailed as possible following [Katz-ST, 1.1 - 1.2.1]. (Alternative proofs can be found in [Me, V, 2.3] or [Ill-DefBT, Cor A 1.3].)

Abolfazl Mohajer

4. Ordinary Abelian varieties and their canonical lift (12.11.09).

Aim of the talk: Define ordinary Abelian varieties over a perfect field k of characteristic $p > 0$ and show that they admit a canonical lift over $W(k)$, the ring of Witt vectors of k.

Details: Give the explanations in [Mum-AV, p. 146-147] and define the p-rank of an Abelian variety over k. Then define an Abelian variety A over k to be ordinary iff $p-\mathrm{rank}(A) = g = \dim A$. (cf. [Me, V, Def (3.1)]). Notice that this is equivalent to $A[p](\bar{k}) \cong (\mathbb{Z}/p\mathbb{Z})^g$. Then state [Mum-AV, p.143, Cor] and show that A is ordinary iff $H^1(X, \mathcal{O}_X)$ is semi simple with respect to F^* iff F^* is bijective on $H^1(X, \mathcal{O}_X)$. (See [Mum-AV, p.147-148], you don't have to prove [Mum-AV, p.148 Thm 3], but maybe explain $\mathrm{Lie}(\hat{A}) \cong H^1(A, \mathcal{O}_A)$: [Mum-AV, p.130, Cor 3].)

Now prove that any ordinary Abelian variety over k has a canonical lift to $W(k)$: [Me, V, Thm (3.3)]. (Notice that [Me, V, Thm (2.3)] was proved in the previous talk.) Give [Me, V, Cor (3.4)].

Holger Partsch

5. The Theorem of Tate (19.11.09).

Aim of the talk: Give a first reduction of Theorem 1 and explain Tate's proof for the case $\mathrm{char}(K) = 0$.

Details: State Theorem 1. Explain why $\mathrm{Hom}_S(G, H) \to \mathrm{Hom}_K(G_K, H_K)$ is always injective and that one can assume $S = \mathrm{Spec}\, R$, with R a complete discrete valuation ring with algebraically closed residue field, in particular if $\mathrm{char}(K) > 0$ we may assume $R = k[[t]]$ with $k = \bar{k}$ (we will need this in Talk 11). See [Tate, (4.2), p.181] or [Be, (4.1), p.254]. Now the case $\mathrm{char}(k) = 0$ is obvious, thus we are left with the two cases

- $\mathrm{char}(k) = p$ and $\mathrm{char}(K) = 0$: Tate.
- $\mathrm{char}(k) = p$ and $\mathrm{char}(K) = p$: de Jong.

In the rest of the lecture try to explain the ingredients of Tate's proof [Tate, Thm 4]. You should take [Tate, §3] as a black box and try to explain/sketch/motivate the rest of the proof. In particular explain why $\mathrm{char}(K) = 0$ is crucial in Tate's approach (see [Tate, p.168 after Cor 2]).

Stefan Kukulies

6. Witt vectors and Witt covectors (26.11.09).

Aim of the talk: Introduce Witt vectors and Witt covectors.

Details: Following [Serre-LF, II, §6] introduce the ring of Witt-vectors (of length n), $W(R)$ (resp. $W_n(R)$) for any ring R. Define the universal polynomials S_n and P_n and the maps V, F and R plus relations. You can skip the proofs. Explain how to view $W_n(R)$ as the R-valued points of the Witt scheme of length n, W_n (see [Gro, I. 1], caution the notation in [Gro] is not standard, T there is our V). Then follow [Be, p.227-230] to define the schemes $CW_{r,s}$ and the fppf sheaf CW. Show that CW is a sheaf of groups with F and V and that if R is a perfect ring, then CW is an fppf sheaf of $W(R)$-modules on the big site of $\operatorname{Spec} R$. Finally introduce the group of unipotent Witt covectors $CW^u = \varinjlim(W_n, V) \subset CW$ (this is what Grothendieck calls the ind-scheme $\varinjlim(W_n, T_n)$). They are characterized by the following property: $(\ldots, a_{-n}, \ldots, a_{-1}, a_0) \in CW(R)$ lies in $CW^u(R)$ iff only finitely many a_{-n} are non-zero (see [Fo, Introduction]). Show that CW^u also has F and V and is a $W(R)$-module if R is perfect.

Tommaso Centeleghe

7. Dieudonné theory for finite - and p-divisible Groups over a perfect field (03.12.09).

Aim of the talk: Explain the functor which gives an equivalence between the category of finite p-torsion group schemes over a perfect field k and the category of $D_k = W(k)[F, V]$-modules which are of finite length as $W(k)$-modules. This functor also gives an equivalence between the category of p-divisible groups over k and the category of D_k-modules which are free and of finite type as $W(k)$-modules.

Details: Go through [Gro, II, 4.] and [Gro, III, 5.6], in particular: Define the category of Dieudonné modules and give the construction of D^* [Gro, II, 4.3, 4.4, 4.5] (Use the notation CW^u instead $\mathbb{W}_{\rightarrow}$). Give [Gro, II, Cor 4.2.1] and [Gro, II, Cor 5.2]. If time permits sketch the construction of an inverse of D^* (see [Gro, II, 6.]). Finally translate these results to p-divisible groups ([Gro, III, 5.6]). At the end you can state (without proof) that if A is an Abelian scheme over k, then $D^*(A(p)) = H^1_{\mathrm{crys}}(A/W)$ (see [Ill-DRW, p. 618, Rem 3.11.2]).

Andre Chatzistamatiou

8. Dieudonné theory for finite - and p-divisible Groups over a perfect valuation ring (10.12.09).

Aim of the talk: Extend the results of the previous talk over a perfect valuation ring (i.e. a valuation ring on which the Frobenius is surjective).

Details: Recall the definition of CW from talk 6. Define $M(G)$ ([Be, p. 237]) and show that it is a D_A-module (follows from talk 6). Use as a black box that $M(G) \otimes_{W(A)} W(K) = M(G_K) =$ the classical Dieudonné functor from talk 7 (see [Be, Cor 2.2.2]). Then explain as much as possible from [Be, Thm 3.4.1] and [Be, Cor 3.4.3]. In particular explain G_M representing E_M.

Dzmitry Doryn

9. Crystals (17.12.09).

Aim of the talk: Give the definition of the crystaline site and crystals. Compute them explicitely over a perfect base.

Details: Introduce divided powers (see [Gro, IV, 1.1]). Be short here (if there is not much time maybe don't write all the axioms, but just say "$\gamma_n(x) = \frac{x^n}{n!}$"). Then give the examples [Gro, IV, 1.3, 1)-3)] ([Gro, IV, 1.3, 5), 6)] only if time allows). Define the pd-envelope: [BeOg, 3.19 Thm], but only in the case $A = \mathbb{Z}_p$, $I = (p)$, γ =standard divided power structure. Don't prove the Theorem, just say $\mathcal{D}_{B,\gamma}(J) =$ "$B[x^{[n]} \mid x \in J]$, $\bar{J} =< x^{[n]} \mid x \in J, n \geq 1 >$ ". Notice that if p is nilpotent in B then $D_{B,\gamma}(J) = \operatorname{Spec} \mathcal{D}_{B,\gamma}(J)$ is a thickening of $\operatorname{Spec} B/J$.

Now define the big crystalline site and crystals (see [Dieu I, p. 21 - 22 (until second paragraph)]). Prove [Be, Prop. 4.2.2.] as detailed as possible. Finally give the definition of an F-crystal ([Be, Def. 4.4.1]) and note that an F-crystal over a perfect scheme is mapped to a Dieudonné module under [Be, Prop. 4.2.2].

Le Dang Thi Nguyen

10. Crystalline Dieudonné theory (07.01.10).

Aim of the talk: Sketch crystalline Dieudonné theory, which for any scheme S on which p is nilpotent gives a functor

$$\mathbb{D} : (p-\text{div-groups}/S) \to (F-\text{crystals}/S), \quad G \mapsto \mathcal{E}\text{xt}^1_{S/\mathbb{Z}_p,\text{crys}}(G, \mathcal{O}_{S/\mathbb{Z}_p}).$$

Details: Go through [Dieu I] middle of page 22 to page 31. Leave out everything concerning $\Delta(G)$ (like [Dieu I, Prop 3, p. 27]). In particular explain that $\mathbb{D}$ is compatible with arbitrary base change, explain the Hodge filtration of $\mathbb{D}(G)$, formula (3.1) on page 27 and the comparison with the classical Dieudonné theory from talk 8. It would be also nice to see how to calculate $\mathcal{E}\text{xt}^i_{S/\mathbb{Z}_p}(G, E)_S$ in easy cases (see [Dieu I, p. 25] or [Dieu II, Ex 2.1.9]). Notice that [Dieu I] is an overview article, details (for which won't be enough time in the talk) can be found in [Dieu II]. An overview of more recent results of crystalline Dieudonné theory can be found in [dJ-ICM].

Jilong Tong

11. In case $\operatorname{char}(K) = p$: Theorem 2 implies Theorem 1 (14.01.10).

Aim of the talk: We see that a crystal over $k[[t]]$ ($k = \bar{k}$) is given by a $W(k)[[t]]$-module with a (quasi-nilpotent) connection. This together with the results from talk 8 and 10 is used to show that Theorem 2 implies Theorem 1.

Details: Prove [Be, Prop. 4.2.3] as detailed as possible. Then explain the description of F-crystals over $k[[t]]$ (resp. $k((t))$) with $k = \bar{k}$ as in the first paragraph on page 262 of [Be].

Now recall Theorem 1 and that we reduced it to the case $S = \operatorname{Spec} k[[t]]$, $k = \bar{k}$ (see talk 5). By the talks 8 and 10 we know that $\mathbb{D}$ commutes with base change and it is an equivalence for perfect valuation rings. This enables us to prove [Be, Thm 4.4.3], which shows that Theorem 1 is implied by the corresponding statement for crystals. Define (F, θ)-modules over $k[[t]]$ ([dJ, 4.4 Def, 4.9 Def]) and explain that by the above we are reduced to prove a version of Theorem 1 for (F, θ)-modules over $\Omega = W(k)[[t]]$. (See also [dJ, §3].) Then state Theorem 2 (=[dJ, Thm 9.1]) and show that it implies Theorem 1 (see [dJ, p. 6, §3]).

Juan Marcos Cervino

12. F-modules and Slopes (21.01.10).

Aim of the talk: As a preparation for the proof of Theorem 2 in the next talk, we investigate F-modules over subrings of $\varprojlim_n (W_n(\overline{k((t))})[\frac{1}{[t]}])$ and their slope filtrations.

Details: Go through [dJ, §4, 5]: The notations on page 6-8 should be provided as a handout (also for the next talk), but should nevertheless be explained. Be short in [dJ, 4.1-4.3] and careful in [dJ, 4.4- 4.7]. (For slopes also see [Katz-Slopes, (1.3)], the "fundamental theorem of Dieudonné " is [Manin, p.32, Thm 2.1], but shouldn't be proved.) Definitions 4.8 and 4.9 are not needed in this talk. Use [dJ, Prop. 5.1] as a black box to give a sketch of the proof of [dJ, Prop. 5.5]. Explain [dJ, Cor 5.7, Prop 5.8]. See [dJ-ICM] for an overview.

Stefan Schröer

13. Proof of Theorem 2 (and hence of Theorem 1) (28.01.10).

Aim of the talk: Proof of Theorem 2 (and hence of Theorem 1).

Details: Recall the definition of (F, θ)-modules ([dJ, 4.9]). Then go through [dJ, 6-9] and explain as much as possible of the proof of [dJ, Thm 9.1]. See [dJ-ICM] for an overview. Recall from talk 11 that this theorem implies Theorem 1 in the case $\operatorname{char}(K) = p$.

Archie Karumbidza

14. Applications to Abelian schemes (04.02.10).

Aim of the talk: The reduction behavior of an Abelian scheme over the generic point of a complete discrete valuation ring of characteristic p is determined by its p-divisible group. If A and B are two Abelian schemes over a field which is finitely generated over $\mathbb{F}_p$, then $\operatorname{Hom}(A, B) \otimes \mathbb{Z}_p = \operatorname{Hom}(A(p), B(p))$.

Details: Go through [dJ, §2], this should be done in great detail. (Notice that [dJ, 2.1 Lem (iii)] is wrong and [dJ, 2.2 Def (ii), a)] has to be modified as in [dJ-Err].)

In case there is some time left one could either give a short overview of [Va] and [VaZ] (in particular [Va, Prop 4.1, Rem 4.2] and [VaZ, Cor 4]) or alternatively explain [Dieu III, 4.5.2 Thm] as an application of crystaline Dieudonné theory.

Moritz Kerz

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